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COMPUTER SIMULATION OF MEASUREMENTS OF TUNE, PHASE AND
DAMPING OF BETATRON OSCILLATIONS*

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SUMMARY

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The influence of sextupole magnets and spontaneous changes in machine tune are studied by means of a simulation computer program. The results are relevant to all measurements using the method of beam excitation.

I. Introduction

Machine studies of PEP revealed a number of discrepancies between real machine and its mathematical model. Although some of them if not all can be attributed to misalignment, instabilities and other errors in machine component, still magnitude of the deviations are somewhat bigger than one can expect from realistic expectations of machine errors. One of the greater concerns is the asymmetry of β -function in different sextants. Another is rather strong deviations of measured β^* values both average and instantaneous from requested ones.

There is a strong believe that these discrepancies decrease the luminosity of the ring which makes it most desirable to find the reason or reasons for such machine behavior.

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Many different approaches have been tried for investigation of these adverse effects. In particular, experiment have been carried out¹ to measure betatron phase differences around the PEP storage ring. The beam feedback system is excited with a tracking oscillator which closely follows the machine tune frequency. The latter one fluctuates as much as ± 0.005 due to power supply drift. The signals from two beam position monitors are then detected, limited, filtered and compared for phase.

Two questions arise now: (1) what is the accuracy of phase measurements and (2) what is the influence of numerous sextupole magnets on these measurements. In present note we address these two questions. Section 1 of the note contains derivation of the phase of the detected signal for ideal linear machine. Section 2 deals with the problem of discrete Fourier analysis. Section 3 describes the computer program written to simulate the process of excitation and detecting of betatron oscillations in presence of machine nonlinearity, radiation damping and random change of the lattice frequency. Results of computation are presented in Section 4. Some conclusions relevant for the measurements using the method of beam excitation are drawn in Section 5. In Appendix one can find the derivation of the frequency shift for nonconservative finite difference equation.

II. Particle Oscillations Excited by External Harmonic Force

Let us assume that the length of the pulse of the device which excites the transverse particle oscillations (the kicker) is much shorter than the betatron wave length. The kicker action on a particle can be approximated than by a sequence of the delta-function kicks spaced in time with the revolution period T . If we denote the kicker frequency ω_k

then equation of transverse motion for coordinate z ($z = x$ or y) is

$$\frac{d^2 z}{dt^2} + 2\gamma \frac{dz}{dt} + K(t) z = F \sum_{n=0}^{N(t)} \delta(t - nT) \cos(\omega_k t + \alpha) \quad (1)$$

Here $K(t)$ is a periodic (horizontal or vertical) focusing function and γ is a corresponding decrement of the radiation damping. F and α are amplitude and phase of the kicker signal at the moment of the particle passage. Since particle moves with constant velocity v the time variable in (1) can be replaced by the path length variable $s = vt$ ($s = 0$ at $t = 0$).

The summation in the right hand side of equation (1) goes from 0 to $N(t)$, where $N(t)$ is the biggest integer which does not exceeds the ratio t/T . We can express $N(t)$ in the following form:

$$N(t) = \frac{t}{T} + \chi(t) , \quad (2)$$

where $\chi(t)$ is periodic function with period T : $\chi(t) = \chi(t + T)$. Absolute value of function $\chi(t)$ does not exceeds 1. For large t $N(t) \approx \frac{t}{T}$ with the accuracy $1/N$.

Since function $\chi(t)$ is periodic we can express it also as function of variable s :

$$\chi(t) = \chi\left(\frac{s}{v}\right) = \chi_1(s) \quad (3)$$

$\chi_1(s)$ is piece-wise linear function of s . It makes jumps $\Delta\chi = 1$ when s is going through the kicker. On the interval $s_k \leq s \leq s_k + L$

$$\chi_1(s) = - (s - s_k)/L \quad (4)$$

where $L = vT$ is the ring circumference.

The solution of (1) is well known and can be expressed in terms of β -function, i.e. the solution of auxiliary equation

$$\frac{1}{2} \beta \frac{d^2}{ds^2} \beta - \frac{1}{4} \left(\frac{d\beta}{ds} \right)^2 + K(s) \beta^2 = 1 \quad (5)$$

The solution satisfying the initial conditions $z(0) = \dot{z}(0) = 0$ is

$$z(t) = \frac{F}{2} \sqrt{\beta(vt) \beta(0)} \sum_{n=0}^{N(t)} e^{-\gamma(t-nT)} (\sin \phi_+ + \sin \phi_-) \quad (6)$$

where

$$\phi_{\pm} = \phi(vt) - \phi(nvT) \pm \omega_k nT \pm \alpha \quad (7)$$

Here

$$\phi(s) = \int_0^s \frac{ds}{\beta(s)} = \frac{2\pi \bar{v}s}{L} + \phi(s) \quad (\phi(0) = 0) \quad (8)$$

Summation in (6) can be performed explicitly. For $N \gg 1$ i.e. for stationary case we get ($\Delta\omega = \omega_k - 2\pi\bar{v}/T$):

$$z(t) = \frac{F}{2} \frac{\sqrt{\beta(vt) \beta(0)}}{T(\Delta\omega^2 + \gamma^2)} \left\{ \gamma \sin(\phi(vt) + NT\Delta\omega + \alpha) - \Delta\omega \cos(\phi(vt) + NT\Delta\omega + \alpha) \right\} \quad (9)$$

or introducing phase $\psi = \arctg \Delta\omega/\gamma$ and substituting (8):

$$z(t) = \frac{F \sqrt{\beta(vt) \beta(0)}}{2T \sqrt{\Delta\omega^2 + \gamma^2}} \sin \left(\frac{2\pi \bar{v}}{T} t + \phi(vt) + NT\Delta\omega + \alpha - \psi \right) \quad (10)$$

We use now the relation $N = \frac{t}{T} + \chi_1(s)$ to get finally:

$$z(t) = \frac{F \sqrt{\beta(s) \beta(0)}}{2T \sqrt{\Delta\omega^2 + \gamma^2}} \sin (\omega_k t + \psi(s) + \chi_1(s) \cdot \Delta\omega + \alpha - \psi) \quad (11)$$

III. Discrete Fourier Transform

Suppose now that at $s = s_A$ we have a detector (pick up) which samples stationary signal (11) at equal time intervals $t = t_A + T \cdot n$. The detector will give a series of values

$$z_A(n) = F_A \sin(\omega_k n T + \phi_A) \quad (12)$$

$$\phi_A = \omega_k t_A + \phi(s_A) + \alpha - \psi + \Delta\omega \cdot \chi_1(s_A), \quad N_1 \leq n \leq N_2, N_2 - N_1 \gg 1.$$

The discrete Fourier transform (DFT) of the sample (12) on the frequency ω_a gives:

$$\begin{Bmatrix} C_A \\ S_A \end{Bmatrix} = \frac{1}{(N_2 - N_1)} \sum_{n=N_1+1}^{N_2} z_A(n) \begin{matrix} \cos \\ \sin \end{matrix} \left\{ \omega_a n T \right\} = \frac{F_A}{2} \text{SINC}(x) \begin{matrix} \sin \\ \cos \end{matrix} \left\{ y_A \right\} \quad (13)$$

where

$$\text{SINC}(x) = \frac{\sin x}{x}, \quad (14)$$

$$x = (\omega_k - \omega_a) T (N_2 - N_1) / 2 \quad (15)$$

$$y_A = (\omega_k - \omega_a) T (N_1 + N_2 + 2) / 2 + \phi_A \quad (16)$$

As we see difference of phases of two pick-ups A and B equals to:

$$\begin{aligned} y_A - y_B &= \phi_A - \phi_B = \frac{\omega_k}{v} (s_A - s_B) + \phi(s_A) - \phi(s_B) \\ &+ \frac{\Delta\omega}{v} [\chi_1(s_A) - \chi_1(s_B)] \end{aligned} \quad (17)$$

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Since the frequency ω_a which we used for DFT is completely arbitrary the value of the phase difference can be found from DFT on any frequency as soon as the kicker frequency ω_k is known. That allows one to heterodyne signal before analysing it what sometimes makes the analysis less expensive to do.

By substituting expression (4) into equation (17) one can easily see that for any ω_k the phase difference is given by

$$y_A - y_B = \frac{2\pi V}{L} s_A + \phi(s_A) - \frac{2\pi V}{L} s_B - \phi(s_B) \quad (18)$$

or conversely

$$y_A - y_B = \phi(s_A) - \phi(s_B) \quad (19)$$

i.e. the phase difference of the signals from two pick-ups equals to the full betatron phase difference for corresponding points of the machine (see equation (8)).

Fourier analysis of signals both in experiment and in computer study inevitably hits one problem — the signal is of finite length or conversely its frequency spectrum is of finite bandwidth.

In computer simulation the finite length of the signal or the finite number of signals to be analysed originate from the finite computing time which is practically available to us.

The function $\text{SINC}(x)$ in (13) appears there as a consequence of jumps in signal magnitude at the beginning and the end of the sample packet. These jumps as any abrupt changes in a signal create additional noiselike harmonics which in turn produce jittering of the phase difference. A simple way to diminish this adverse effect is to use one or another kind of windows² with smooth transition edges.

Windows are weighting functions designed in such a way as to annihilate as many derivatives at the boundary as practical. From this point of view the summation in (13) is equivalent to the usage of the rectangular (Dirichlet) window:

$$F_D(n) = \begin{cases} 0 & n \leq N_1 \\ 1 & N_1 < n \leq N_2 \\ 0 & n > N_2 \end{cases} \quad (20)$$

Apart from phase factor function $\text{SINC}(x)$ appearing in (13) is nothing else as spectral density of the rectangular window.

In our computations we make use also of two other simple windows.

One is Hanning window:

$$F_H(n) = \begin{cases} 0 & n \leq N_1 \\ \frac{1}{2} [1 - \cos(2\pi \frac{n-N_1-1}{N_2-N_1-1})] & N_1 < n \leq N_2 \\ 0 & n > N_2 \end{cases} \quad (21)$$

Another is a modification of Hanning window (so called Tukey window), which is useful for very long samples:

$$F_T(n) = \begin{cases} 0 & n \leq N_1 \\ \frac{1}{2} [1 - \cos(2\pi \frac{n-N_1-1}{\Delta N})] & N_1 < n \leq N_1 + \Delta N + 1 \\ 1 & N_1 + \Delta N < n < N_2 - \Delta N \\ \frac{1}{2} [1 + \cos(2\pi \frac{n-N_2+\Delta N}{\Delta N})] & N_2 - \Delta N \leq n \leq N_2 \\ 0 & n > N_2 \end{cases} \quad (22)$$

Tukey window is an intermediary between Dirichlet and Hanning windows.

It smoothes the signal only at its edges and leaves the signal unperturbed away from the edges.

The use of the smoothing windows greatly improves the accuracy of measurements (in our case in sense of computational results) of phase differences.

Fig. 1 illustrates the influence of windowing in signal processing. The frequency spectrum of finite sample of pure cosine signal is presented here for rectangular (1) and Hamming (2) windows. The spectrum for the latter one has broader main pick, but side maxima are reduced significantly.

IV. Computer Program BPH

Simulation of the phase measurements have been done with the help of a program BPH (user SYK). The program includes:

- a) Excitation of transverse particle oscillations on frequency
 $\omega_k = 2\pi\bar{\nu}/T + \Delta\omega_k$
- b) Damping of the oscillations
- c) Nonlinearities in the lattice (sextupole magnets)
- d) Random variations of the eigenfrequency of betatron oscillations
 $\omega_b = 2\pi\bar{\nu}/T$ in magnitude with random varying period of such changes
- e) Sampling of particle displacements at two given locations in the ring
- f) Use of one of the three above mentioned windows and
- g) DFT of both samples on several different frequencies ω_a .

The output of the program is the phase difference (modulo 2π) of the betatron oscillations for two given pick-up locations.

The input for the program is a list of the starting and the closing points of the circumference of the ring, all sextupoles and all beam position monitors (BPM) of the machine. For each element the list contains the element number in the lattice (as it is assigned to it by program PETROS³),

its name, its distance from the starting point in mm (rounded up to integer value), values of β -function (in m), its derivative $\alpha = -\beta'/2$ and phase and ID number (ID = 101 for the first and the last points, 105 for a BPM and 111 for a sextupole).

For sextupoles their nonintegrated strength value (in m^{-3}) is also given in the list. The first line contains three integers: NLIST equal to the number of elements listed, 0 and NPHS. The latter is used to distinguish units in which phase is given (if NPHS = 1 the phase is understood to be actually the tune, i.e. the phase divided by 2π). Example of such input data for configuration 212Z35 (i.e. $\beta_x^* = 4.2m$, $\beta_y^* = 0.35m$, $\eta^* = 0.0m$, $\bar{v}_x = 21.32$, $\bar{v}_y = 18.23$) is filed under the name PHASE (user SYK) on BAT001. This input is similar to one for program BB described in work⁴.

The program uses also a number of constants which allow one to choose different options.

First of all ISEX = 0 turns off all sextupoles making the machine linear. For any other integer ISEX program includes into calculations sextupole kicks. NWND = 1 makes the window rectangular. NWND = 2 and 3 introduces the Hanning and Tukey windows correspondingly. NREV determines full number of revolutions for particle tracking (maximum value 60,000). NDEL determines the lengths of samples (i.e. the number $N_2 - N_1$ in DFT (see (13)), N_2 being NREV). Three numbers M1, M2 and M3 determine locations of two pick-ups and an effective perturbing quadrupole magnet for tune variation. These numbers should be chosen from the BPM element numbers in the file PHASE. Let us denote $\Delta z'$ an angular kick in particle trajectory produced by the kicker:

$$\Delta z'(n) = F \cos [(2\pi\bar{v} + T\Delta\omega)n + \alpha] \quad (23)$$

Parameters F , $ALEF = \alpha$ and $DOMDR = T\Delta\omega$ allow one to vary the values of amplitude, phase and frequency of the angular kick. The decrement of the radiation damping of betatron oscillations is determined by the variable GM . The way in which the damping is introduced into tracking program and consequences to which it brings is discussed in Appendix.

At last several parameters steer random tune variations in time. If $NRND = 0$ the tune variation is switched off completely. The tune itself is equal to $\bar{\nu} + \Delta\nu$, where $\Delta\nu$ is a variable parameter DNO . If $NRND \neq 0$ variation takes place. In this case the tune ν is randomly changed in interval $\bar{\nu} \pm \Delta\nu$ with time period which is randomly changed between $T \cdot NDEL2$ and $T \cdot (NDEL2 + NDEL3)$. The jumps of the time changes are smoothed up by appropriate slow cosine functions to avoid unrealistic fast tune changes.

V. Results of Computations

First of all the program was checked for trivial errors by observing for linear machine rate of signal increase for $\gamma = 0$, the excitation level for $\gamma \neq 0$, the tune change $\nu - \bar{\nu}$ and so on. All these numbers comply with formulae in Section 1. Next we examine the response curves of the oscillator for different excitation levels with and without sextupole magnets. Figs. 2, 3 and 4 present stationary amplitude of horizontal particle displacement at one of the pick ups as function of the kicker frequency displacement $T\Delta\omega$ for three different levels of excitation. Each figure presents two curves for linear (1) and nonlinear (2) machines. The abrupt jump on curve 2 of Fig. 4 illustrates the appearance of the second order effect of quadratic nonlinear field. At large amplitudes of oscillations frequency dependence on amplitude makes the response not uniquely determined in certain frequency interval.

Fig. 4 illustrates in addition the frequency shift due to damping in finite difference equations (see Appendix). Curve 3 is calculated with twice as big damping decrement.

As one can see the displacement of the maximum of the oscillator response is shifted from zero point by the amount approximately two times bigger than the displacement of the maximum for curve 1.

Fig. 5 represents the same results for the phase of the excited oscillations. The symmetric character of the curve describing the phase as function of $\Delta\omega$ for linear machine is broken by the nonlinearity. The slope of the curves at the crossing point increases with the increase of the excitation level. The crossing point itself deviates from the point for which the corresponding response curve reaches its maximum.

To illustrate formula (19) we have done simulations for a number of different locations of the pick-up B (the location of the pick-up A was kept one and the same). Table 1 contains the results. The first three columns of the table give the location, betatron phase with respect to the kicker ϕ_{mod} and betatron phase with respect to the pick-up A ($\Delta\phi_{\text{mod}}$). Next columns contain the same phase as measured from simulation $\Delta\psi_{\text{meas}}$ (integer number of 2π are added to this quantity to make up for the model) and the difference $\Delta\Delta\psi = \Delta\psi_{\text{meas}} - \Delta\phi_{\text{mod}}$. For linear machine the use of the Hanning window significantly increases the accuracy of phase measurements. The last two columns give $\Delta\psi_{\text{meas}}$ and $\Delta\Delta\psi$ for nonlinear machine (using again the Hanning window).

Fig. 6 illustrates phase difference for two pick-ups as function of the machine tune. This curve is calculated with the Hanning window

and it looks the same for linear and nonlinear machine (at least for low level of excitation).

At last Fig. 7 presents the accuracy of the phase difference measurements in function of the analysing frequency ω_a in DFT for two different windows. Large picks at $\omega_a - \omega_L = \pm 1.6 \cdot 10^{-3}/T$ on curves for rectangular window are caused by sharp decrease of amplitudes of corresponding harmonics at these points. On the other hand the phase difference for the Hanning window remains constant in the frequency interval under consideration. These results do not depend on tune variation in time. Actually each curve represents the results of computations both with and without tune variation.

VI. Some Conclusions

Harmonic excitation of the beam oscillations is widely used technique for measuring tune, damping and phase of betatron oscillations in accelerators. As is shown in this study one should be careful with the level of excitation. Indeed nonlinearity of the machine could drastically distort the results of measurements at high level of excitations. Signal processing both in experiments and in computations should be done with great deal of caution since the finite length of a signal or the finite bandwidth of filters could significantly influence the results and accuracy of measurements.

But on the other hand properly processed the phase measurements give very reliable and accurate results which are independent on many factors in real machine such as influence of sextupole magnets and variation of machine tune with time.

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Appendix

Frequency of oscillations for nonconservative finite difference equations.

We shall estimate the frequency shift due to damping in finite difference equations on a simple example of harmonic oscillator.

Let us denote Ω and Γ the frequency of undamped oscillations and the decrement of the damping. The undamped oscillations are described by equation

$$d^2Z/dt^2 + \Omega^2 Z = 0 \quad (A-1)$$

or introducing variable $\tau = \Omega t$

$$Z'' + Z = 0 \quad (A-2)$$

Equivalent finite difference equations look like

$$Z_{n+1} = Z_n + p_n, \quad (A-3)$$

$$p_{n+1} = -Z_n + p_n, \quad (A-4)$$

where $p = Z'$

We include now damping into consideration in the following way:

$$Z_{n+1} = Z_n + p_n \quad (A-5)$$

$$p_{n+1} = (-Z_n + p_n)(1 - 2\Gamma/\Omega) \quad (A-6)$$

Returning to differential equation we find

$$Z' = p \quad (A-7)$$

$$p' = -Z(1 - 2\Gamma/\Omega) - p \, 2\Gamma/\Omega \quad (A-8)$$

This system of the first order differential equations is equivalent to following second order equation:

$$z'' + 2 \frac{\Gamma}{\Omega} z' + (1 - 2\Gamma/\Omega) z = 0 \quad (\text{A-9})$$

or returning to time variable

$$\ddot{z} + 2\Gamma\dot{z} + \Omega^2(1 - 2\Gamma/\Omega) z = 0 \quad (\text{A-10})$$

The last equation shows that effective frequency of damped oscillations described by (A-5), (A-6) is

$$\omega = \Omega \left(1 - \frac{\Gamma}{\Omega}\right) \quad (\text{A-11})$$

or the frequency shift

$$\omega - \Omega = -\Gamma \quad (\text{A-12})$$

Contrary to the frequency shift due to damping in differential equation (A-1) which is quadratic in Γ , the frequency shift of finite difference equations (A-5), (A-6) is linear in Γ .

Figure Captions

Fig. 1 - Frequency spectra of a pure cosine signal of a finite length for rectangular (1) and Hanning (2) windows. Vertical scale in arbitrary units. Both samples contain 10,000 pulses. Note broadening of the main line of the second spectrum and absence of sidelobes in considered frequency interval.

Fig. 2 - Response curves of horizontal betatron oscillations for linear (1) and nonlinear (2) machine. Amplitude of oscillations in mm versus the difference of the kicker ω_k and linear betatron ω_b frequencies. Amplitude of the angular kick $F = 0.3 \cdot 10^{-6}$ radians, the radiation damping decrement $\gamma = 0.8 \cdot 10^{-3}/T$.

Fig. 3 - The same as on Fig. 2, but for the excitation $F = 1.0 \cdot 10^{-6}$ radians.

Fig. 4 - The same as on Fig. 2, but for the excitation $F = 3.0 \cdot 10^{-6}$ radians.

Curve 3, calculated for $\gamma = 1.6 \cdot 10^{-3}/T$, shows twice as big frequency shift (see Appendix).

Fig. 5 - Phase characteristics of resonances corresponding to plots on Fig. 2-4. 1 - linear machine, 2, 3 and 4 - nonlinear machine for three excitation levels.

Fig. 6 - Phase difference of two pick-ups as function of the machine tune $\bar{\nu}$. This plot is the same for linear and nonlinear machine. Hanning window. No random tune variations.

Fig. 7 - Phase difference of two pick-ups as function of the frequency ω_a used for DFT.

- 1 - linear machine, rectangular window
- 2 - linear machine, Hanning window
- 3 - nonlinear machine, rectangular window
- 4 - nonlinear machine, Hanning window

Table 1

Pick-up Location	ϕ_{mod}	$\Delta\psi_{\text{mod}}$	ISEX = 0 (rectangular)		ISEX = 0 (Hanning)		ISEX = 1 (Hanning)	
			$\Delta\psi_{\text{meas}}$	$\Delta\psi$	$\Delta\psi_{\text{meas}}$	$\Delta\psi$	$\Delta\psi_{\text{meas}}$	$\Delta\psi$
1009	125.3392	116.7389	116.7398	.0009	116.7381	-.0008	116.7412	.0023
1016	126.3093	117.7090	117.7115	.0025	117.7087	-.0003	117.7121	.0031
1025	127.3017	118.7014	118.7009	-.0005	118.7012	-.0002	118.7026	.0012
1035	128.3033	119.7030	119.7033	.0003	119.7023	-.0007	119.7049	.0019
1042	129.4448	120.8445			120.8442	-.0003	120.8476	.0031
1050	131.1075	122.5072	122.5065	-.0007	122.5066	-.0006	122.5083	.0011
1063	131.8636	123.2633	123.2650	.0017	123.2625	-.0008	123.2660	.0027
1064	132.4970	123.8967	123.8993	.0026	123.8963	-.0004	123.9000	.0033
1066	132.6524	124.0521	124.0546	.0025	124.0520	-.0001	124.0553	.0032
1071	132.7888	124.1885	124.1966	.0021	124.1883	-.0002	124.1915	.0030
1072	133.0707	124.4704	124.4716	.0012	124.4703	-.0001	124.4729	.0025
3	0.8730	-7.7273	-7.7246	.0027	-7.7267	.0006	-7.7259	.0014
4	1.1563	-7.4440	-7.4420	.0020	-7.4431	.0009	-7.4429	.0011
9	1.2933	-7.3071	-7.3054	.0017	-7.3061	.0010	-7.3062	.0009
10	1.4491	-7.1512	-7.1501	.0011	-7.1503	.0009	-7.1506	.0006
11	2.0811	-6.5192	-6.5192	.0000	-6.5188	.0004	-6.5191	-.0001
24	2.8338	-5.7665	-5.7653	.0012	-5.7670	-.0005	-5.7657	.0009
32	4.5033	-4.0970	-4.0956	.0014	-4.0961	.0009	-4.0960	.0010

Table 1 (cont'd)

Pick-up Location	ϕ_{mod}	$\Delta\psi_{\text{mod}}$	ISEX = 0 (rectangular)		ISEX = 0 (Hanning)		ISEX = 1 (Hanning)	
			$\Delta\psi_{\text{meas}}$	$\Delta\Delta\psi$	$\Delta\psi_{\text{meas}}$	$\Delta\Delta\psi$	$\Delta\psi_{\text{meas}}$	$\Delta\Delta\psi$
39	5.6393	-2.9610	-2.9607	.0003	-2.9613	-.0003	-2.9608	.0002
49	6.6347	-1.9656	-1.9627	.0029	-1.9658	-.0002	-1.9639	.0017
58	7.6312	-0.9691	-0.9678	.0013	-0.9682	.0009	-0.9683	.0008
65	8.6003	0	—	—	—	—	—	—
72	9.6035	1.0032	1.0058	.0026	1.0028	-.0004	1.0047	.0015
82	10.5959	1.9956	1.9975	.0019	1.9964	.0008	1.9969	.0013
90	11.1008	2.5005	2.5009	.0004	2.5013	.0008	2.5010	.0005
92	11.2234	2.6231	2.6232	.0001	2.6238	.0007	2.6234	.0003
100	11.7281	3.1278	3.1278	.0000	3.1278	.0000	3.1279	.0001
110	12.7202	4.1199	4.1224	.0025	4.1194	-.0005	4.1214	.0015
117	13.7237	5.1234	5.1253	.0019	5.1241	.0007	5.1248	.0014
124	14.6928	6.0925	6.0924	-.0001	6.0928	.0003	6.0927	.0002
133	15.6891	7.0888	7.0909	.0021	7.0882	-.0006	7.0901	.0013
143	16.6865	8.0862	8.0885	.0024	8.0868	.0006	8.0878	.0016
150	17.8238	9.2235	9.2234	-.0001	9.2238	.0003	9.2237	.0002

AA
mm

461510

3

100% MAX. IN TO THE CENTIMETER

2

1

Fig. 2

1 A

mm

461510

8

6

4

2

$10^3 \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{d\omega'}{\omega'}$

$10^3 (\omega - \omega_0) T$

F. 3

60
50
40
30
20
10
0
-10
-20
-30
-40

$10^4 (\Delta\psi - \Delta\psi_{mod})$

$10^3 (x - \bar{x})$

-3 -2 -1 0 1 2 3