①

JULY 1981

Q. 2875

PPPL-1816

uc-20g

B-6040

MASTER

ELECTROSTATIC WAVES IN GENERAL
MAGNETIC FIELD CONFIGURATION

BY

LIU CHEN AND SHIH-TUNG TSAI

PLASMA PHYSICS LABORATORY

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

This work was supported by the U.S. Department of Energy
Contract No. DE-AC02-76-CHO 3073. Reproduction, transla-
tion, publication, use and disposal, in whole or in part,
by or for the United States government is permitted.

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

Printed in the United States of America.

Available from:

National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151

Price: Printed Copy \$ * ; Microfiche \$3.50

<u>*PAGES</u>	<u>NTIS Selling Price</u>	
1-25	\$5.00	
26-50	\$6.50	
51-75	\$8.00	
76-100	\$9.50	
101-125	\$11.00	
126-150	\$12.50	
151-175	\$14.00	
176-200	\$15.50	
201-225	\$17.00	
226-250	\$18.50	
251-275	\$20.00	
276-300	\$21.50	
301-325	\$23.00	
326-350	\$24.50	
351-375	\$26.00	
376-400	\$27.50	
401-425	\$29.00	
426-450	\$30.50	
451-475	\$32.00	
476-500	\$33.50	
500-525	\$35.00	
526-550	\$36.50	
551-575	\$38.00	
576-600	\$39.50	

For documents over 600 pages, add \$1.50 for each additional 25 page increment.

Electrostatic Waves in General Magnetic Field Configurations

by

Lin Chen

Plasma Physics Laboratory, Princeton University

Princeton, New Jersey 08544 USA

and

Shih-Tung Tsai

Institute of Physics, Chinese Academy of Sciences

Beijing, The People's Republic of China

ABSTRACT

A scheme for investigating linear electrostatic waves in general magnetically confined plasmas is presented. The scheme is a generalization of the low-frequency (less than the cyclotron frequency) gyrokinetic theory of Rutherford and Frieman¹ as well as Taylor and Hastie² to arbitrary frequencies. Governing integral wave equations for slab plasmas with magnetic shear as well as axisymmetric tokamaks are then derived to illustrate the applications.

I. INTRODUCTION

Linear wave properties in realistic magnetically confined plasmas can often be qualitatively different from those predicted by either simple extensions of infinite-medium theories or simplified model calculations due to effects such as particle magnetic drifts, trapping, magnetic shear, plasma inhomogeneities, and various other geometrical effects. Motivated by these considerations, Rutherford and Frieman¹ as well as Taylor and Hastie² have, independently, developed a formalism for treating electrostatic waves in general plasma equilibria. While their formalism, now known as the gyrokinetic formalism, contains the crucial finite Larmor-radius effects, it is limited to frequencies below cyclotron frequencies. It is to be noted that this low-frequency gyrokinetic formalism has provided the impetus for later successful applications to, notably, linear microinstability analyses in tokamaks.^{3,4} Recently, it has also been extended to include the full electromagnetic effects.^{5,6}

In this work, we remove the low-frequency restriction and generalize the gyrokinetic formalism to arbitrary frequencies. Our results are, thus, only limited by the existence of the small parameter $\epsilon = L_p/L_\perp$, where L_p is the Larmor radius and $L_\perp$ is the scale length of the equilibrium magnetic field. Therefore, we believe they may have wide applications in laboratory and space plasmas.⁷

Section II contains the theoretical approach and derivations of the generalized gyrokinetic equation valid for arbitrary frequencies. To illustrate its possible applications, an integral wave equation is derived in Section III for slab plasmas in a sheared magnetic field. In Section IV, we consider the case of axisymmetric tokamaks using, in addition, the ballooning-mode representation.⁸⁻¹⁰ Final conclusions and discussions are given in Section V.

II. THEORETICAL ANALYSES

We adopt here Catto's approach to the gyrokinetic formalism.¹¹ Thus, the analyses are carried out in the guiding-center phase space, (X, Y) , which is related to the particle phase space, (x, y) , via the following guiding-center transformation

$$X = x + y \times \hat{e}_\parallel / R, \quad (1)$$

$$Y = (y, z, t) \quad (2)$$

where $\hat{e}_\parallel = \nabla \Phi \times \hat{e}_\parallel / B$, Φ is the equilibrium electric potential, $\omega_c = qB/m$ is the cyclotron frequency, $\hat{e}_\parallel = \nabla \Phi \times \hat{e}_\parallel / B$, α is the pitch angle and,

$$\hat{e}_\parallel = \nabla \Phi \times \hat{e}_\parallel / B \sin \alpha, \quad (3)$$

with $\hat{e}_1, \hat{e}_2$ and $\hat{e}_\parallel$ being the local orthogonal unit vectors.

The unperturbed Vlasov operator in the (X, Y) phase space

$$\mathcal{L}_0 = \partial/\partial t + \hat{e}_\parallel \cdot \nabla_X + (\hat{e}_\parallel \times \nabla_Y) \cdot \nabla_Y + (q/m) \hat{e}_\parallel \cdot \nabla_Y, \quad (4)$$

becomes, in the (X, Y) phase space,

$$\begin{aligned} \mathcal{L}_0 = & \partial/\partial t + \hat{e}_\parallel \cdot \nabla_X + \hat{e}_\parallel \cdot (\hat{e}_1 \times \nabla_{Y1} + \hat{e}_2 \times \nabla_{Y2}) + (q/m) \hat{e}_\parallel \cdot \nabla_Y \\ & + \hat{e}_\parallel \cdot \nabla_X + (q/m) \hat{e}_\parallel \cdot (\hat{e}_1 \times \nabla_{Y1} + \hat{e}_2 \times \nabla_{Y2}) + (q/m) \hat{e}_\parallel \cdot \nabla_Y, \end{aligned} \quad (5)$$

where

$$\lambda_{B1} = v \times \nabla_{\mathbf{x}} (e_1 / \Omega) \cdot \nabla_{\mathbf{x}} , \quad (6)$$

$$\lambda_{B2} = (\nabla_{\mathbf{x}} u) (\partial / \partial u) + (\nabla_{\mathbf{x}} a) (\partial / \partial a) , \quad (7)$$

$$\nabla_{\mathbf{x}} u = -(\nabla u \cdot \mathbf{B} + v_1 \nabla_{\mathbf{x}} e_1 \cdot \nabla_{\mathbf{x}}) / B , \quad (8)$$

$$\nabla_{\mathbf{x}} \mathbf{I} = (\nabla_{\mathbf{x}} a_2) \cdot \mathbf{e}_1 - (v_1 / v^2) \nabla_{\mathbf{x}} e_1 \cdot (\nabla_{\mathbf{x}} \times \mathbf{e}_1) , \quad (9)$$

$$\mathbf{E}_0 = -\nabla_{\mathbf{x}} \phi_0 , \quad v_1 = v \cdot \mathbf{e}_1 , \quad v_{E0} = c \mathbf{E}_0 \times \mathbf{e}_1 / B , \text{ and } \mathbf{e}_1 = \mathbf{e}_1 \sin t + \mathbf{e}_2 \cos t .$$

With $\lambda = B / L_0 \ll 1$, the equilibrium distribution function, F , is that given in Refs. 1 and 2, i.e.,

$$F = F_0 + F_1 + \dots , \quad (10)$$

where

$$F_0 = F_0(\mathbf{E}, u, \mathbf{x}_{\perp}) , \quad (11)$$

$$\text{i.e., } \mathbf{e}_1 \cdot \nabla_{\mathbf{x}} F_0 = 0 ,$$

$$F_1 = (P/B) (\partial F_0 / \partial u) , \quad (12)$$

$$P = -\frac{1}{2} v_{\perp}^2 \nabla_0 + \int^{\infty} (da'/\Omega) [v_1 (\nabla_{\mathbf{x}} e_1 \cdot \nabla_{\mathbf{x}}) - v_{\perp}^2 \nabla_{\mathbf{x}} e_1 / 2] , \quad (13)$$

$$\nabla_0 = \nabla_d + \nabla_{z0} , \quad (14)$$

and

$$\gamma_1 = \gamma_0 \times ((v_1^2/2) \gamma_X \ln B + v_1^2 e_1 \cdot \gamma_{e_1}^2 / 2) . \quad (15)$$

For the perturbed distribution function, δf , we let

$$\delta f = \frac{1}{\gamma} \delta \psi + \frac{\lambda}{\gamma_0} + \frac{1}{B} \frac{\lambda}{\gamma_u} F_0 + \delta \phi , \quad (16)$$

where $\delta \psi$ is the perturbed electrostatic potential. The linearized Vlasov equation

$$L_V \delta f = (-i\omega) \delta f + \gamma_V F , \quad (17)$$

can be solved, after some algebraic manipulations,¹

$$L_V \delta f = -i\omega \delta f = -(q/\gamma) (R_1 + R_2 + R_3) , \quad (18)$$

where

$$R_1 = -(\partial \delta f / \partial \psi) (\partial F_0 / \partial \psi) = (\gamma_X \gamma_0 \times e_1 / \gamma) \cdot \gamma_X F_0 , \quad (19)$$

$$R_2 = -(1/B) (\partial F_0 / \partial \psi) (\lambda / \gamma_0 + v_1 e_1 \cdot \gamma_X) \delta \psi , \quad (20)$$

$$\begin{aligned} R_3 &= (\partial (\gamma_1 e_1 + \gamma_D) \cdot \gamma_X - \partial (\gamma_1 \cdot \gamma_X \ln B)) \\ &= (\gamma_0 \gamma_1 \gamma_X \gamma_0 + \gamma_0 \gamma_D \gamma_X) \left(\frac{\lambda}{\gamma_0} + \frac{1}{B} \frac{\lambda}{\gamma_u} \right) + \frac{1}{B} \frac{\lambda \gamma_D}{\gamma_u} , \end{aligned} \quad (21)$$

$$\hat{v}_y = v_y + (v_{\perp}^2/2c) \mathbf{e}_y \cdot \nabla \times \mathbf{e}_y, \quad (22)$$

and $(a)'_1 \equiv \partial a / \partial x$.

At this point, our analysis deviates from the usual low-frequency treatment, where in L_g one takes $\partial \phi / \partial x$ to be $O(1)$ and the rest of the terms to be $O(\lambda)$. We, instead, adopt the following orderings for the perturbed quantities:

$$|\partial/\partial t| \sim v_{\perp} |\mathbf{e}_y \cdot \nabla| \sim v_{\perp} |\mathbf{e}_y \times \nabla| \sim |\partial| \sim O(1); \quad (23)$$

here, $v_{\perp}$ is the characteristic (thermal) velocity. Thus, both the perpendicular and parallel (to $\mathbf{B}$) wavelengths are formally of the order of the characteristic Larmor radius.

Fourier decomposing the perturbed quantities in gyrophase χ , i.e.,

$$\{\delta g, \delta \phi\} = \sum_{\ell=-\infty}^{\infty} [\langle \delta g \rangle_{\ell}, \langle \delta \phi \rangle_{\ell}] \exp(-i\ell\chi), \quad (24)$$

where

$$\langle \delta g \rangle_{\ell} = (1/2\pi) \int_0^{2\pi} d\chi \delta g(\chi, u, e, x) \exp(i\ell\chi).$$

Eq. (18) then gives, assuming $\exp(-i\omega t)$ time dependence,

$$\langle L_{\mp} \delta g \rangle_{\ell} = \langle L_{\alpha} \rangle_{\ell} \langle \delta g \rangle_{\ell} + \langle L_{\alpha} \tilde{\delta g} \rangle_{\ell} = -(q/n) \langle R \rangle_{\ell}, \quad (25)$$

where

$$\langle L_g \rangle_i = (\hat{v}_g e_1 + \hat{v}_0) \cdot \hat{z}_x - i(\omega - \Omega + \Omega_{gi}) , \quad (26)$$

$$v_i = \langle \hat{v} \cdot \hat{z}_x \rangle_0 = v_1 (\hat{e}_1 \cdot \hat{e}_1 \cdot \hat{z}_x \cdot \hat{e}_2) = \hat{e}_1 \cdot (\hat{z}_x \times \hat{e}_1) / 2 . \quad (27)$$

then corresponds to the gyrophase-averaged correction to the cyclotron frequency.¹² We note that equilibrium quantities in $\langle L_g \rangle_i$ are to be evaluated at the guiding center position $\hat{x}$ and $O(\lambda^2)$ corrections in $\langle L_g \rangle_i$ are ignored. Also,

$$\langle \hat{L}_g \rangle_i = \langle \hat{L}_g \rangle_i \exp(-i\lambda x) ; \quad (28)$$

$\langle L_g \rangle_i \langle \hat{L}_g \rangle_j$, hence, corresponds to couplings between the neighboring cyclotron harmonics due to $O(\lambda)$ terms in L_g which are sinusoidal in x .

Now, since the perpendicular wavelengths are of $O(\lambda)$, the $\langle R \rangle_i^1$ are of the same order. We then have, for $\langle L_g \rangle_i \sim O(1)$, i.e., away from resonances with the cyclotron harmonics (including $i = 0$), $|\langle L_g \rangle_i \langle \hat{L}_g \rangle_j| \sim O(\lambda) |\langle L_g \rangle_i \langle \hat{g} \rangle_j|$ and, therefore, can be neglected. Near the i th resonance, $|\langle L_g \rangle_i| \sim O(\lambda)$ and $|\langle L_g \rangle_{j \neq i}| \sim O(1)$; solution is then dominated by $\langle \hat{g} \rangle_j$, i.e., $|\langle \hat{g} \rangle_{j \neq i}| / |\langle \hat{g} \rangle_i| \sim O(\lambda)$. Again,

$|\langle \hat{L}_g \rangle_i| \sim O(\lambda) |\langle L_g \rangle_i \langle \hat{g} \rangle_j|$ and we can neglect it. Hence, under quite general conditions, the cyclotron-harmonics coupling term, $\langle \hat{L}_g \hat{L}_g \rangle_i$, is negligible and Eq. (25) reduces to

$$\langle L_g \rangle_i \langle \hat{g} \rangle_j = -(q/\pi) \langle R \rangle_i . \quad (29)$$

Equation (29) can be written in the more conventional form by letting

$$\langle Sg \rangle_l = -(q/m) \langle S\phi \rangle_l \nabla F_0 / B \partial u + \langle Sh \rangle_l, \quad (30)$$

and $\langle Sh \rangle_l$ is given by

$$\langle L_g \rangle_l \langle S\phi \rangle_l = -(q/m) (\langle S_1 \rangle_l + \langle S_2 \rangle_l \delta_{l,0}), \quad (31)$$

where

$$\langle S_1 \rangle_l = -i \nabla F_0 / \partial \epsilon + (q \nabla F_0 / B \partial u + i \nabla_X F_0 \cdot (e_{\parallel} / Q) \times \nabla_X) \langle S\phi \rangle_l, \quad (32)$$

$$\begin{aligned} \langle S_2 \rangle_l &= (i \nabla_X \cdot \nabla_X \langle S\phi \rangle_l - \nabla_X \cdot \nabla_X \langle S\phi \rangle_l) \\ &\quad - \nabla_X \cdot \nabla_X \langle S\phi \rangle_l - \langle \nabla_X \cdot \nabla_X \rangle_l \cdot \nabla_X \ln B - \langle \nabla_X \cdot \nabla_X \rangle_l \langle S\phi \rangle_l \\ &\quad + i \omega_X \langle S\phi \rangle_l + 2 \langle (S\phi \partial) \rangle_l (3 / \partial \epsilon + 3 / B \partial u) (\nabla F_0 / B \partial u), \end{aligned} \quad (33)$$

and $\delta_{l,0}$ is the Kronecker delta. In deriving Eq. (31) we have noted that $\langle S_2 \rangle_l$ is only important for the $l = 0$ component, that is, for $l \neq 0$, $|\langle S_2 \rangle_l| / |\langle S_1 \rangle_l| \sim O(\lambda)$. Also, the last two terms in $\langle S_2 \rangle_l$ vanish for $l \neq 0$.

Equation (31) with $\langle L_g \rangle_l$ given by Eq. (26) is the gyrokinetic equation generalized to arbitrary frequencies and, therefore, is the most important result of the present work. Equations (16), (24), (30), and (31) thus completely determine the linear response. Clearly, the advantage one has gained from the preceding analyses is that Eq. (31) is much simpler to solve than the original Vlasov equation, Eq. (17). In passing, we remark that the

low-frequency results of Refs. 1 and 2 can be easily recovered by noting that, in this limit, $|\hat{v}_1 \cdot \hat{e}_1 / \hat{v}_X - i\omega| \sim 0(\epsilon)$ and $\langle Sg \rangle_0$ dominates; or alternatively,

$$\langle Sg \rangle_0 \exp(-i\omega t) = \langle Sg \rangle_0 \exp(-i\omega t) \quad (20)$$

$$\langle \delta h \rangle_{\neq 0} = (q/m) \langle Sg \rangle_0 \beta F_0 / B \beta \mu \quad \text{and}$$

Before we proceed to calculations with specific magnetic configurations, it is instructive to examine the solutions via the following WKB ansatz

$$\begin{aligned} \psi(x, y) &= \tilde{\psi}(x, y) \exp[iS(x_1)] \\ &= \tilde{\psi}(x, y) \exp[iS(x_1) - iLt] \end{aligned} \quad (34)$$

where $L = \frac{1}{2} \hat{v}_1 \cdot \hat{e}_1 / \omega$, $x_1 = \hat{v}_X S$, and $\tilde{\psi}$ contains the slow variations in $x_1(x_2)$. Equation (31) then reduces to

$$(\hat{v}_1 \cdot \nabla) \langle \delta h \rangle_{\neq 0} \equiv (\hat{v}_1 \cdot \nabla) \langle \delta h \rangle_{\neq 0} = i(q/m) J_1(\gamma) \tilde{\psi} P_1(F_0) \quad (35)$$

where

$$Q_1 = \omega - \frac{1}{2} \hat{v}_1 \cdot \hat{e}_1 - \lambda(Q - \omega_\alpha) \quad (36)$$

$$P_2(F_0) = (\omega^2 / \lambda^2 + (2\beta / B \beta \mu + \frac{1}{2} \hat{v}_1 \cdot \hat{e}_1 / \omega) F_0) F_0 \quad (37)$$

J_1 is the Bessel function, $\gamma = |\hat{v}_1 \cdot \hat{e}_1 / \omega|$ and $\lambda / \beta Z \equiv \hat{e}_1 \cdot \hat{v}_X$. In obtaining Eq. (35), we have noted that $\langle Sg \rangle_0 = J_1 \tilde{\psi}(x_1) \exp[iS(x_1)]$ and that, as in the low frequency case,^{1,2} $\langle S_2 \rangle_0$ can be shown to contribute only higher-order terms in $\langle \delta h \rangle_0$. $\langle \delta h \rangle_{\neq 0}$ can be solved by integrating along Z (i.e., along the

field line) with circulating and trapped particles differentiated via their respective boundary conditions. The solution is the same as that given in Refs. 1 and 2 with appropriate replacements and, for the present purpose, it will be symbolically written as

$$\langle \delta \tilde{n} \rangle_l = i(q/m) P_l(F_0) \langle \tilde{L}_g \rangle_l^{-1} [J_l(\gamma) \delta \tilde{\phi}] . \quad (18)$$

The perturbed distribution function, $\langle \tilde{f} \rangle$, is then given by

$$\langle \tilde{f} \rangle = (q/m) \langle \tilde{L}_g \rangle_l^{-1} P_l(F_0) \delta \tilde{\phi} + i \frac{\pi}{2} \exp(iL - i\gamma) \langle \tilde{L}_g \rangle_l^{-1} [J_l(\gamma) \delta \tilde{\phi}] .$$

$$P_l(F_0) \langle \tilde{L}_g \rangle_l^{-1} [J_l(\gamma) \delta \tilde{\phi}] . \quad (19)$$

The corresponding wave equation can then be obtained by substituting $\langle \tilde{f} \rangle$ into the Poisson equation;

$$\begin{aligned} \left(k_z^2 - \frac{\gamma^2}{\beta^2 z} \right) \delta \tilde{\phi} &= 8\pi^2 \sum_j \frac{q_j^2}{m} \left(\frac{\pi}{|v_j|} \right) \\ &\cdot \left[\delta \tilde{\phi} \frac{\partial F_0}{\partial v} + i \frac{\pi}{2} J_l(\gamma) P_l(F_0) \langle \tilde{L}_g \rangle_l^{-1} [J_l(\gamma) \delta \tilde{\phi}] \right] , \end{aligned} \quad (20)$$

where $\beta/z \equiv e_{\parallel} \cdot \nabla_x$ and j = species. Finally, observe that the usual cyclotron-resonance condition $\omega - \Omega + i v_j \beta/z = 0$ is now modified to $\omega - \Omega + i v_j \beta/z - k_{\perp} \cdot v_D + i \omega_x = 0$ and, hence, there appear additional channels for wave-particle resonance, particularly since v_D and ω_x are both velocity dependent.

III. SLAB PLASMA WITH FINITE MAGNETIC SHEAR

In this section we apply the results obtained in Section II to the specific model of a plane slab plasma in a sheared magnetic field, $\vec{B} = B_0 \vec{e}_y + \vec{e}_x \times \nabla_y \psi$. The case of axisymmetric tokamaks will be considered in the next section.

For simplicity, we assume $B_0 = 0$ and no equilibrium temperature variations. Hence

$$\vec{B} = B_0 \vec{e}_y + \vec{e}_x \times \nabla_y \psi. \quad (41)$$

Furthermore, perturbations are taken to be of the following form

$$f(x, y) = \tilde{f}(x, y) \exp(i\omega t). \quad (42)$$

Since we are interested in solutions which are localized about the resonant surface (i.e., $x = 0$), therefore, (x^2, y^2) corrections will be ignored and $\nabla_X = \nabla_y \cdot \hat{x} = \nabla_y \cdot \hat{x}$. Under these assumptions, Eq. (31) yields

$$\omega_p^2 = \frac{q^2 m_0}{4\pi n_0} G_T^{-1}(x) \langle \tilde{f} \rangle_y (\omega^2 / c^2 + \nabla_y^2 \psi) = \frac{q^2}{4\pi n_0} \tilde{f}_0, \quad (43)$$

and

$$\nabla_y^2 \tilde{f} = \tilde{f} \nabla^2 \psi = \tilde{f} \nabla_y^2 \psi. \quad (44)$$

Let $\tilde{f} = \tilde{f}_x \tilde{f}_y$ and $\tilde{f}_y = \tilde{f}(x = 0)$. Denoting the Fourier transform of $\tilde{f}(x, y)$ to be $\tilde{F}(k_x, k_y)$ with $k_x = k_x \frac{a}{x} = k_x \frac{a}{y}$, it is then straightforward to

derive

$$\begin{aligned} \int d^3y \, \delta \bar{n}(\underline{k}, \underline{y}) &\equiv \delta \bar{n}(\underline{k}) = n_0 \frac{2\pi}{\hbar} \frac{1}{v_z} \frac{1}{v_z} \frac{1}{v_z} \\ &\cdot \left(\delta \bar{n}(\underline{k}) \frac{\partial^2}{\partial k^2} - \frac{1}{\hbar} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \right) \frac{1}{\partial k^2} - \frac{1}{\hbar} \frac{\partial}{\partial y} \frac{1}{\partial k^2} \\ &\cdot \left(\frac{\partial}{\partial k_x} \delta \bar{n}(\underline{k}) \right) \frac{1}{\partial y} \frac{\partial}{\partial y} \frac{1}{\partial k_x} (\underline{k}_x - \underline{k}_x') \end{aligned} \quad (45)$$

here, $\underline{k}' = k'_x \underline{e}_x + k'_y \underline{e}_y$ and $\underline{y}, \underline{y}' = (x, y) \underline{e}_z$. Substitution of (45) into the Poisson's equation

$$\nabla^2 \bar{n}(\underline{k}) = 4\pi \int d^3y \, \delta \bar{n}(\underline{k}) \quad (46)$$

yields the desired wave equation, which is general in its integral equation and must be solved numerically. If, to be more specific, we further assume a Maxwellian plasma, $\delta \bar{n} = \delta n_0 \exp(-\frac{1}{2} \frac{v^2}{v_e^2})$, Eq. (45) then becomes the more familiar expression

$$\begin{aligned} \delta \bar{n}(\underline{k}) &= -(\hbar_0 / 2) \left(\delta \bar{n}(\underline{k}) + \frac{1}{\hbar} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \right) \\ &\cdot \left(\frac{\partial}{\partial k_x} \delta \bar{n}(\underline{k}) \right) \frac{1}{\partial k_x} \frac{1}{\partial k_x} \frac{1}{\partial k_x} (\underline{k}_x - \underline{k}_x') \end{aligned} \quad (47)$$

where

$$\bar{n}(\underline{k}, \underline{k}') = I_0(\bar{b}) \exp(-\bar{b}), \quad (48)$$

$$\bar{b} = k k' v_e^2 / 2 \Omega_0, \quad \hat{b} = (k^2 + k'^2) v_e^2 / 4 \Omega_0, \quad \omega_0 = e v_0 / \hbar B,$$

$\mathbf{E} = -\nabla\phi - \frac{1}{c} \nabla\psi \times \mathbf{e}_z$, and Z is the plasma dispersion function.

The high results of this section have been derived without the MKS assumption. Similar to the low-frequency limit,¹¹ adopting the MKS ansatz $\mathbf{E} = -\nabla\phi - \frac{1}{c} \nabla\psi \times \mathbf{e}_z$ for $\mathbf{E}$, $\mathbf{E} = -\nabla\phi - \frac{1}{c} \nabla\psi \times \mathbf{e}_z$ in Eq. (47) and dropping the non-regular nature of the problem, may not be satisfactory.

III. AXISYMMETRIC TOKAMAKS

For axisymmetric tokamaks, poloidal angle, θ (poloidal angle), and ψ (poloidal flux) are coordinates. The equilibrium magnetic field is given

$$\mathbf{B} = B_0 \mathbf{e}_z + \nabla\psi \times \mathbf{e}_z \quad (48)$$

$$\mathbf{B} = B_0 \mathbf{e}_z + \nabla\psi \times \mathbf{e}_z \quad (49)$$

whereas, we are interested in perturbations with large poloidal and toroidal mode numbers but long parallel wavelengths and, therefore, the ballooning-mode representation¹⁻¹⁰ may be used to express $\delta h(x, y)$ as¹⁴

$$\delta h(x, y) = \sum_n \exp(-inx/y_0) \int_{-\infty}^{\infty} \hat{h} \exp(im\hat{\theta}/y_0) \exp[ia(\tau - \int_0^{\hat{\theta}} \gamma d\hat{\theta} + \int_0^{\hat{\theta}} k d\hat{\theta})] \quad (51)$$

here, y_0 is the period in y , $y = IJ/R^2$, $J = (\nabla_x \times \nabla_x \cdot \nabla_x)^{-1}$ is the

Jacobian, R is the major radius, $k(\gamma)$ is the radial wavenumber, and $\tilde{h}$ contains the slow variations in γ and $\hat{a}$.

At this point, we note that the low-frequency analyses of Refs. 14 and 15 can be readily extended to obtain solutions of the present generalized gyrokinetic equation, Eq. (31). Hence, we shall omit the details and give only the final results. Defining

$$I_1(a, b) = \frac{b}{a} \int_0^{\hat{a}} d\hat{a} (JB/|\hat{v}_1|) (\omega - \gamma\omega_1 + \gamma\omega_2 + \hat{a}_1 - \gamma\omega_2), \quad (52)$$

$$\hat{a}_1 = \frac{nB}{PB_Y} (\mathbf{e}_1 \times \mathbf{e}_2) = nRB_Y \frac{\hat{a}}{a_b} \frac{\partial \hat{a}}{\partial b} = k^2 \mathbf{e}_1, \quad (53)$$

and

$$P_1'(F_0) = (JA^2/2e + (QA^2/B\hat{a}) + nBA/2\hat{a}\hat{a}) F_0, \quad (54)$$

we find, for circulating particles with $\hat{v}_1 > 0$,

$$\begin{aligned} \langle \tilde{h} \rangle_{1,+} &= i(q/m) P_1'(F_0) \int_{-\infty}^{\hat{a}} d\hat{a} J_1(\gamma') \tilde{h}(JB/|\hat{v}_1|) \\ &\quad \cdot \exp[-iI_1(\hat{\theta}, \hat{\theta}')] \quad , \end{aligned} \quad (55)$$

and, with $\hat{v}_1 < 0$,

$$\begin{aligned} \langle \tilde{h} \rangle_{2,-} &= i(q/m) P_2'(F_0) \int_{\hat{\theta}}^{\infty} d\hat{\theta}' J_2(\gamma') \tilde{h}(JB/|\hat{v}_1|) \\ &\quad \cdot \exp[iI_2(\hat{\theta}, \hat{\theta}')] \quad , \end{aligned} \quad (56)$$

where $\hat{a}_1 = \hat{a}_1(y)$, $\hat{a}_2 = \hat{a}_2(y)$. As to trapped particles, we find, summing over $\hat{a}$

$$\begin{aligned} \hat{a}_{1,2} &= -2\pi \int_{\hat{a}_1}^{\hat{a}_2} d\hat{a} \sin(I_1(\hat{a}_1, \hat{a}_2))^{-1} \cdot \frac{1}{2} \frac{d\hat{a}}{d\hat{a}} \\ &= J_1(y) \langle \hat{a}_1(y) \rangle \cos(I_1(\hat{a}_1, \hat{a}_2)) \\ &= \cos(I_1(\hat{a}_1, \hat{a}_2)) \end{aligned} \quad (57)$$

where

$$\hat{a}_{1,2} = (\hat{a}_1, \hat{a}_2) \text{ ,}$$

and $\hat{a}_1$ and $\hat{a}_2$ are the two turning points closest to $\hat{a}$. The density distribution is then given by

$$\begin{aligned} \hat{a}_{1,2} &= 2\pi \int_{\hat{a}_1}^{\hat{a}_2} d\hat{a} \frac{\partial u}{\partial y} \frac{d\hat{a}}{d\hat{a}} \frac{1}{m} \frac{\partial}{\partial \hat{a}} \frac{\partial F}{\partial \hat{a}} \\ &= \int_{\hat{a}_1}^{\hat{a}_2} J_1(y) (\langle \hat{a}_1 \rangle_{\hat{a}_1, \hat{a}_2} + \langle \hat{a}_2 \rangle_{\hat{a}_1, \hat{a}_2} + \langle \hat{a}_3 \rangle_{\hat{a}_1, \hat{a}_2}) \end{aligned} \quad (58)$$

where integrations over $\hat{a}$ must be appropriately defined for circulating and trapped particles. The governing integral wave equation is then determined

by Poisson's equation

$$\nabla^2 \hat{a} = 4\pi q \hat{a} \quad (59)$$

We now make some general remarks. First, we note that the vanishing of $\sin(\hat{A}_1, \hat{A}_2)$ in Eq. (57) clearly exhibits the cyclotron, bounce, and drift wave-particle resonances. As to the circulation particles, these resonances are imbedded within the integrals of Eqs. (55) and (56), which can become more transparent if simplifying assumptions such as purely circulating particles (i.e., $v_\parallel = \text{constant}$) and/or large aspect-ratio tokamaks with concentric, circular magnetic surfaces are adopted. In fact, in these and other limiting cases, the wave equation, Eq. (50), itself may be sufficiently simplified to make analytical or numerical treatments more tractable. These more detailed considerations are currently being pursued and the results will be reported in the future.

V. CONCLUSIONS AND DISCUSSIONS

In the present work, we have developed a systematic scheme for the investigations of linear electrostatic waves in plasmas with general magnetic field configurations. Our scheme removes the low-frequency restriction on the gyrokinetic formalism developed by Rutherford and Frieman¹ as well as Taylor and Hastie.² A generalized gyrokinetic equation valid for the entire frequency range is then derived, which can, in principle, be solved by integrating along the magnetic field lines to determine the linear responses and, hence, the corresponding wave equation. As specific examples, we have applied this scheme to derive the governing integral wave equations for plane slab plasmas in a sheared magnetic field as well as axisymmetric tokamaks. Detailed properties predicted by these equations are, however, beyond the scope of this work and will be left for future investigations.

We note that although only collisionless electrostatic waves are considered here to simplify the presentations, extensions to include additional effects, e.g., electromagnetic fields and collisions, are straightforward and will be reported in future works.

Finally, we emphasize again that the results obtained here are valid as long as the Larmor radius is small compared with the equilibrium scale lengths and, hence, they could have wide applications in both laboratory and space plasmas.

ACKNOWLEDGMENTS

The work done by one of us (L. C.) was supported by United States Department of Energy Contract No. DE-AC02-76-CH03073. Some initial work was done when L. C. visited Japan under the Japan Society for Promotion of Science fellowship. The hospitality and stimulating discussions provided to him by Professor Kyoji Nishikawa and his colleagues at Hiroshima University are highly appreciated. It is also a pleasure to thank J. D. Callen, P. K. Kaw, R. M. Kulsrud, C. R. Oberman, L. D. Pearlstein, P. H. Rutherford, and H. Sanuki for useful discussions.

REFERENCES

- ¹P. H. Rutherford and E. A. Frieman, Phys. Fluids 11, 562 (1968).
- ²J. G. Taylor and R. J. Hastie, Plasma Phys. 10, 479 (1968).
- ³B. B. Kadomtsev and O. P. Pogutse, Nucl. Fusion 11, 67 (1971).
- ⁴W. M. Tang, Nucl. Fusion 18, 1089 (1978).
- ⁵T. M. Antonsen and S. Lane, Phys. Fluids 23, 1205 (1981).
- ⁶P. J. Catto, W. M. Tang, and D. E. Baldwin, Science Application Inc., Report SAI-254-80-566-LJ PRI-14, to be published.
- ⁷For current-free axisymmetric mirror geometries, our results can be shown to agree with those of C. V. Horton, Jr., J. D. Callen, and M. V. Rosenbluth, Phys. Fluids 14, 2019 (1971), where the linearized Vlasov equation is solved by orbital integrations in the particle phase space.
- ⁸J. W. Connor, R. J. Hastie, and J. B. Taylor, Proc. R. Soc. Lond. A365, 1 (1979).
- ⁹Y. C. Lee and J. W. van Dam, in Proceedings of the Finite-Beta Theory Workshop, Varenna, Italy, edited by B. Coppi and W. Sadowski (National Technical Information Service, Springfield, VA., 1979), p. 93.
- ¹⁰A. H. Glasser, in Proceedings of the Finite-Beta Theory Workshop, Varenna, Italy, edited by B. Coppi and W. Sadowski (National Technical Information Service, Springfield, VA., 1979), p. 55. Also, R. L. Dewar, M. S. Chance, A. H. Glasser, J. M. Greene, and E. A. Frieman, Princeton Plasma Physics Laboratory Report PPPL-1587 (1979).
- ¹¹P. J. Catto, Plasma Phys. 20, 719 (1978).
- ¹²R. M. Kulsrud (Private communication).
- ¹³L. Chen, M. S. Chance, C. Z. Cheng, E. A. Frieman, P. K. Kaw, Y. C. Lee, H. Okuda, G. M. Rewoldt, P. H. Rutherford, W. M. Tang, P. N. Guzdar, Y. C. Lee, R. Marchand, W. M. Nevins, K. T. Tsang, D. J. Sigmar, and

J. T. Whitson, Princeton Plasma Physics Laboratory Report PPPL-1673

1980, to be published.

14. G. Connor, R. J. Hastie, and J. B. Taylor, Plasma Phys. 22, 757 (1980).

15. E. A. Frieman, G. M. Rewoldt, W. M. Tang, and A. H. Glasser, Phys. Fluids
22 (1979) (1980).

ALL CATEGORIES

R. Asawa, Auburn University, Alabama
 S. T. Aziz, Univ. of Alabama
 Geophysical Institute, Univ. of Alaska
 G. L. Johnston, Sonoma State Univ., California
 H. H. Kuen, Univ. of S. California
 Institute for Energy Studies, Stanford University
 H. D. Campbell, University of Florida
 N. L. Olsson, University of South Florida
 W. M. Stacey, Georgia Institute of Technology
 Benjamin M., Iowa State University
 Magne Kristiansen, Texas Tech. University
 W. L. Wise, Nat'l Bureau of Standards, Wash., D.C.
 Australian National University, Canberra
 C. N. Watson-Munro, Univ. of Sydney, Australia
 F. Cap, Inst. for Theo. Physics, Austria
 Dr. M. Heindler, Institute for Theoretical Physics
 Technical University of Graz
 Ecole Royale Militaire, Bruxelles, Belgium
 D. Paimbois, C. European Comm., B-1049-Brussels
 P. H. Sakinaka, Instituto de Fisica, Campinas, Brazil
 M. P. Bachinski, MPB Techn., Ste. Anne de Bellevue,
 Quebec, Canada
 C. R. James, University of Alberta, Canada
 T. A. Johnston, INRS-Energie, Vearenes, Quebec
 H. M. Skarsgard, Univ. of Saskatchewan, Canada
 Inst. of Physics, Academia Sinica, Peking,
 People's Republic of China
 Inst. of Plasma Physics, Hefei,
 Anhwei Province, People's Republic of China
 Library, Tsing Hua Univ., Peking, People's
 Republic of China
 Zhengwu Li, Southwestern Inst. of Phys., Leshan,
 Sichuan Province, People's Republic of China
 Librarian, Culham Laboratory, Abingdon, England (2)
 A. M. Dupas Library, C.E.N.-G. Grenoble, France
 Central Res. Inst. for Physics, Hungary
 S. R. Sharma, Univ. of Rajasthan, JAIPUR-4, India
 R. Shingai, Meerut College, India
 A. K. Sundaram, Phys. Res. Lab., India
 Biblioteca, Frascati, Italy
 Biblioteca, Milano, Italy
 G. Rostagni, Univ. Di Padova, Padova, Italy
 Preprint Library, Inst. de Fisica, Pisa, Italy
 Library, Plasma Physics Lab., Gokasho, Uji, Japan
 S. Mori, Japan Atomic Energy Res. Inst., Tokai-Mura
 Research Information Center, Nagoya Univ., Japan
 S. Shiota, Tokyo Inst. of Tech., Japan
 Inst. of Space & Aero. Sci., Univ. of Tokyo
 T. Uchida, Univ. of Tokyo, Japan
 H. Yamato, Toshiba R. & D. Center, Japan
 M. Yoshikawa, JAERI, Tokai Res. Est., Japan
 Dr. Tsuneto Nakakita, Toshiba Corporation,
 Kawasaki-Ku Kawasaki, 210 Japan
 N. Yajima, Kyushu Univ., Japan
 R. England, Univ. Nacional Auto-noma de Mexico
 B. S. Liley, Univ. of Waikato, New Zealand
 S. A. Mørre, Saab Univas Norge, Norway
 J. A. C. Cabral, Univ. de Lisboa, Portugal
 O. Petrus, A.L.I. C.U.L.A Univ., Romania
 J. de Villiers, Atomic Energy Bd., South Africa
 A. Maurech, Comisaria De La Energy y Recursos
 Minerales, Spain
 Library, Royal Institute of Technology, Sweden
 Cent. de Res. En Phys. Des Plasmas, Switzerland
 Librarian, Form-Instituut Voor Plasma-Fysica,
 The Netherlands

Bibliothek, Stuttgart, West Germany
 R. D. Butler, Univ. of Stuttgart,
 West Germany
 Max-Planck-Institut für Plasmaphysik,
 W. Germany
 Nucl. Res. Estab., Julich, West Germany
 K. Schindler, Inst. Für Theo. Physik,
 W. Germany

EXPERIMENTAL
THEORETICAL

M. H. Brennan, Flinders Univ. Australia
 H. Barnard, Univ. of British Columbia, Canada
 S. Srinivasan, Univ. of Calgary, Canada
 J. Radet, C.E.N.-B.P., Fontenay-aux-Roses,
 France
 Prof. Schnitzler, Observatoire de Nice,
 France
 S. C. Sharma, Univ. of Cape Coast, Ghana
 R. N. Aiyer, Laser Section, India
 B. Bhatt, Physical Res. Lab., India
 L. K. Chavda, S. Gujarat Univ., India
 I. M. Das Das, Banarus Hindu Univ., India
 S. Cooperman, Tel Aviv Univ., Israel
 E. Greenspan, Nucl. Res. Center, Israel
 P. Rosenau, Israel Inst. of Tech., Israel
 Int'l. Center for Theo. Physics, Trieste, Italy
 J. Kawakami, Nihon University, Japan
 T. Nakayama, Ritsumeikan Univ., Japan
 S. Nagao, Tohoku Univ., Japan
 J. I. Sakai, Tohama Univ., Japan
 S. Tjørtta, Univ. i Bergen, Norway
 M. A. Heilberg, Univ. of Natal, South Africa
 H. W. Lundgren, Chalmers Univ. of Tech.,
 Sweden
 Astro. Inst., Sonnenborgh Obs.,
 The Netherlands
 T. J. Boyd, Univ. College of North Wales
 G. Huener, Univ. Hohenberg, W. Germany
 H. J. Kaeppeler, Univ. of Stuttgart,
 West Germany
 K. H. Spatschek, Univ. Essen, West Germany

EXPERIMENTAL
ENGINEERING

B. Grek, Univ. of Quebec, Canada
 P. Lukac, Komenského Univ., Czechoslovakia
 G. Horikoshi, Nat'l Lab for High Energy Physics,
 Tsukuba-Gun, Japan

EXPERIMENTAL

F. J. Paoloni, Univ. of Wollongong, Australia
 J. Kistemaker, Rom Inst. for Atomic
 & Molec. Physics, The Netherlands

THEORETICAL

E. Verneest, Inst. Vor Theo. Mech., Belgium
 J. Teichmann, Univ. of Montreal, Canada
 T. Kahan, Univ. Paris VII, France
 R. K. Chhajlani, India
 S. K. Trehan, Panjab Univ., India
 T. Namikawa, Osaka City Univ., Japan
 H. Narumi, Univ. of Hiroshima, Japan
 Korea Atomic Energy Res. Inst., Korea
 E. T. Karlson, Uppsala Univ., Sweden
 L. Stenflo, Univ. of UMEA, Sweden
 J. R. Saraf, New Univ., United Kingdom