Compressible Flow of a Multiphase Fluid Between Two Vessels: Part I — Ideal Carrier Gas

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ABSTRACT

The transfer of a multiphase fluid from a high pressure vessel to one initially at lower pressure is investigated. The fluid is composed of two phases which do not undergo any change. The phases consist of an ideal gas, and solid particles (or liquid droplets) having constant density. The mixture is assumed to be stagnant and always perfectly mixed as well as at thermal equilibrium in each constant volume vessel. The fluid also remains homogeneous and at equilibrium while flowing between vessels. The transport properties of the mixture are taken to be zero. One important finding is that the expanding mixture or pseudo-fluid behaves similar to a polytropic Abel-Noble gas. The mixture thermodynamic properties, the end state in each vessel at pressure equilibrium, the critical parameters, and time dependent results are given for the adiabatic and isothermal limiting cases. The results include both initially sonic and initially subsonic transfer. No mathematical restriction is placed on the particle concentration, although some limiting results are given for small particle volume fraction. The mass transferred at adiabatic pressure equilibrium can be significantly less than that when thermal equilibrium is also reached. Furthermore, the adiabatic pressure equilibrium level may not be the same as that obtained at thermal equilibrium, even when all initial temperatures are the same. Finally, it is shown that the transfer times can be very slow compared to those of a pure gas due to the large reduction possible in the mixture sound speed.

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87

Contents

1	Introduction	1
2	Mixture Thermodynamic Properties	2
2.1	Equation of State	3
2.2	Specific Heat Ratio	4
2.3	Isentropic Relations	4
2.4	Adiabatic Sound Speed	5
3	End-State Results	10
3.1	Thermal Equilibrium	10
3.2	Adiabatic System	11
4	Critical Parameters	17
5	Transient Results	19
5.1	Negligible Particle Volume Fraction	20
5.1.1	Sonic Solution	21
5.1.2	Subsonic Solutions	22
5.2	Isothermal Supply Choked Flow	25
5.3	Numerical Results	23
6	Conclusions	27
7	References	28

List of Figures

- 1 Mixture ratio of specific heats versus particle mass fraction for $\gamma_g = 5/3$ with δ as parameter: —, $\delta = 10^{-2}$; ·····, $\delta = 10^{-1}$; ---, $\delta = 1$; — · —, $\delta = 10$ 30
- 2 Mixture sound speed versus particle volume fraction for $\gamma_g = 5/3$ with r as parameter: (a) $\delta = 10^{-2}$, (b) $\delta = 1$: —, $r = 10^{-1}$; ·····, $r = 10^{-2}$; ---, $r = 10^{-3}$; — · —, $r = 10^{-4}$ 31
- 3 (a) Minimum sound speed, and (b) corresponding particle volume fraction versus r for $\gamma_g = 5/3$ with δ as parameter: —, $\delta = 10$; ·····, $\delta = 1$; ---, $\delta = 10^{-1}$; — · —, $\delta = 10^{-2}$ 32
- 4 Equilibrium pressure versus initial pressure ratio with V' as parameter: (i) thermal equilibrium and (ii) adiabatic limit for $\alpha = \beta$: —, $V' = 1$; ·····, $V' = 3$; ---, $V' = 10$; — · —, $V' = 100$ 33
- 5 Equilibrium supply volume fraction versus initial supply volume fraction: (i) isothermal limit with $\bar{P}(\infty)/\bar{T}(\infty)$ as parameter, and (ii) adiabatic limit with $\bar{P}_1^{1/\gamma_1}(t_{eq})$ as parameter: —, $\bar{P}(\infty)/\bar{T}(\infty)$ or $\bar{P}_1^{1/\gamma_1}(t_{eq}) = 0.8$; ·····, $\bar{P}(\infty)/\bar{T}(\infty)$ or $\bar{P}_1^{1/\gamma_1}(t_{eq}) = 0.6$; ---, $\bar{P}(\infty)/\bar{T}(\infty)$ or $\bar{P}_1^{1/\gamma_1}(t_{eq}) = 0.4$; — · —, $\bar{P}(\infty)/\bar{T}(\infty)$ or $\bar{P}_1^{1/\gamma_1}(t_{eq}) = 0.2$ 34
- 6 Mass defect versus adiabatic equilibrium pressure for $\gamma_1 = 7/5$, $\alpha = \beta$, and $\bar{T}_2(0) = 1$ with $\epsilon_1(0)$ as parameter: —, $\epsilon_1(0) = 1$; ·····, $\epsilon_1(0) = 0.8$; ---, $\epsilon_1(0) = 0.6$; — · —, $\epsilon_1(0) = 0.4$; — — —, $\epsilon_1(0) = 0.2$; — — —, $\epsilon_1(0) \ll 1$ 35
- 7 Mass defect versus initial pressure ratio with V' as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = \bar{T}(\infty) = 1$; $\epsilon_1(0) \ll 1$: (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$; $\epsilon_1(0) = 1/2$: (d) $\bar{\phi}_2(0) = 1$, (e) $\bar{\phi}_2(0) \ll 1$, (f) $\epsilon_2(0) = 1/2$: —, $V' = 0.3$; ·····, $V' = 1$; ---, $V' = 3$; — · —, $V' = 10$; — — —, $V' = 100$ 36
- 8 Equilibrium pressure ratio versus initial pressure ratio with V' as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = \bar{T}(\infty) = 1$; $\epsilon_1(0) \ll 1$: (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$; $\epsilon_1(0) = 1/2$: (d) $\bar{\phi}_2(0) = 1$, (e) $\bar{\phi}_2(0) \ll 1$, (f) $\epsilon_2(0) = 1/2$: —, $V' = 0.3$; ·····, $V' = 1$; ---, $V' = 3$; — · —, $V' = 10$; — — —, $V' = 100$ 37

- 9 Critical pressure ratio versus supply volume fraction with γ_1 as parameter; inset is comparison with equation (79): —, $\gamma_1 = 1$; ·····, $\gamma_1 = 1.2$; - - - -, $\gamma_1 = 7/5$; — · — ·, $\gamma_1 = 5/3$ 38
- 10 Sonic time scale factor versus supply mass fraction for $\gamma_g = 5/3$ with δ_1 as parameter: —, $\delta_1 = 10^{-2}$; ·····, $\delta_1 = 10^{-1}$; - - - -, $\delta_1 = 1$; — · — ·, $\delta_1 = 10$ 39
- 11 Time of adiabatic pressure equilibrium versus initial pressure ratio for $\epsilon_1(0) \ll 1$ and $\epsilon_2(0) \ll 1$ with γ_1 as parameter for (a) $\bar{V}_2 \ll 1$ and (b) $\bar{V}_2 \gg 1$, and corresponding volume fraction versus time with γ_1 and $\bar{P}_2(0)$ as parameters for (c) $\bar{V}_2 \ll 1$ and (d) $\bar{V}_2 \gg 1$: —, $\gamma_1 = 1$; ·····, $\gamma_1 = 1.2$; - - - -, $\gamma_1 = 7/5$; — · — ·, $\gamma_1 = 5/3$ 40
- 12 Supply volume fraction for sonic isothermal discharge versus normalized time with $\epsilon_1(0)$ as parameter when $\delta_1 \gg 1$: —, $\epsilon_1(0) \ll 1$; ·····, $\epsilon_1(0) = 0.15$; - - - -, $\epsilon_1(0) = 0.3$; — · — ·, $\epsilon_1(0) = 0.45$; — — —, $\epsilon_1(0) = 0.6$. 41
- 13 Time of adiabatic pressure equilibrium versus initial pressure ratio for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, and $\bar{T}_2(0) = 1$ with $\bar{V}_2$ as parameter; $\epsilon_1(0) \ll 1$: (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$; $\epsilon_1(0) = 1/2$: (d) $\bar{\phi}_2(0) = 1$, (e) $\bar{\phi}_2(0) \ll 1$, (f) $\epsilon_2(0) = 1/2$: —, $\bar{V}_2 \ll 1$; ·····, $\bar{V}_2 = 1$; — · — ·, $\bar{V}_2 = 3$; - - - -, $\bar{V}_2 \gg 1$ 42
- 14 Volume fraction versus time for $\epsilon_1(0) \ll 1$ and $\bar{P}_2(0) = 1/2$ with $\bar{V}_2$ as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = 1$ and (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$: —, $\bar{V}_2 \ll 1$; ·····, $\bar{V}_2 = 1$; — · — ·, $\bar{V}_2 = 3$; - - - -, $\bar{V}_2 \gg 1$ 43
- 15 Volume fraction versus time for $\epsilon_1(0) = 1/2$ and $\bar{P}_2(0) = 1/2$ with $\bar{V}_2$ as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = 1$ and (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$: —, $\bar{V}_2 \ll 1$; ·····, $\bar{V}_2 = 1$; — · — ·, $\bar{V}_2 = 3$; - - - -, $\bar{V}_2 \gg 1$ 44
- 16 Volume fraction versus time for $\epsilon_1(0) \ll 1$ and $\bar{P}_2(0) = 0.01$ with $\bar{V}_2$ as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = 1$ and (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$: —, $\bar{V}_2 \ll 1$; ·····, $\bar{V}_2 = 1$; — · — ·, $\bar{V}_2 = 3$; - - - -, $\bar{V}_2 \gg 1$ 45
- 17 Volume fraction versus time for $\epsilon_1(0) = 1/2$ and $\bar{P}_2(0) = 0.01$ with $\bar{V}_2$ as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = 1$ and (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$: —, $\bar{V}_2 \ll 1$; ·····, $\bar{V}_2 = 1$; — · — ·, $\bar{V}_2 = 3$; - - - -, $\bar{V}_2 \gg 1$ 46

1 Introduction

The rapid transfer of both ideal and non-ideal gas, between finite volume reservoirs connected in a variety of ways, has been studied extensively by Chenoweth (1974, 1979, 1983). Many physical effects can independently alter the behavior in such problems. In order to understand how each parameter enters the problem, and which ones are dominant for particular cases, great care must be exercised in the analysis. The use of limiting analyses helps bound the results, and prevents attributing peculiar behaviors to the wrong physical causes. This is even more important when the transfer involves a multiphase fluid, because the additional parameters which enter the problem can have first order effects which are not always expected or easy to explain. The effect of the mixture equation of state and the corresponding sound speed behavior are examples of this sort of phenomenon.

Only homogeneous equilibrium two-phase mixtures are investigated here. It is shown that such a mixture, where the carrier gas is behaving as an Abel-Noble gas (non-ideal effects are described by the first term of a co-volume virial expansion in pressure), also behaves as an Abel-Noble fluid. The mixture co-volume depends linearly on the constant particle mass fraction to combine the gas and particle effects. This Abel-Noble fluid has a modified gas constant and undergoes a polytropic expansion from the supply because the adiabatic index (mixture ratio of specific heats) is then constant and lies between the isothermal limit of unity and the gas ratio of specific heats. The transfer of a pure Abel-Noble gas is well understood (Chenoweth 1983). Although we present detailed results only for an ideal carrier gas, the Abel-Noble fluid mixture does have some features in common with similar pure gas results.

The mixture sound speed can have a very complicated behavior, compared to the relatively simple Abel-Noble gas case, even when the carrier gas is ideal (the gas density is negligible compared to the gas co-density constant). It can, in fact, have values much smaller than the gas sound speed for many combinations of parameters. Some aspects of this physical phenomenon were originally discussed by Sewell (1910). For transfer problems controlled by an orifice or nozzle between the reservoirs (rather than a long pipe) the supply sound speed is the dominant fluid property. Thus the importance of fully understanding its effect is paramount to the transient study of two-phase fluid transfer. The final state, of course, is determined from the equation of state without ever considering the sound speed.

A wide variety of expressions for sound speed have been derived for different limiting conditions. Marble (1970) discusses several limits and gives different expressions, all valid for small particle volume fractions applicable to dusty-gas. von Schubert (1969) gives analogous expressions without restrictions on volume fraction. For the case of gas bubbles within a liquid carrier, Wood (1941, 1955), Karplus (1958), Wallis (1969), Henry, *et al.* (1971), and van Wijngaarden (1972) give expressions for arbitrary gas volume fraction in the isothermal limit, as well as more restricted expressions valid in other limits which they discuss. Henry, *et al.* (1971) give generalized expressions also valid for

phase change and discuss the limitations of the various approximations, along with data comparisons, in perhaps the greatest detail of any of the references given here. Recent papers by Gumerov, *et al.* (1988) and Ruggles, *et al.* (1988) review and summarize much of the past work and examine the role of the excitation frequency in the transition from equilibrium to the frozen limit. Apparently Ackeret (1930) originally derived some of these limiting expressions for application to gas-water mixtures, while others first appear in Wood (1941). Heinrich (1942) gives the equilibrium sound speed for a homogeneous mixture undergoing an isentropic expansion, and discusses some of its characteristics. Tangren, *et al.* (1949) rederive the same expressions and apply them to de Laval nozzle flow. Additionally, they obtain critical property ratios in the isothermal limit and present tables for the parameter range applicable to homogeneous water-gas mixture flow. They give limited experimental evidence to support the nozzle flow analysis using the critical properties derived there.

Most of the existing experimental data on sound speed is for air-water and vapor-water mixtures (Karplus 1958, 1961, Semenov and Kosterin 1964, and Silberman 1957), and the existence of minimum values much less than those of the gas has been demonstrated in these cases. Known expressions adequately predict the large changes in sound speed over a wide range of volume fractions. Additionally, Nguyen, *et al.* (1981) compare several *ad hoc* sound speed equations to the above data and obtain reasonable agreement in most cases, particularly near the minimum. The only experimental data that we are aware of which deals with solid-particle/gas mixtures is that of Zink and Delsasso (1958). They only investigate dilute concentrations in a variety of carrier gases, thus their results have limited usefulness in verifying the sound speed variation over a wide range of volume fractions.

In this work, the equilibrium isentropic sound speed expression is applied to the transfer of a homogeneous gas-particle mixture between two finite vessels joined by a nozzle or an orifice to control the flow. The effects of non-isothermal flow on the critical properties, the supply and receiving vessel end states, as well as the transfer time to pressure equilibrium are investigated for a wide range of parameters. Analytical limiting solutions are given for some special parameter values, but generally the governing equations must be solved numerically. No attempt is made here to include the many other flow regimes which are possible (see Henry, *et al.* 1971) or to define the physical limitations of the model used; these difficult tasks are clearly beyond the scope of the current paper.

2 Mixture Thermodynamic Properties

The mixture thermodynamic properties, analogous to those given by Heinrich (1942) and Tangren, *et al.* (1949), are given in this section to establish nomenclature and form the foundation for the following sections. Furthermore, many of the properties are extended in several areas to better clarify the nature of the mixture behavior.

2.1 Equation of State

The mixture density ρ can be written as

$$\rho = [(1 - \phi)/\rho_g + \phi/\rho_p]^{-1} = (1 - \epsilon) \rho_g + \epsilon \rho_p \quad (1)$$

in terms of the particle mass fraction ϕ or the particle volume fraction ϵ . Note that $\epsilon = \phi \rho / \rho_p$. The particle density ρ_p is taken to be constant, and the gas density ρ_g is initially taken to be non-ideal

$$P_g = Z_g \rho_g R_g T_g, \quad (2)$$

where the pressure P , the temperature T , the compressibility factor Z , and the gas constant R are identified in the gas phase by the subscript g . Since

$$\rho_g = \rho(1 - \phi)/(1 - \epsilon), \quad (3)$$

then for $P = P_g$ (no particle contribution to pressure) and $T = T_p = T_g$ (no thermal lag), the mixture equation of state can be written as

$$P = Z \rho R T. \quad (4)$$

The above equation of state is written in terms of a mixture gas constant

$$R = (1 - \phi) R_g, \quad (5)$$

and “compressibility” factor

$$Z = (1 - \rho/d)^{-1} = Z_g / (1 - \epsilon), \quad (6)$$

where

$$d = [(1 - \phi)/d_g + \phi/d_p]^{-1}, \quad (7)$$

provided the gas compressibility factor is given by the Abel-Noble expression (van der Waals gas with negligible intermolecular force term)

$$Z_g = (1 - \rho_g/d_g)^{-1}, \quad (8)$$

which is valid when $T \gg T_c$ (T_c is the critical gas temperature). The co-densities d_g and d for the gas and the mixture, respectively, are the inverse of the usual co-volumes and the particle co-density d_p is identified to be ρ_p . Obviously the mixture described by (6) and (7) also behaves as an Abel-Noble fluid if ϵ is not negligible compared to unity. In the following work it is assumed that $\rho_g \ll d_g$, so that an ideal carrier gas results ($Z_g = 1$). It should be noted that d_g should be determined from PVT data (Chenoweth 1983), since it cannot be reliably obtained from critical property information. The effects of $Z_g \neq 1$ will be examined in Part II where an Abel-Noble carrier gas is used.

2.2 Specific Heat Ratio

The mixture specific heat ratio $\gamma = C_P/C_V$ can be written in terms of the particle mass fraction, the gas specific heat ratio $\gamma_g = C_{Pg}/C_{Vg}$, and $\delta = C/C_{Pg}$, where C is the particle heat capacity:

$$\gamma = \gamma_g \left[\frac{1 - \phi(1 - \delta)}{1 - \phi(1 - \delta\gamma_g)} \right], \quad (9)$$

using the mixture specific heats at constant pressure and volume

$$C_P = \phi C + (1 - \phi)C_{Pg}, \quad (10)$$

$$C_V = \phi C + (1 - \phi)C_{Vg}, \quad (11)$$

where C and the gas heat capacities are taken to be independent of T (calorically perfect gas). The behavior of the mixture specific heat ratio is shown on figure 1; note that $1 \leq \gamma \leq \gamma_g$. When $\delta \ll 1$ or $\phi\delta \ll 1$, $\gamma \rightarrow \gamma_g$ results, while the isothermal limit $\gamma \rightarrow 1$ is obtained when $\gamma_g \rightarrow 1$, $\phi \rightarrow 1$, or $\phi\delta \gg 1$.

2.3 Isentropic Relations

In a reversible adiabatic expansion with constant particle mass fraction (no slip velocity between gas and particles) Heinrich (1942) and Tangren, *et al.* (1949) obtain the following relations

$$T\rho_g^{-(\gamma-1)} = C_1, \quad (12)$$

$$P\rho_g^{-\gamma} = C_2, \quad (13)$$

$$PT^{-\gamma/(\gamma-1)} = C_3, \quad (14)$$

where C_1 , C_2 , and C_3 are constants. Only the presence of γ instead of γ_g alters these relations from the well known ideal gas expressions. Since γ is constant when ϕ is constant, then γ corresponds to a true polytropic exponent which is bounded by unity and γ_g .

2.4 Adiabatic Sound Speed

When (1) is differentiated with respect to P (with ϕ constant) we obtain

$$\frac{d\rho}{dP} = \left[\frac{(1-\epsilon)^2}{(1-\phi)} \frac{d\rho_g}{dP} + \frac{\epsilon^2}{\phi} \frac{d\rho_p}{dP} \right]. \quad (15)$$

Often $dP/d\rho_g$ and $dP/d\rho_p$ are identified or interpreted as pseudo-isothermal gas and particle sound speeds squared, respectively, and $dP/d\rho$ is the corresponding mixture value. This interpretation is only valid for very slow processes which involve low frequencies. More generally, for rapid expansions the derivatives in (15) should be evaluated at constant entropy. Using (13) we get

$$\frac{d\rho_g}{dP} = \frac{\rho_g}{\gamma P} = \frac{\gamma_g}{\gamma a_g^2}, \quad (16)$$

in terms of the adiabatic gas sound speed a_g . It is obvious that the “isothermal” and “adiabatic” limits of Wallis (1969) are recovered from (16) when $\gamma \rightarrow 1$ and $\gamma \rightarrow \gamma_g$, respectively. In this work, the last term in (15) vanishes since ρ_p is constant. In the more realistic case where ρ_p is not constant, the necessary approximation to neglect the last term is

$$\frac{dP}{d\rho_p} \gg \left(\frac{\gamma}{\gamma_g} \right) \left(\frac{\epsilon}{1-\epsilon} \right)^2 \left(\frac{1-\phi}{\phi} \right) a_g^2. \quad (17)$$

The mixture sound speed $a = \sqrt{dP/d\rho}$ can then be written as

$$a = \frac{a_g}{(1-\epsilon)} \left[\frac{\gamma}{\gamma_g} (1-\phi) \right]^{1/2}, \quad (18)$$

where

$$a_g = (\gamma_g R_g T_g)^{1/2}. \quad (19)$$

This adiabatic equilibrium sound speed represents the low frequency limit (Gumerov, *et al.* 1988). The high frequency (frozen) limit results in $a \rightarrow a_g$ and is not included in the analysis below. Note that the mixture adiabatic sound speed only changes in an expansion due to the dependence on ϵ and a_g (via T), since ϕ and γ remain fixed. However, since the initial supply sound speed controls the transfer time scale (Chenoweth 1974, 1983), it is important to understand how (18) is affected by all parameters. For the purpose of examining the behavior of (18), the density ratio

$$r = \frac{\rho_g}{\rho_p} = \left(\frac{1-\phi}{\phi} \right) \left(\frac{\epsilon}{1-\epsilon} \right) \quad (20)$$

is introduced; then (18) becomes

$$\left(\frac{a}{a_g} \right)^2 = \frac{r[r-\epsilon(r-\delta)]}{(1-\epsilon)[r-\epsilon(r-1)][r-\epsilon(r-\delta\gamma_g)]}. \quad (21)$$

The pure gas and solid limits are recovered since if $\epsilon \rightarrow 0$, $a \rightarrow a_g$, while for $\epsilon \rightarrow 1$, $a \rightarrow \infty$. It can be shown that $a_g < a < \infty$ only when $r > [1 + \delta(\gamma_g - 1)]/2$. However, for constant r in the range $0 \leq r < [1 + \delta(\gamma_g - 1)]/2$, a minimum sound speed exists such that $a_{\min} < a_g$ in the physical region $0 \leq \epsilon \leq 1$. This behavior is illustrated in figure 2 where a/a_g is plotted versus ϵ with r as a parameter for $\gamma_g = 5/3$, $\delta = 0.01$, and $\delta = 1$. It appears that there is often a wide range of ϵ values near 1/2 having sound speed close to the minimum, so this value is quite useful to know. For some purposes, when the volume fraction remains in this range, the sound speed can be taken to be independent of ϵ and assumed to be at its minimum level.

The explicit value of volume fraction at which the minimum sound speed occurs is given by

$$c_3 \epsilon^3 + c_2 \epsilon^2 + c_1 \epsilon + c_0 = 0. \quad (22)$$

The interval for r resulting in $a < a_g$ arises from the fact that c_0 must be negative in order for the location of the minimum given by the above equation be in the physical region. The coefficients in (22) are given by

$$c_0 = r^2[2r - 1 - \delta(\gamma_g - 1)], \quad (23)$$

$$c_1 = -2r[3r^2 - 2r(1 + \delta\gamma_g) + \delta\gamma_g], \quad (24)$$

$$c_2 = 6r^3 - 5r^2(1 + \frac{3\delta}{5} + \delta\gamma_g) + 2\delta r(1 + 2\gamma_g + \delta\gamma_g) - \delta^2\gamma_g, \quad (25)$$

$$c_3 = -2(r - 1)(r - \delta)(r - \delta\gamma_g), \quad (26)$$

and the cubic discriminant $D = Q^3 + R^2$ is specified by

$$Q = \frac{1}{3}c_1c_3 - \frac{1}{9}c_2^2, \quad (27)$$

$$R = \frac{c_3}{6}(c_1c_2 - 3c_0c_3) - \frac{1}{27}c_2^3. \quad (28)$$

Note that

$$Q = -\frac{1}{9} \left[(1 - 3\delta + \delta\gamma_g)r^2 + 2\delta(1 - \gamma_g + \delta\gamma_g)r - \delta^2\gamma_g \right]^2 \quad (29)$$

is always less than zero, so that (22) has three real roots and they must in general be examined to determine the physical one. Before this is done, a number of special cases are given to establish the fundamental behavior of the mixture sound speed.

1. $\delta = 1$:

For $\delta = 1$, $D = 0$ and a minimum

$$\left(\frac{a}{a_g} \right)^2 = \frac{4r}{\gamma_g} \left(1 - \frac{r}{\gamma_g} \right) \quad (30)$$

exists only when $r \leq \gamma_g/2$. The volume fraction at which this minimum occurs has an upper bound of $1/2$, and is given by

$$\epsilon = \frac{1}{2} \left[\frac{1 - 2r/\gamma_g}{1 - r/\gamma_g} \right]. \quad (31)$$

This minimum is evident in figure 2(b) where (21) is plotted as a function of ϵ for $\gamma_g = 5/3$ and $\delta = 1$. Notice the broad range of volume fraction which has a sound speed close to the minimum value given by (30). This fact makes (30) a valuable expression especially for small r values. The corresponding minimum values are given as a function of r in figure 3(a) as the dotted curve. The upper limit $a = a_g$ occurs when $r = \gamma_g/2$ at $\epsilon = 0$. Also, (31) appears as the dotted curve in figure 3(b) giving the value of ϵ corresponding to the minimum sound speed versus r .

2. $r \ll \delta$:

When $r \ll \delta$, the minimum sound speed is given by

$$\left(\frac{a}{a_g} \right)^2 = \frac{4r}{\gamma_g} \left\{ 1 - [1 + (\delta - 1)\gamma_g] \left(\frac{r}{\delta\gamma_g} \right) + \gamma_g(\delta - 1)(\gamma_g - 1) \left(\frac{r}{\delta\gamma_g} \right)^2 + \dots \right\}, \quad (32)$$

and occurs when

$$\epsilon = \frac{1}{2} \left\{ 1 - [1 + (\delta - 1)\gamma_g] \left(\frac{r}{\delta\gamma_g} \right) - [1 + (\delta - 1)\gamma_g(4 + \delta\gamma_g - 3\gamma_g)] \left(\frac{r}{\delta\gamma_g} \right)^2 + \dots \right\}. \quad (33)$$

The first terms in (32) and (33) agree with those given by Heinrich (1942), and with those obtained from (30) and (31) when $r \ll 1$. The minimum is smaller than that predicted by the first term, and its location is bounded to be less than $1/2$ only if $\delta > (1 - 1/\gamma_g)$ (since the next order term inside the braces is the same in each equation); when the inequality is reversed, the minimum is greater than that predicted by the first term and it is possible for the corresponding volume fraction to be somewhat above $1/2$. This can be seen on figure 3(b). Clearly any time $r \ll 1$, the minimum in a^2 is always much less than the gas value a_g^2 .

3. $c_3 = 0$:

When $c_3 = 0$, (22) becomes a quadratic equation in ϵ and the complexity of the cubic solution is avoided. This occurs for three special values of r . In presenting these results the following constants are defined:

$$\begin{aligned} A &= \sqrt{FG}, & B &= \sqrt{F^2 - \gamma_g DG}, & C &= \sqrt{\gamma_g DF}, \\ D &= \delta - 1, & E &= \delta\gamma_g + 1, & F &= \delta\gamma_g - 1, & G &= \gamma_g - 1. \end{aligned} \quad (34)$$

(a) $r = 1$:

This case corresponds to neutrally buoyant particles. Since $0 \leq \epsilon \leq 1$, then $1/G \leq \delta < \infty$ is required for the minimum

$$\left(\frac{a}{a_g}\right)^2 = \frac{AD^2}{\delta(G-A)(A-F)} \quad (35)$$

to exist at

$$\epsilon = \frac{\delta A - F}{DF}. \quad (36)$$

(b) $r = \delta$:

For $r = \delta$ and $0 \leq \delta \leq (2 - G)^{-1}$, then

$$\left(\frac{a}{a_g}\right)^2 = \frac{27\delta D^2 G^2}{(2\gamma_g - B - E)(G + B - 2F)(G + B + F)} \quad (37)$$

and

$$\epsilon = \frac{2E - B - \gamma_g - 3\delta}{3DG} \quad (38)$$

give the minimum and its location.

(c) $r = \delta\gamma_g$:

For $r = \delta\gamma_g$ and $0 \leq \delta \leq (2 + G)^{-1}$, the minimum sound speed and the particulate volume fraction at which it occurs are given by

$$\left(\frac{a}{a_g}\right)^2 = \frac{\delta\gamma_g^2 C}{(F + C)(G - F - C)}, \quad (39)$$

and

$$\epsilon = \frac{C + \gamma_g F}{FG}. \quad (40)$$

4. $c_3 \neq 0$:

Finally when $c_3 \neq 0$ the solution to the cubic equation (22) is found to be

$$\epsilon = \frac{Y - c_2/3}{c_3}, \quad (41)$$

where

$$Y = \begin{cases} 2 \operatorname{sgn}(R)\sqrt{-Q} \cosh \left\{ \frac{1}{3} \cosh^{-1} \left[|R|/\sqrt{-Q^3} \right] \right\}, & \text{when } D \geq 0, \\ 2 \operatorname{sgn}(R)\sqrt{-Q} \cos \left\{ \frac{1}{3} \cos^{-1} \left[|R|/\sqrt{-Q^3} \right] + (i-1)\frac{2\pi}{3} \right\}, & \text{when } D \leq 0, \end{cases} \quad (42)$$

with $i = 1, 2, 3$. Note that when $D \leq 0$ only one value of i yields a physical solution with $0 \leq \epsilon \leq 1$. The volume fraction from (41) is then substituted into (21) to find the minimum of $(a/a_g)^2$ at that value. Figures 3(a) and 3(b) give results for that minimum and its location versus r for various values of δ which do require the use of (41), (42), and (21). Although for small r , $(a/a_g)_{\min}^2 \rightarrow 4r/\gamma_g$ as shown by (32) and its location $\epsilon \rightarrow 1/2$ agreeing with (33), significantly different behavior is obtained for larger values of r .

3 End-State Results

Considerable information can be obtained without solving the time-dependent problem by examining the asymptotic end-states at pressure equilibrium under adiabatic and isothermal conditions in terms of the initial data. The importance of such results are relevant to rapid mass transfer since then a quasi-static pressure equilibrium occurs while significant expansive cooling and compressive heating effects still exist in the supply and receiver, respectively. Complete thermal equilibrium of the system is usually approached at a much slower rate than that for pressure, and it is often not allowed to occur due to transfer time constraints. In some cases, a significant amount of mass is transferred following the initial pressure equilibrium, by the ensuing heat transfer, during the approach to the final state. Furthermore, even though the vessel pressures remain approximately equal during this final approach, they may change together significantly while thermal equilibrium is being reached. The maximum possible effects which can occur, can be estimated without considering any details of the convective heat transfer processes. This can be done by comparing the conditions existing at adiabatic pressure equilibrium with those existing at the final end-state. Such a comparison is formulated below using the following normalization by initial supply quantities

$$\bar{f}_j = f_j/f_1(0), \quad (43)$$

where f is any dependent variable, and the subscript j is 1 or 2 denoting supply and receiver, respectively.

3.1 Thermal Equilibrium

The final state, where $t \rightarrow \infty$ and both pressure and thermal equilibrium exist, is very easy to obtain since the final gas density is known in terms of the constant total gas mass and volume in the system. Thus from (2) we have

$$\bar{P}(\infty) = \bar{T}(\infty) \left(\frac{1 + \rho' V'}{1 + V'} \right), \quad (44)$$

where the final system temperature $\bar{T}(\infty)$ is known and it is not necessarily equal to $\bar{T}_2(0)$,

$$V' = \frac{V_{g2}(0)}{V_{g1}(0)} = \bar{V}_2 \left[\frac{1 - \epsilon_2(0)}{1 - \epsilon_1(0)} \right], \quad (45)$$

using the normalized receiver volume $\bar{V}_2 = V_2/V_1$, and

$$\rho' = \frac{\rho_{g2}(0)}{\rho_{g1}(0)} = \frac{\bar{P}_2(0)}{\bar{T}_2(0)}. \quad (46)$$

Figure 4(i) is a plot of $\bar{P}(\infty)/\bar{T}(\infty)$ versus $\bar{P}_2(0)/\bar{T}_2(0)$ with V' as parameter. Similarly, using (2) and (20), the final supply volume fraction is

$$\bar{\epsilon}_1(\infty) = \left\{ \epsilon_1(0) + [1 - \epsilon_1(0)] \frac{\bar{T}(\infty)}{\bar{P}(\infty)} \right\}^{-1}, \quad (47)$$

Figure 5(i) is a plot of $\bar{\epsilon}_1(\infty)$ versus $\epsilon_1(0)$ with $\bar{P}(\infty)/\bar{T}(\infty)$ as parameter. Notice that $\bar{\epsilon}_2(\infty)$ is obtained from $\bar{\epsilon}_1(\infty)$ using (50) below if $\bar{V}_2 \neq 0$.

3.2 Adiabatic System

In this section results are expressed in terms of the normalized supply particle volume fraction $\bar{\epsilon}_1$. The supply expansion is assumed to be isentropic, so that the pressure and temperature can be written as

$$\bar{P}_1 = \bar{\epsilon}_1^\gamma \left[\frac{1 - \epsilon_1(0)}{1 - \bar{\epsilon}_1 \epsilon_1(0)} \right]^\gamma, \quad (48)$$

$$\bar{T}_1 = \bar{P}_1^{1-1/\gamma}. \quad (49)$$

Since the supply and receiver volumes as well as the total mass are constant, the receiver particle volume fraction and mass fraction can be written as

$$\bar{\epsilon}_2 = \bar{\epsilon}_2(0) + (1 - \bar{\epsilon}_1) / \bar{V}_2, \quad (50)$$

$$\bar{\phi}_2 = \left[\frac{1 - \bar{\epsilon}_1 + \bar{V}_2 \bar{\epsilon}_2(0)}{(1 - \bar{\epsilon}_1) + \bar{V}_2 \bar{\rho}_2(0)} \right]. \quad (51)$$

Here the definition $\bar{\rho}_2(0) = \bar{\epsilon}_2(0) / \bar{\phi}_2(0)$ requires special attention as discussed below. When the system is adiabatic, its total internal energy remains constant, and it is independent of the external temperature. The receiver temperature is then obtained from the condition $\bar{E} = \bar{E}_1 + \bar{E}_2$ and the mixture specific heat (11) so that

$$\bar{T}_2 = \left(\frac{\bar{E} - \bar{\epsilon}_1 \bar{T}_1}{1 - \bar{\epsilon}_1 + \bar{V}_2 \bar{\rho}_2(0)} \right) \left(\frac{1 + \phi_1 F_1}{1 + \phi_1 \bar{\phi}_2 F_1} \right), \quad (52)$$

where the system energy $\bar{E}$ is given by

$$\bar{E} = 1 + \bar{V}_2 \bar{T}_2(0) \bar{\rho}_2(0) \left[\frac{1 + \phi_1 \bar{\phi}_2(0) F_1}{1 + \phi_1 F_1} \right], \quad (53)$$

and F_1 is defined by (34). Finally the receiver pressure is obtained from (2) and (20) as

$$\bar{P}_2 = \bar{T}_2 \left[\frac{1 - \epsilon_1(0)}{1/\bar{\epsilon}_2 - \epsilon_1(0)} \right] \left[\frac{1/\bar{\phi}_2 - \phi_1}{1 - \phi_1} \right]. \quad (54)$$

Equations (48)-(54) involve $\bar{\epsilon}_1$ in addition to nine parameters describing the initial state of the system. Some of these parameters can be related using (2) and (20) so that in effect only eight dimensionless parameters are independent. These can be related by

$$\bar{\rho}_2(0) = \frac{\bar{P}_2(0)}{\bar{T}_2(0)} \left[\frac{1 - \epsilon_1(0) \bar{\epsilon}_2(0)}{1 - \epsilon_1(0)} \right] \left[\frac{1 - \phi_1}{1 - \phi_1 \bar{\phi}_2(0)} \right] \quad (55)$$

except when $\bar{P}_2(0) = 0$ and there are particles present in the receiver (since $\phi_2(0)$ is unity in this limit), in which case

$$\bar{\rho}_2(0) = \phi_1 \bar{\epsilon}_2(0). \quad (56)$$

The system of equations (48)-(56) describe the state of the adiabatic system at all times in terms of $\bar{\epsilon}_1$. At pressure equilibrium, where $\bar{P}_1 = \bar{P}_2$, we can write

$$\left[\frac{1 - \epsilon_1(0)}{1/\bar{\epsilon}_1 - \epsilon_1(0)} \right]^\gamma = \frac{[1 - \epsilon_1(0)] [1 - \bar{\epsilon}_1 + \alpha \bar{V}_2] [1 + \beta \bar{V}_2 \bar{T}_2(0)]}{[1 - \bar{\epsilon}_1 + \alpha \bar{V}_2] + [\bar{V}_2(1 - \epsilon_2(0)) - \epsilon_1(0)] [1 - \bar{\epsilon}_1 + \beta \bar{V}_2] + [(\beta - \alpha) \bar{V}_2 \epsilon_1(0) \bar{\epsilon}_1]}, \quad (57)$$

where

$$\alpha = \rho' \left[\frac{1 - \epsilon_2(0)}{1 - \epsilon_1(0)} \right], \quad (58)$$

and

$$\beta = \bar{p}_2(0) \left[\frac{1 + \phi_2(0) F_1}{1 + \phi_1 F_1} \right]. \quad (59)$$

Now (57) can be solved for $\bar{\epsilon}_1 = \bar{\epsilon}_1(t_{eq})$ to obtain the adiabatic end-state analogous to the isothermal expression given by (47). There is an advantage obtained by using γ_1 as a parameter rather than δ_1 . When (9) is used to replace δ_1 by γ_1 , and (55) is used to eliminate $\bar{\phi}_2(0)$ from (59),

$$\beta = \alpha \left(\frac{\gamma_1 - 1}{\gamma_g - 1} \right) + \bar{\epsilon}_2(0) \left(\frac{\gamma_g - \gamma_1}{\gamma_g - 1} \right) \quad (60)$$

so that all explicit mass fraction dependence has been removed from the solution to (57). That is, any combination of ϕ_1 and δ_1 which yields γ_1 for a given γ_g is allowed without altering $\bar{\epsilon}_1(t_{eq})$, when $\epsilon_1(0)$, $\epsilon_2(0)$, $\bar{p}_2(0)$, $\bar{T}_2(0)$, and $\bar{V}_2$ are specified (fixed parameters). If instead, $\bar{\epsilon}_2(0)$ is eliminated via (55),

$$\beta = \alpha \left[1 - \left(\frac{\gamma_g - \gamma_1}{\gamma_g - 1} \right) \left(\frac{1 - \bar{\phi}_2(0)}{1 - \phi_1 \bar{\phi}_2(0)} \right) \right] \quad (61)$$

and then explicit ϕ_1 dependence is removed only if $\bar{\phi}_2(0) \equiv 1$ or $\bar{\phi}_2(0) \ll 1$, although it appears to be weak unless $\bar{\phi}_2(0) > 1$. These cases prove to be very useful in the examples discussed below. Generally, a numerical solution of (57) is required; however, there are some useful special cases where exact results can be obtained.

1. $\gamma_1 \rightarrow 1$ (isothermal limit):

There are three independent ways to cause $\gamma_1 \rightarrow 1$: $\gamma_g \rightarrow 1$, $\phi_1 \rightarrow 1$, and $\phi_1 \delta_1 \gg 1$. When $\gamma_1 \rightarrow 1$ (supply vessel isothermal), (57) becomes quadratic in $\bar{\epsilon}_1(t_{eq})$. The

case $\gamma_g \rightarrow 1$ results in the trivial no flow limit $\bar{\epsilon}_1(t_{eq}) \rightarrow 1$. The no flow limit also occurs whenever $\bar{P}_2(0) \rightarrow 1$, $\bar{V}_2 \rightarrow 0$, or $\epsilon_2(0) \rightarrow 1$. For $\phi_1 \rightarrow 1$ or $\phi_1 \delta_1 \gg 1$ the non-trivial solution $\bar{\epsilon}_1(t_{eq}) = \bar{\epsilon}_{1f}$ results, where $\bar{\epsilon}_{1f}$ is given by (47). In this case

$$\bar{P}_1(t_{eq}) = \left(\frac{1 + \rho' V'}{1 + V'} \right), \quad (62)$$

which is the same as that given by (44) only when $\bar{T}(\infty) = 1$, and then these results are already included in figures 4(ii) and 5(ii).

2. $V' \ll 1$ (charging limit):

When the initial receiver gas volume is much smaller than the initial supply gas volume, the supply conditions will remain essentially unchanged during the charging of the receiver to the supply pressure level, so that $\bar{P}_1 \approx 1$ and $\bar{\epsilon}_1 \approx 1$.

The conditions in the receiver do change and can be investigated via (48)-(55) if the limiting results are carefully derived. The procedures and results are similar to those of a pure gas (see Chenoweth 1974, Sections C and D). When $\phi_2 = \phi_1$ and using $\bar{P}_2(t_{eq}) = 1$,

$$\bar{\epsilon}_2(t_{eq}) = \bar{\epsilon}_2(0) + \left[\frac{1 - \bar{P}_2(0)}{\gamma_1} \right] [1 - \epsilon_1(0)\bar{\epsilon}_2(0)], \quad (63)$$

and

$$\bar{T}_2(t_{eq}) = \left\{ \frac{\gamma_1 - \epsilon_1(0)(1 - \bar{P}_2(0))}{1 + \bar{P}_2(0) \left[\gamma_1 / \bar{T}_2(0) - 1 \right] - \epsilon_1(0) [1 - \bar{P}_2(0)]} \right\}. \quad (64)$$

When the receiving vessel is initially evacuated $\bar{T}_2(t_{eq}) \rightarrow [\gamma_1 - \epsilon_1(0)] / [1 - \epsilon_1(0)]$. This limit gives the maximum effect of compressive heating during the charging process, which for a pure gas reduces to the well known result of γ_g . Clearly, temperatures exceeding the pure gas limit are possible when $\epsilon_1(0) \neq 0$. Also notice from (63) that when an evacuated vessel, initially containing no particles, is used to rapidly draw a sample from its surroundings, the sample volume fraction may be less than that of its surroundings by a factor as large as γ_1^{-1} .

3. $V' \gg 1$ (discharging limit):

In this limit the receiver conditions remain essentially unchanged, so that the supply discharges to the receiver pressure level. Therefore from (48) or (57) we obtain the supply volume fraction at the adiabatic pressure equilibrium $\bar{P}_1(t_{eq}) = \bar{P}_2(0)$:

$$\bar{\epsilon}_1(t_{eq}) = \left\{ \epsilon_1(0) + [1 - \epsilon_1(0)] \bar{P}_2(0)^{-1/\gamma_1} \right\}^{-1}. \quad (65)$$

Figure 5 is a plot of $\bar{\epsilon}_1(t_{eq})$ versus $\epsilon_1(0)$ with $\bar{P}_2(0)^{1/\gamma_1}$ as the parameter. For a pure gas this limit is given by Chenoweth (1974 Sections E and H).

4. $\alpha = \beta$ (single fluid or negligible particle heat capacity limit):

Another solution of (57) occurs when $\alpha = \beta$, giving

$$\bar{\epsilon}_1(t_{eq}) = \left\{ \epsilon_1(0) + [1 - \epsilon_1(0)]\bar{P}_1(t_{eq})^{-1/\gamma_1} \right\}^{-1}, \quad (66)$$

where

$$\bar{P}_1(t_{eq}) = \left[\frac{1 + V' \bar{P}_2(0)}{1 + V'} \right]; \quad (67)$$

then $\bar{P}_1(t_{eq}) = \bar{P}(\infty)$ only if $\bar{T}(\infty) = \bar{T}_2(0) = 1$. Here Figure 4 is a plot of $\bar{P}_1(t_{eq})$ versus $\bar{P}_2(0)$ with V' as the parameter. Similarly Figure 5 is a plot of $\bar{\epsilon}_1(t_{eq})$ versus $\epsilon_1(0)$ with $\bar{P}_1(t_{eq})^{1/\gamma_1}$ as the parameter. The results from (66) shown in figure 5 are valid for any arbitrary adiabatic case, although figure 4 from (67) requires that $\alpha = \beta$ in order to relate $\bar{\epsilon}_1(t_{eq})$ to V' and $\bar{P}_2(0)$. This special solution can occur in three ways:

- (a) receiver initially evacuated with no particles, $\bar{P}_2(0) = 0$, $\epsilon_2(0) = \phi_2(0) = 0$,
- (b) mass fraction of particles the same in each vessel (single fluid) $\phi_1 = \phi_2(0)$, or
- (c) particle heat capacity negligible versus that of the gas, $\delta_1 \ll 1$, $\gamma_1 \rightarrow \gamma_g$.

Obviously cases (a) and (b) both result in single fluid transfer problems. For case (a) $\alpha = \beta = 0$, while for cases (b) and (c) we have

$$\alpha = \beta = \rho' \left[\frac{1 - \epsilon_2(0)}{1 - \epsilon_1(0)} \right]. \quad (68)$$

In the limiting solutions given above, $\bar{\epsilon}_1(t_{eq}) \neq \bar{\epsilon}_1(\infty)$ except for very special cases. In order to examine the magnitude of the difference, a defect parameter is defined analogous to that defined by Chenoweth (1974 Section A, and 1983 Section H):

$$\Delta \equiv \frac{\bar{\epsilon}_1(t_{eq}) - \bar{\epsilon}_1(\infty)}{1 - \bar{\epsilon}_1(\infty)}. \quad (69)$$

Since $\bar{\epsilon}_1 = \bar{\rho}_1$, this parameter represents the maximum fraction of the mass transferred at thermal equilibrium, which may be retained in the supply when pressure equilibrium is first reached. This mass defect is due to departures from thermal equilibrium and can be bounded in some cases. For example, using (47) and (66) in (69) when $\bar{T}_2(0) = 1$ so that $\bar{P}_1(t_{eq}) = \bar{P}(\infty)/\bar{T}(\infty)$ when $\alpha = \beta$, we get

$$\Delta = \frac{\bar{P}_1(t_{eq})^{1/\gamma_1} - \bar{P}_1(t_{eq})}{[1 - \bar{P}_1(t_{eq})] \left\{ 1 - \epsilon_1(0)[1 - \bar{P}_1(t_{eq})^{1/\gamma_1}] \right\}} \quad (70)$$

which is bounded by

$$\Delta_0 \leq \Delta \leq \Delta_1 = \Delta_0 / \bar{P}_1(t_{eq})^{1/\gamma_1}. \quad (71)$$

The subscript on Δ refers to the values of $\epsilon_1(0)$. That is, for all $0 \leq \epsilon_1(0) \leq 1$, Δ is always greater than that of pure gas ($\epsilon_1(0) = 0$) for any given $\bar{P}_1(t_{eq})$. For $\epsilon_1(0) \leq 1/2$, a lower upper bound can be obtained

$$\Delta \leq \Delta_{1/2} \leq 1 - 1/\gamma_1. \quad (72)$$

The value $1 - 1/\gamma_1$ is approached for all $\epsilon_1(0)$, when $\bar{P}_1(t_{eq})$ (given by (67)) approaches unity; clearly this is possible only when $V' \ll 1$ or $\bar{P}_2(0) \rightarrow 1$. When $\gamma_1 \rightarrow \gamma_g$ and $V' \rightarrow \bar{V}_2$, the results agree with those for a pure gas given by Chenoweth (1974 Section A and 1983 Section H). For $1/2 \leq \epsilon_1(0) < 1$ a maximum value occurs when $d\Delta/d\bar{P}_1(t_{eq}) = 0$ in the interval $0 < \bar{P}_1(t_{eq}) \leq 1$ with the value $1 - 1/\gamma_1 \leq \Delta_{max} < \Delta_1 \leq 1$, where unity is approached by Δ_1 as $\bar{P}_1(t_{eq}) \rightarrow 0$. Figure 6 is a plot of the mass defect given by (70) versus $\bar{P}_1(t_{eq})$ with $\epsilon_1(0)$ as a parameter when $\gamma_1 = 7/5$, $\bar{T}_2(0) = 1$, and $\alpha = \beta$. The behavior described above can be easily observed on this plot. The expressions (70) and (72) are valid whenever $V' \gg 1$, $\gamma_1 \rightarrow 1$, or $\alpha = \beta$ with $\bar{T}_2(0) = 1$, as separate cases or in combination, and then $\bar{P}(\infty) = \bar{P}_1(t_{eq})$ if $\bar{T}(\infty) = 1$.

It is of interest to examine the mass defect (as defined by (69)), and the pressure ratio

$$P_r \equiv \bar{P}_1(t_{eq}) / \bar{P}(\infty) \quad (73)$$

using the numerical solution of (57) when $\alpha \neq \beta$, $\gamma_1 \neq 1$, and V' is finite, to determine if significant departures from the special cases can exist. For such general cases, if $\bar{T}(\infty)$ is included, nine independent parameters affect the results for Δ and P_r . In order to study their variation in the three examples given below, we will fix $\bar{T}_2(0) = \bar{T}(\infty) = 1$, $\gamma_g = 5/3$, and $\gamma_1 = 7/5$. Also, as previously noted, explicit ϕ_1 and δ_1 dependence is not present for fixed γ_1 and γ_g values for several useful cases. These are the cases given here as examples. In figures 7-8, Δ and P_r are plotted versus $\bar{P}_2(0)$ with V' as a parameter for $\epsilon_1(0) \ll 1$ and $1/2$. The first case, given in figures 7(a,d) and 8(a,d), is for $\bar{\phi}_2(0) = 1$ and is given as a reference case since it is already included in the exact results shown in figures 4-6. Figures 7(b,e) and 8(b,e) show the results for the case where negligible particles are initially present in the receiver ($\bar{\phi}_2(0)$ and $\bar{\epsilon}_2(0) \ll 1$). For $\bar{P}_2(0) \neq 0$ there are significant effects on Δ and P_r , and furthermore we find that the pressure ratio is greater than unity. Finally, figures 7(c,f) and 8(c,f) apply to the case where $\epsilon_2(0) = 1/2$; in this case we observe somewhat larger effects on Δ and very significant deviations of P_r from unity even for $\bar{P}_2(0) = 0$. Here $P_r \leq 1$ for most parameter values. Note that the behavior for the pressure ratio $\bar{P}_2(0) = 0$ is very different from that of the other cases.

The behavior resulting if $P_r \neq 1$ is analogous in some respects to that found for ideal gas mixing involving different temperatures and different internal molecular structure (see Chenoweth and Paolucci 1989).

4 Critical Parameters

In order to be able to determine if the flow is sonic and the sonic flow rate, the determination of critical parameters is necessary. It is assumed that the flow control point is located at the minimum area between the supply and receiver vessels.

Since viscous effects are neglected here, the one-dimensional Euler equation

$$u \, du = -dP/\rho \quad (74)$$

can be written in integrated form as

$$u^2 - u_1^2 = 2 \frac{P_1}{\rho_1} \left\{ \epsilon_1 \left(1 - \frac{P}{P_1} \right) + (1 - \epsilon_1) \left(\frac{\gamma_1}{\gamma_1 - 1} \right) \left[1 - \left(\frac{P}{P_1} \right)^{1-1/\gamma_1} \right] \right\}. \quad (75)$$

The above equation gives the velocity u at any point where the pressure P is in terms of the supply reservoir conditions indicated by the subscript 1. Equation (75) was given by Tangren, Dodge, and Seifert (1949). They show that it contains the pure gas isentropic expansion as well as the incompressible Bernoulli equation in the limiting cases of $\epsilon_1 \rightarrow 0$ and $\epsilon_1 \rightarrow 1$, respectively.

The sound speed can also be written in terms of the pressure ratio P/P_1 as

$$a^2 = \left[\frac{\gamma_1 P_1}{(1 - \epsilon_1) \rho_1} \right] \left(\frac{P}{P_1} \right)^{1-1/\gamma_1} \left[1 - \epsilon_1 + \epsilon_1 \left(\frac{P}{P_1} \right)^{1/\gamma_1} \right]^2, \quad (76)$$

so that when $u_1^2 \ll u^2$, and $u^2 = a^2$ and $P = P_*$, the critical pressure ratio

$$K = \frac{P_*}{P_1} \quad (77)$$

is related to γ_1 and ϵ_1 by

$$\epsilon_1(\gamma_1 - 1) \left[1 - (1 + \gamma_1)K - \frac{\gamma_1}{2} \left(\frac{\epsilon_1}{1 - \epsilon_1} \right) K^{1+1/\gamma_1} \right] + (1 - \epsilon_1)\gamma_1 \left[1 - \left(\frac{1 + \gamma_1}{2} \right) K^{1-1/\gamma_1} \right] = 0. \quad (78)$$

The above equation shows that K is not constant, but may change rapidly with ϵ_1 (except when $\epsilon_1 \ll 1$) during the transfer. Equation (78) is valid for arbitrary γ_1 and is more general than the expression

$$\left(1 + \frac{K\epsilon_1}{1 - \epsilon_1} \right)^2 \approx 2 \left[\frac{\epsilon_1}{1 - \epsilon_1} (1 - K) - \ln K \right] \quad (79)$$

given by Tangren, *et al.* (1949) which is only valid for $\gamma_1 \rightarrow 1$. They tabulated the results for $1/3 \leq \epsilon_1 \leq 0.952$ since their interest was in a mixture of small gas bubbles in water. The more general results given by (78) are shown in figure 9 for $1 \leq \gamma_1 \leq 5/3$ and $0 \leq \epsilon_1 \leq 1$. In the figure, K/K_0 is plotted versus ϵ_1 with γ_1 as the parameter, where

$$K_0 = \left(\frac{2}{\gamma_1 + 1} \right)^{\gamma_1/(\gamma_1 - 1)} \quad (80)$$

corresponds to the negligible volume fraction limit for K when $\epsilon_1 \ll 1$, which reduces to the pure gas value only if $\phi_1 \ll 1$ is also present. Note that with this normalization $0 \leq K/K_0 \leq 1$. We observe that although there is significant dependence on γ_1 , it is not a large effect.

In order to obtain explicit analytical results, K/K_0 can be expanded in ϵ_1 and $(\gamma_1 - 1)$ as

$$K = K_0 \left[1 + \bar{A}_1 \epsilon_1 + \bar{A}_2 \epsilon_1^2 + \dots \right], \quad (81)$$

$$\bar{A}_1 = [1 - (1 + \gamma_1)K_0], \quad (82)$$

$$\bar{A}_2 = 4 \left[K_0^{1/\gamma_1} + b_0 + b_1(\gamma_1 - 1) + b_2(\gamma_1 - 1)^2 + \dots \right]. \quad (83)$$

An accurate approximation is obtained for $\epsilon_1 \leq 0.6$ if the expansions are truncated above the quadratic terms, and the coefficients are adjusted such that $b_0 = -0.6505$, $b_1 = -0.0941$, and $b_2 = 0.0213$. The figure 9 inset gives the approximate results (81)-(83) as

dotted curves for the same γ_1 values used on the larger figure and given now as solid curves to show that the approximation is sufficiently accurate to be used for analytical purposes if $\epsilon_1 \leq 0.6$.

The other relevant critical parameters can be expressed in terms of K , γ_1 and ϵ_1 as follows:

$$\begin{aligned} \phi_*/\phi_1 = 1, \quad T_*/T_1 = K^{1-1/\gamma_1}, \quad r_*/r_1 = K^{1/\gamma_1}, \quad \epsilon_*/\epsilon_1 = [\epsilon_1 + (1 - \epsilon_1)K^{-1/\gamma_1}]^{-1}, \\ a_*/a_1 = [(1 - \epsilon_1) + \epsilon_1 K^{1/\gamma_1}] K^{(\gamma_1-1)/2\gamma_1}, \quad \rho_* a_*/\rho_1 a_1 = K^{(\gamma_1+1)/2\gamma_1}. \end{aligned} \quad (84)$$

5 Transient Results

Bounding end-state results were presented in section 3 for several transfer conditions. In the preceding section we presented the critical flow properties necessary to determine the sonic transfer rate as well as specifying when the transfer is in fact sonic. In this section transient results are obtained, and the time scale t_{eq} which relates to the previous results in the adiabatic limit is found.

Since $\bar{\tau}_1 = \bar{\rho}_1$, the time rate of change of the supply volume fraction is

$$\frac{d\bar{\tau}_1}{d\tau} = -\bar{\rho}_e \bar{u}_e \bar{V}_2 / (1 + \bar{V}_2), \quad (85)$$

where the velocity at the minimum flow area A_e

$$\bar{u}_e = \frac{u_e}{a_{g1}(0)}, \quad (86)$$

and the non-dimensional time

$$\tau = \left[\frac{A_e a_{g1}(0)}{V_2} \right] (1 + \bar{V}_2) t \quad (87)$$

are scaled by the initial supply gas sound speed. A more natural scaling which occurs involves the initial supply mixture sound speed, but the gas value is used here to better show the large effect which the presence of particles have on the actual transfer times. The reason for including $\bar{V}_2$ in the time scaling becomes obvious when the limits $\bar{V}_2 \ll 1$

and $\bar{V}_2 \gg 1$ are considered. The nondimensional mass flux can be evaluated using (4), (5), (48), (49), (75) and (84) assuming $\bar{u}_1^2 \ll \bar{u}_e^2$,

$$\bar{\rho}_e \bar{u}_e = \left[\frac{\bar{\epsilon}_1}{\epsilon_1 + (1 - \epsilon_1)y^{-1/\gamma_1}} \right] \left[2 \frac{(1 - \phi_1)}{\gamma_g} \left(\frac{\epsilon_1}{\epsilon_1(0)} \right)^{\gamma_1 - 1} \left(\frac{1 - \epsilon_1(0)}{1 - \epsilon_1} \right)^{\gamma_1 - 1} \right]^{1/2} \left[(1 - y) \left(\frac{\epsilon_1}{1 - \epsilon_1} \right) + \left(\frac{\gamma_1}{\gamma_1 - 1} \right) (1 - y^{1-1/\gamma_1}) \right]^{1/2}, \quad (88)$$

depending on the pressure ratio y so that

$$y = \begin{cases} K & \text{when } \bar{P}_2/\bar{P}_1 \leq K, \\ \bar{P}_2/\bar{P}_1 & \text{when } \bar{P}_2/\bar{P}_1 > K, \end{cases} \quad (89)$$

corresponding to sonic or subsonic exit velocity at minimum flow area. In general (85) with (88) must be integrated, from $\bar{\epsilon}_1(0) = 1$ until $\bar{\epsilon}_1$ approaches the end-state $\bar{\epsilon}_1(t_{eq})$. At the same time (78) is solved for $K(\gamma_1, \epsilon_1)$, and $\bar{P}_2/\bar{P}_1$ is obtained via (48)-(56) so that y can be computed from (89). Alternately, the approximate expression for K given by (81) can be used in (85) and (88) so that (85) can then be integrated directly for K while $\bar{P}_2/\bar{P}_1 \leq K$. In this sonic flow limit, the dimensional time history of $\bar{\epsilon}_1$ depends on four independent dimensionless parameters γ_g , δ_1 , ϕ_1 , and $\epsilon_1(0)$ which characterize the initial supply conditions. Note that although $\bar{V}_2$ appears via the scaling of τ , it is not present in the sonic dimensional time history. However, when $\bar{P}_2/\bar{P}_1 > K$, $\bar{V}_2$ enters, and three other independent parameters describing the initial receiver conditions must be selected from $\bar{T}_2(0)$, $\bar{P}_2(0)$, $\bar{\epsilon}_2(0)$, and $\bar{\phi}_2(0)$. Note that the fourth parameter is related to the other three via (55). Since transient solutions must asymptotically approach the end-state results as $\tau \rightarrow \tau_{eq}$, the need of eight independent dimensionless parameters is consistent with the requirement previously found necessary to compute the end-state. From figure 9 it is clear that increasing the supply volume fraction enlarges the subsonic region in pressure ratio space for a given γ_1 , whereas the opposite is true for increasing the supply mass fraction for a fixed ϵ_1 .

5.1 Negligible Particle Volume Fraction

When $\epsilon_1(0) \ll 1$, then

$$\frac{d \left[\bar{\epsilon}_1^{(1-\gamma_1)/2} \right]}{d\tau} = \left\{ (1 - \phi_1) \left(\frac{\gamma_1 - 1}{2} \right) \left(\frac{\gamma_1}{\gamma_g} \right) \left[1 - y^{(\gamma_1 - 1)/\gamma_1} \right] \right\}^{1/2} \left(\frac{\bar{V}_2}{1 + \bar{V}_2} \right) y^{1/\gamma_1}. \quad (90)$$

5.1.1 Sonic Solution

The above equation has the sonic or choked flow solution

$$\bar{\epsilon}_1 = \left[1 + \left(\frac{\bar{V}_2}{1 + \bar{V}_2} \right) \frac{\tau}{\tau_c} \right]^{2/(1-\gamma_1)}, \quad (91)$$

when $y = K \rightarrow K_0$, where the time scale factor

$$\tau_c^{-1} = (1 - \phi_1)^{1/2} \left(\frac{\gamma_1}{\gamma_g} \right)^{1/2} \left(\frac{\gamma_1 - 1}{2} \right) \left(\frac{2}{\gamma_1 + 1} \right)^{(\gamma_1 + 1)/2(\gamma_1 - 1)} \quad (92)$$

reduces to the well known pure gas result if $\phi_1 \ll 1$ and $\phi_1 \delta_1 \ll 1$ so that $\gamma_1 \rightarrow \gamma_g$. Since τ is the dimensionless gas time scale, then the effects of particles enter the solution via τ_c and the exponent $2/(1 - \gamma_1)$; the polytropic exponent (adiabatic index) is specified by the mixture ratio of specific heats γ_1 in the supply which together with the supply mass fraction ϕ_1 , determine the supply mixture sound speed relative to that of the gas. Figure 10 gives τ_c versus ϕ_1 with δ_1 as parameter for $\gamma_g = 5/3$, a form comparable to figure 1 for γ_1 . It is obvious that transfer times much longer than those for a pure gas are easily obtained, since even when $\epsilon_1(0) \ll 1$ it is still possible for ϕ_1 and γ_1 to take on their normal range of values. In addition if $\phi_2 = \phi_1$ and $\epsilon_2(0) \ll 1$

$$\bar{\tau}_1 = \left[\frac{1 + \bar{V}_2 \bar{P}_2(0)}{1 + \bar{V}_2 \bar{P}_2 / \bar{P}_1} \right]^{1/\gamma_1} \quad (93)$$

When the minimum area flow first becomes subsonic (denoted by an asterisk) we have

$$\bar{\tau}_1(\tau_*) = \left[\frac{1 + \bar{V}_2 \bar{P}_2(0)}{1 + \bar{V}_2 K_0} \right]^{1/\gamma_1}, \quad (94)$$

where

$$\frac{\tau_*}{\tau_c} = \left\{ \left[\frac{1 + \bar{V}_2 \bar{P}_2(0)}{1 + \bar{V}_2 K_0} \right]^{(1-\gamma_1)/2\gamma_1} - 1 \right\} \left(\frac{1 + \bar{V}_2}{\bar{V}_2} \right) \quad (95)$$

specifies the time at which this occurs.

5.1.2 Subsonic Solutions

As noted, the subsonic solution requires additional information about the receiver conditions. Analytical results can be obtained in this case if $\epsilon_2(0) \ll 1$ and $\phi_2 = \phi_1$. Then using (93) for $\bar{\epsilon}_1$ and $y = \bar{P}_2/\bar{P}_1$, solutions can be obtained in the charging limit as well as the discharging limit.

1. $\bar{V}_2 \ll 1$ (charging limit):

In this limit $\bar{T}_1 \approx 1$, $\bar{P}_1 \approx 1$, and $\bar{\epsilon}_1 \approx 1$, a new variable w can be introduced such that

$$y = \bar{P}_2 = (1 - w^2)^{\gamma_1/(\gamma_1-1)}. \quad (96)$$

The new variable w is bounded by

$$0 \leq w \leq w_* = \left(\frac{\gamma_1 - 1}{\gamma_1 + 1} \right)^{1/2} \quad (97)$$

and is the minimum flow area Crocco number. It physically represents the fraction of the total enthalpy converted to kinetic energy at that point. The nondimensional time for the problem becomes

$$\tau = \left[\frac{A_e a_{g1}(0)}{V_2} \right] t \quad (98)$$

and then (90) can be integrated to give

$$\tau = \tau_0 + (w_0 - w) \tau_w, \quad (99)$$

where the subscript 0 identifies the starting subsonic conditions defined by

$$\tau_0 = \begin{cases} \tau_* & \text{when } \bar{P}_2(0) \leq K_0, \\ 0 & \text{when } \bar{P}_2(0) > K_0, \end{cases} \quad (100)$$

where K_0 is given by (80),

$$\frac{\tau_*}{\tau_c} = \left(\frac{\gamma_1 - 1}{2\gamma_1} \right) [K_0 - \bar{P}_2(0)], \quad (101)$$

and

$$w_0 = \begin{cases} w_* & \text{when } \bar{P}_2(0) \leq K_0, \\ [1 - \bar{P}_2(0)^{(\gamma_1-1)/\gamma_1}]^{1/2} & \text{when } \bar{P}_2(0) > K_0. \end{cases} \quad (102)$$

The time scale factor τ_w is related to τ_c by

$$\frac{\tau_w}{\tau_c} = \left(\frac{\gamma_1 - 1}{2} \frac{K_0}{w_*} \right). \quad (103)$$

At pressure equilibrium $\bar{P}_2 = 1$ and $w = 0$, so at that time

$$\tau_{eq} = \tau_0 + w_0 \tau_w \quad (104)$$

and the time dependent solution is then given by

$$\bar{P}_2 = \left[1 - \left(\frac{\tau_{eq} - \tau}{\tau_w} \right)^2 \right]^{\gamma_1/(\gamma_1-1)}, \quad (105)$$

$$\bar{\epsilon}_2 = \bar{\epsilon}_2(0) + \frac{1}{\gamma_1} [\bar{P}_2 - \bar{P}_2(0)], \quad (106)$$

$$\bar{T}_2 = \left\{ \frac{\gamma_1 \bar{P}_2}{\bar{P}_2 + \bar{P}_2(0) [\gamma_1/\bar{T}_2(0) - 1]} \right\}. \quad (107)$$

Note that when the receiving vessel is initially evacuated $\bar{T}_2 \rightarrow \gamma_1$. Of course, for solutions where $\bar{P}_2(0) < K_0$, the subsonic solution (105) must be mated with the sonic solution

$$\bar{P}_2 = \bar{P}_2(0) + \left(\frac{2\gamma_1}{\gamma_1 - 1} \right) \frac{\tau}{\tau_c} \quad (108)$$

at $\bar{P}_2 = K_0$. Results for the time of adiabatic pressure equilibrium and the receiver volume fraction are given on figures 11(a) and 11(c), respectively, for various γ_1 and $\bar{P}_2(0)$ parameter values. Only the subsonic portion of the $\bar{P}_2$ or $\bar{\epsilon}_2$ solutions is nonlinear in time. This form of the results is useful, since when δ_1 , ϕ_1 , and γ_g are given, γ_1 is found from figure 1 or equation (9) and τ_c is obtained from figure 10 or equation (92); then for a given $\bar{P}_2(0)$, actual times can be associated with the results when A_e , $a_{g1}(0)$, and V_2 are provided using (98), and ϵ_2 is found once $\epsilon_1(0)$ and $\epsilon_2(0)$ are selected. Notice that the greatest sensitivity to γ_1 occurs for $\bar{P}_2(0) \ll 1$.

2. $\bar{V}_2 \gg 1$ (discharging limit):

In this discharging limit $\bar{T}_2 = \bar{T}_2(0)$, $\bar{P}_2 = \bar{P}_2(0)$, and $\bar{P}_1 = \bar{\epsilon}_1 \bar{T}_1 = \bar{\epsilon}_1^{\gamma_1}$ so that by defining a new variable x proportional to the minimum area Mach number, we can write the pressure ratio as

$$y = \bar{P}_2(0) \bar{\epsilon}_1^{-\gamma_1} = (1 + x^2)^{\gamma_1/(1-\gamma_1)} \quad (109)$$

so that (90) can then be integrated to give

$$\tau = \tau_0 + (I_{n_0} - I_n) \tau_x, \quad (110)$$

where

$$I_n = \int (1 + x^2)^{n/2} dx, \quad (111)$$

$$\tau_0 = \begin{cases} \tau_* & \text{when } \bar{P}_2(0) \leq K_0, \\ 0 & \text{when } \bar{P}_2(0) > K_0, \end{cases} \quad (112)$$

$$x_0 = \begin{cases} \left(\frac{\gamma_1 - 1}{2}\right)^{1/2} & \text{when } \bar{P}_2(0) \leq K_0, \\ [\bar{P}_2(0)^{(1-\gamma_1)/\gamma_1} - 1]^{1/2} & \text{when } \bar{P}_2(0) > K_0. \end{cases} \quad (113)$$

and

$$\frac{n}{2} = \frac{2 - \gamma_1}{\gamma_1 - 1}. \quad (114)$$

The subscript 0 identifies the beginning subsonic conditions. The time required for the flow to unchoke is

$$\frac{\tau_*}{\tau_c} = \left(\frac{2}{\gamma_1 + 1}\right)^{1/2} \bar{P}_2(0)^{(1-\gamma_1)/2\gamma_1} - 1. \quad (115)$$

The time scale factor τ_x is related to τ_w by

$$\frac{\tau_x}{\tau_w} = \bar{P}_2(0)^{(1-\gamma_1)/2\gamma_1}. \quad (116)$$

At pressure equilibrium $\bar{P}_1 \rightarrow \bar{P}_2(0)$, $x \rightarrow 0$, and $I_n \rightarrow 0$, so that the time to pressure equilibrium is

$$\tau_{eq} = \tau_0 + I_{n_0} \tau_x, \quad (117)$$

and the time dependent solution is given by

$$\tau = \tau_{eq} - I_n \tau_x, \quad (118)$$

which allows τ to be calculated explicitly for any pressure ratio.

Exact analytical results are obtained by using the recursion formula

$$I_n = \frac{1}{1+n} \left\{ x (1+x^2)^{n/2} + n I_{n-2} \right\} \quad (119)$$

with $n = 0, 1, 2, \dots$, where $I_0 = x$, and $I_1 = \{x(1+x^2)^{1/2} + \ln[x + (1+x^2)^{1/2}]\}/2$ terminate the recursion for any positive even or odd integer value of n , respectively. The mixture ratio of specific heats is given by

$$\gamma_1 = \frac{n+4}{n+2}, \quad (120)$$

so that the results are valid at an infinite number of discrete values of $\gamma_1 = 2, 5/3, 6/4, 7/5, 8/6, \dots$, which asymptotically approaches unity for large n .

Similar results were obtained for the pure gas case by Chenoweth (1974, section E), who gave explicit expressions only for $\gamma_g = 5/3$ and $7/5$. As opposed to the pure gas case, here the other values of γ_1 between γ_g and unity are more physically significant because γ_1 can actually reach all those values by a proper choice of ϕ_1 .

These discharging results are given on figure 11(b) and 11(d) in a form similar to the charging case, where the time of adiabatic pressure equilibrium and the supply volume fraction are plotted versus $\bar{P}_2(0)$ and τ/τ_{eq} , respectively, for various values of γ_1 and $\bar{P}_2(0)$. Here $\tau_{eq} \rightarrow \infty$ as $\bar{P}_2(0) \rightarrow 0$. For charging, the most sensitivity of $\bar{\epsilon}_2$ to γ_1 occurred for $\bar{P}_2(0) \ll 1$, while here that limit for $\bar{\epsilon}_1$ shows the least sensitivity to γ_1 on the τ/τ_{eq} scale. However, due to the fact τ_{eq} has its greatest variation with γ_1 in that limit, there are still very large γ_1 effects present on the $\bar{\epsilon}_1$ dimensional time histories.

5.2 Isothermal Supply Choked Flow

When $\bar{P}_2/\bar{P}_1 \leq K$, the approximate expressions (80)-(83), truncated above the quadratic terms, may be used to integrate (85)-(89) provided $\epsilon_1(0) \leq 0.6$. This becomes particularly easy in the isothermal limit $\gamma_1 \rightarrow 1$ since then $K_0 = e^{-1/2}$, $\bar{A}_1 \approx -0.2131 = -\bar{a}$ and $\bar{A}_2 \approx -0.1760 = -\bar{b}$. At least one of the conditions $\gamma_g \rightarrow 1$, $\phi_1 \rightarrow 1$, or $\phi_1 \delta_1 \gg 1$ must be true so that $\gamma_1 \rightarrow 1$. The resulting supply volume fraction is then given implicitly as a function of time by

$$\tau = \left(\frac{\gamma_g}{1-\phi_1}\right)^{1/2} \left(\frac{1+\bar{V}_2}{\bar{V}_2}\right) \left\{ \bar{e} \ln \left[\left(\frac{1+\bar{c}\epsilon_1}{1+\bar{c}\epsilon_1(0)}\right) \left(\frac{1-\bar{d}\epsilon_1(0)}{1-\bar{d}\epsilon_1}\right) \right] - \bar{f} \ln \left[\left(\frac{\epsilon_1}{\epsilon_1(0)}\right)^2 \left(\frac{1-\bar{a}\epsilon_1(0)-\bar{b}\epsilon_1^2(0)}{1-\bar{a}\epsilon_1-\bar{b}\epsilon_1^2}\right) \right] \right\}, \quad (121)$$

where $\bar{c} = 0.3264$, $\bar{d} = 0.5394$, $\bar{e} = 1.7015$, and $\bar{f} = 0.8244$ specify the other constants. The point in time where (121) ceases to be valid (*i.e.* $\bar{P}_2/\bar{P}_1 = K$) is determined from

$$K \approx e^{-1/2} (1 - \bar{a}\epsilon_1 - \bar{b}\epsilon_1^2). \quad (122)$$

Figure 12 is a plot of $\bar{\epsilon}_1$ versus $\tau(1 - \phi_1)^{1/2}\bar{V}_2/[\gamma_g^{1/2}(1 + \bar{V}_2)]$ with $\epsilon_1(0)$ as a parameter. It shows that when $\epsilon_1(0)$ is not negligible, significant effects occur in this physically realistic limit. Obviously these results are valid for all times in the limit of a sonic discharge to vacuum, $\bar{V}_2 \gg 1$ and $\bar{P}_2(0) \ll 1$, when $\gamma_1 \approx 1$. It does not appear that explicit analytical results are easily obtained in this limit in a form comparable to (121) when the flow is unchoked.

5.3 Numerical Results

The detailed transient solutions valid for $\epsilon_1(0) \ll 1$, $\epsilon_2(0) \ll 1$, $\bar{\phi}_2(0) = 1$, and $\bar{V}_2 \ll 1$ or $\bar{V}_2 \gg 1$ are very useful, but they leave a number of questions unanswered. In particular, what are the effects of non-negligible reservoir volume fractions, unequal reservoir mass fractions, and reservoir volume ratios of the order of unity, both individually and in combination? These questions are examined in this section using exactly the same conditions previously used in the end-state calculations given in figures 7 and 8. That is, $\gamma_g = 5/3$, and $\gamma_1 = 7/5$. The same three cases, $\bar{\phi}_2(0) = 1$, $\bar{\phi}_2(0) \ll 1$, and $\epsilon_2(0) = 1/2$, are compared for various $\bar{V}_2$ and $\bar{P}_2(0)$ values for both $\epsilon_1(0) \ll 1$ and $\epsilon_1(0) = 1/2$ with $\bar{T}_2(0) = 1$. The results are then presented in the same form used for the special charging and discharging exact solutions already discussed. These are valid for any ϕ_1 and δ_1 giving $\gamma_1 = 7/5$ when $\gamma_g = 5/3$, similar to the end-state results.

First, the times of adiabatic pressure equilibrium are given in figure 13. Figure 13(a) is for $\bar{\phi}_2(0) = 1$ and $\epsilon_1(0) \ll 1$ so that $\alpha = \beta$ and the upper and lower curves are in fact the exact solutions; thus the only new information on this plot is the $\bar{V}_2 = 1$ and 3 curves. Notice that the $\bar{V}_2 = 1$ curve is much closer to the charging limit than it is to the discharging curve. In figure 13(b) we see that for the case where there is negligible particles $\bar{\phi}_2(0) \ll 1$ in the receiver initially, it appears that there is little difference from figure 13(a). In fact, even when $\epsilon_2(0) = 1/2$, from figure 13(c) we see that the upper curve for discharging appears to be the same; here the other curves are lowered as might be expected, since the gas volume ratio V' has been decreased. When $\bar{P}_2(0) < K$, there are two contributions to τ_{eq} : the time to unchoked flow τ_* and the time spent with subsonic flow. The sonic solution to (90) depends on γ_1 , $\epsilon_1(0)$, and $(\tau/\tau_c)\bar{V}_2/(1 + \bar{V}_2)$ so that it is independent of the receiver conditions which are being altered here. Obviously, any time the sonic portion dominates the subsonic portion, little effect of the receiver parameters will be observed. Also, since the subsonic portion of the solution approaches the end-state results as $t \rightarrow t_{eq}$, these results give a good indication of the parameter dependence for the final part of the time histories. In the discharging limit, the end-state results for $\bar{\epsilon}_1(t_{eq})$ show a dependence only on γ_1 , $\epsilon_1(0)$, and $\bar{P}_2(0)$. Thus one would expect to observe only the dependence on $\epsilon_1(0)$ for τ_{eq}/τ_c in the $\bar{V}_2 \gg 1$ limit and independent of $\bar{\phi}_2(0)$ and $\bar{\epsilon}_2(0)$. Indeed, this is seen in figures 13(d-f) which are for $\epsilon_1(0) = 1/2$, where again

the dashed curves appear to be the same but substantially different from the $\epsilon_1(0) \ll 1$ cases. In these figures all of the $\epsilon_1(0) = 1/2$ curves are substantially below the $\epsilon_1(0) \ll 1$ results. Thus, it appears that increasing the volume fractions decrease transfer times, while increasing mass fractions increase the transfer times via τ_c .

Time histories for $\bar{\epsilon}_1$ and $\bar{\epsilon}_2 - \bar{\epsilon}_2(0)$ are given on figures 14-17 for the cases corresponding to those on figure 13, but for just two specific values of $\bar{P}_2(0)$; these values are $\bar{P}_2(0) = 1/2$ and 0.01, representing mostly subsonic flow, and flow which remains sonic for significant periods prior to the unchoking, respectively. The behavior just discussed for the discharging limit $\bar{V}_2 \gg 1$ is readily observable here; similarly, the sonic solution portions for $\bar{P}_2 < K$ display the behavior already discussed. This explains why there appears to be much less effect of $\bar{\phi}_2(0)$ and $\bar{\epsilon}_2(0)$ when $\bar{P}_2(0) = 0.01$ than for $\bar{P}_2(0) = 1/2$. Also, for $\bar{P}_2(0) = 0.01$ it appears that the histories in terms of τ/τ_{eq} are nearly independent of $\epsilon_1(0)$; thus the effects of $\epsilon_1(0)$ are primarily contained in the normalization of $\bar{\epsilon}_1$, $\bar{\epsilon}_2$, and τ/τ_{eq} . This is not the case for $\bar{P}_2(0) = 1/2$, where the entire flow is predominantly subsonic.

6 Conclusions

The equilibrium flow of homogeneous two-phase mixtures between two vessels with no phase change and an ideal carrier gas has been investigated. It is found that the mixture behaves similar to a polytropic Abel-Noble gas. The mixture thermodynamic properties, the end-state in each vessel at pressure equilibrium, the critical parameters, and time dependent results are given for adiabatic and isothermal limiting cases. The mass transferred at adiabatic pressure equilibrium can be substantially less than that when thermal equilibrium is also reached. It is also found that the adiabatic pressure equilibrium level may not be the same as that reached at thermal equilibrium. Furthermore, it is shown that the transfer times can be very slow compared to those of a pure gas, due to the large reduction possible in the mixture sound speed.

The results given here form the base solutions which can be used for comparisons as the more restrictive assumptions are removed. The limits of validity of these base solutions must be determined more rigorously than it was possible in this investigation. The end-state results do show the conditions under which finite reservoir heat transfer rates are necessary to determine mass transfer and pressure levels. The effects of a non-ideal carrier gas, non-zero mixture transport properties, phase change, thermal slip, and velocity slip were beyond the scope of this study and must be examined in future investigations. In any event, the parameter dependence of the time to adiabatic pressure equilibrium should prove to be useful for scaling estimates even when many of the assumptions are not strictly valid. In the pure gas limit, such a result has been shown to be true in many cases.

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Figure 1: Mixture ratio of specific heats versus particle mass fraction for $\gamma_g = 5/3$ with δ as parameter: —, $\delta = 10^{-2}$; ·····, $\delta = 10^{-1}$; ---, $\delta = 1$; - · - · -, $\delta = 10$.

Figure 2: Mixture sound speed versus particle volume fraction for $\gamma_g = 5/3$ with r as parameter: (a) $\delta = 10^{-2}$ (b) $\delta = 1$: —, $r = 10^{-1}$; ·····, $r = 10^{-2}$; ---, $r = 10^{-3}$; — · —, $r = 10^{-4}$.

Figure 3: (a) Minimum sound speed, and (b) corresponding particle volume fraction versus r for $\gamma_g = 5/3$ with δ as parameter: —, $\delta = 10$; ·····, $\delta = 1$; ---, $\delta = 10^{-1}$; — · —, $\delta = 10^{-2}$.

Figure 4: Equilibrium pressure versus initial pressure ratio with V' as parameter: (i) thermal equilibrium and (ii) adiabatic limit for $\alpha = \beta$: —, $V' = 1$; ·····, $V' = 3$; ---, $V' = 10$; — · —, $V' = 100$.

Figure 5: Equilibrium supply volume fraction versus initial supply volume fraction: (i) isothermal limit with $\bar{P}(\infty)/\bar{T}(\infty)$ as parameter, and (ii) adiabatic limit with $\bar{P}_1^{1/\gamma}(t_{eq})$ as parameter: —, $\bar{P}(\infty)/\bar{T}(\infty)$ or $\bar{P}_1^{1/\gamma}(t_{eq}) = 0.8$; ·····, $\bar{P}(\infty)/\bar{T}(\infty)$ or $\bar{P}_1^{1/\gamma}(t_{eq}) = 0.6$; ---, $\bar{P}(\infty)/\bar{T}(\infty)$ or $\bar{P}_1^{1/\gamma}(t_{eq}) = 0.4$; - · - · - , $\bar{P}(\infty)/\bar{T}(\infty)$ or $\bar{P}_1^{1/\gamma}(t_{eq}) = 0.2$.

Figure 6: Mass defect versus adiabatic equilibrium pressure for $\gamma_1 = 7/5$, $\alpha = \beta$, and $\bar{T}_2(0) = 1$ with $\epsilon_1(0)$ as parameter: —, $\epsilon_1(0) = 1$; ·····, $\epsilon_1(0) = 0.8$; ---, $\epsilon_1(0) = 0.6$; — · —, $\epsilon_1(0) = 0.4$; — — —, $\epsilon_1(0) = 0.2$; — — — —, $\epsilon_1(0) \ll 1$.

Figure 7: Mass defect versus initial pressure ratio with V' as parameter for $\gamma_g = 5/3, \gamma_1 = 7/5, \bar{T}_2(0) = \bar{T}(\infty) = 1; \epsilon_1(0) \ll 1$: (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$; $\epsilon_1(0) = 1/2$: (d) $\bar{\phi}_2(0) = 1$, (e) $\bar{\phi}_2(0) \ll 1$, (f) $\epsilon_2(0) = 1/2$: —, $V' = 0.3$; ·····, $V' = 1$; ---, $V' = 3$; — · —, $V' = 10$; — — —, $V' = 100$.

Figure 8: Equilibrium pressure ratio versus initial pressure ratio with V' as parameter for $\gamma_g = 5/3, \gamma_1 = 7/5, \bar{T}_2(0) = \bar{T}(\infty) = 1$; $\epsilon_1(0) \ll 1$: (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$; $\epsilon_1(0) = 1/2$: (d) $\bar{\phi}_2(0) = 1$, (e) $\bar{\phi}_2(0) \ll 1$, (f) $\epsilon_2(0) = 1/2$: —, $V' = 0.3$; ·····, $V' = 1$; ---, $V' = 3$; — · — ·, $V' = 10$; — — — —, $V' = 100$.

Figure 9: Critical pressure ratio versus supply volume fraction with γ_1 as parameter; inset is comparison with equation (79): —, $\gamma_1 = 1$; ·····, $\gamma_1 = 1.2$; ---, $\gamma_1 = 7/5$; — · —, $\gamma_1 = 5/3$.

Figure 10: Sonic time scale factor versus supply mass fraction for $\gamma_g = 5/3$ with δ_1 as parameter: —, $\delta_1 = 10^{-2}$; ·····, $\delta_1 = 10^{-1}$; ---, $\delta_1 = 1$; — · —, $\delta_1 = 10$.

Figure 11: Time of adiabatic pressure equilibrium versus initial pressure ratio for $\epsilon_1(0) \ll 1$ and $\epsilon_2(0) \ll 1$ with γ_1 as parameter for (a) $\bar{V}_2 \ll 1$ and (b) $\bar{V}_2 \gg 1$, and corresponding volume fraction versus time with γ_1 and $\bar{P}_2(0)$ as parameters for (c) $\bar{V}_2 \ll 1$ and (d) $\bar{V}_2 \gg 1$: —, $\gamma_1 \rightarrow 1$; ·····, $\gamma_1 = 1.2$; ---, $\gamma_1 = 7/5$; — · —, $\gamma_1 = 5/3$.

Figure 12: Supply volume fraction for sonic isothermal discharge versus normalized time with $\epsilon_1(0)$ as parameter when $\delta_1 \gg 1$: —, $\epsilon_1(0) \ll 1$; ·····, $\epsilon_1(0) = 0.15$; - - -, $\epsilon_1(0) = 0.3$; — · — ·, $\epsilon_1(0) = 0.45$; — — —, $\epsilon_1(0) = 0.6$.

Figure 13: Time of adiabatic pressure equilibrium versus initial pressure ratio for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, and $\bar{T}_2(0) = 1$ with $\bar{V}_2$ as parameter: $\epsilon_1(0) \ll 1$: (a) $\bar{\phi}_2(0) \ll 1$, (b) $\bar{\phi}_2(0) = 1/2$; $\epsilon_1(0) = 1/2$: (d) $\bar{\phi}_2(0) = 1$, (e) $\bar{\phi}_2(0) \ll 1$, (f) $\bar{\phi}_2(0) = 1/2$: $\bar{V}_2 \ll 1$:; $\bar{V}_2 = 3$: ---; $\bar{V}_2 \gg 1$: —, — — —.

Figure 14: Volume fraction versus time for $\epsilon_1(0) \ll 1$ and $\bar{P}_2(0) = 1/2$ with $\bar{V}_2$ as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = 1$ and (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$: —: $\bar{V}_2 \ll 1$;: $\bar{V}_2 = 1$; —: $\bar{V}_2 = 3$; - - -: $\bar{V}_2 \gg 1$.

Figure 15: Volume fraction versus time for $\epsilon_1(0) = 1/2$ and $\bar{P}_2(0) = 1/2$ with $\bar{V}_2$ as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = 1$ and (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\bar{\epsilon}_2(0) = 1/2$: —, $\gamma_2 = 1$; $\cdots$, $\gamma_2 = 3$; $---$, $\gamma_2 = 5$; $- \cdot -$, $\gamma_2 = 7$.

Figure 16: Volume fraction versus time for $\epsilon_1(0) \ll 1$ and $\bar{P}_2(0) = 0.01$ with $\bar{V}_2$ as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $\bar{T}_2(0) = 1$ and (a) $\bar{\sigma}_2(0) = 1$, (b) $\bar{\sigma}_2(0) \ll 1$, (c) $\bar{\sigma}_2(0) = 1/2$; —: $\bar{V}_2 = 1$;: $\bar{V}_2 = 3$; ---: $\bar{V}_2 \gg 1$.

Figure 17: Volume fraction versus time for $\epsilon_1(0) = 1/2$ and $\bar{P}_2(0) = 0.01$ with $\bar{V}_2$ as parameter for $\gamma_g = 5/3$, $\gamma_1 = 7/5$, $T_2(0) = 1$ and (a) $\bar{\phi}_2(0) = 1$, (b) $\bar{\phi}_2(0) \ll 1$, (c) $\epsilon_2(0) = 1/2$; — $\bar{V}_2 = 1$; $\bar{V}_2 = 3$; - - - $\bar{V}_2 \gg 1$.

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