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A Computational Investigation of the Limits to Pease-Braginskii Collapse of a Z-Pinch

Paul Douglas Nielsen
(Ph.D. Thesis)

June 1981

Lawrence
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A Computational Investigation of the Limits to
Pease-Braginskii Collapse of a Z-Pinch

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ABSTRACT

A z-pinch is a magnetohydrodynamic phenomenon that consists of a current-carrying cylinder of plasma confined by its self-induced azimuthal magnetic field. The interaction of the z-directed current and the azimuthal field produces a $\mathbf{j} \times \mathbf{B}$ force that is directed radially inward. Ignoring two-dimensional effects, the plasma compresses radially until its internal pressure balances the magnetic pressure. This pressure equilibrium was first described by Bennett in 1934 and is called a Bennett equilibrium.

In the late 1950's, Pease and Braginskii independently demonstrated that the Bennett equilibrium was not sufficient for a true equilibrium. Radiative cooling and resistive heating could perturb the plasma's internal pressure. Due to these effects, a plasma otherwise at pressure equilibrium could contract or expand. The current at which a plasma is in Bennett equilibrium and radiative cooling equals ohmic heating is called the Pease-Braginskii current. For an optically thin

plasma, a current higher than the Pease-Braginskii current would cause a catastrophic collapse -- internal energy continually would be lost through radiation faster than it could be developed through joule heating and the plasma's radius would approach zero.

This dissertation investigate the one-dimensional limits to such a radiation enhanced collapse through the use of a Lagrangian simulation code, LASNEX. The code includes the effects of a wide range of phenomena -- opacity, ionization, experimentally determined equations of state, magnetic effects on transport coefficients, and external electrical circuits. Special attention was given to the magnetic field subroutines. They were revised to include ion acoustic and lower hybrid drift induced resistivity and to increase accuracy and efficiency. The magnetic pressure term was differenced in a manner that eliminates any influence of zone size, allowing large, low density zones outside the plasma column. In these large zones, magnetic flux and energy were determined by direct integration instead of summation to ensure overall conservation. With these changes, the computational timesteps were determined by phenomena in the plasma including the Alfvén velocity in the low density region. These modifications improved the accuracy of the code on Z-pinch problems by a factor of 10-100 depending on the minimum pinch radius reached.

The one-dimensional simulations contained in this report demonstrate limits to radiation enhanced collapse that depend on the initial plasma density and the rate of current rise. First, there is a

low density, fast current rise limit due to strong shock heating and poor electron-ion and electron-radiation coupling. Second, there is a high density, fast current rise limit due to the relative scaling of the pinch time and the LC circuit's quarter cycle time. Third, there is a high density, slow current rise limit due to the plasma's inertia and opacity.

Despite these limits, these simulations indicate a significant regime where one-dimensional radiation enhanced collapse does occur, where radiation fluences are prodigious, and where the collapse is terminated only when the plasma becomes optically thick, trapping energy in the interior of the pinch. Apparent discrepancies between these simulations and experiments are explainable by dimensional and geometric arguments and by the effects of enhanced resistivity. Resistivities of only 5-10 times classical values significantly affect pinch dynamics and reduce pinch densities and radiation fluences.

In one dimension, however, neither the ion acoustic nor the lower hybrid drift instabilities significantly affect pinch behavior despite the fact that their thresholds are exceeded. Two-dimensional simulations demonstrate the development of axial flow that reduces densities, energy coupling and radiation while increasing instability induced resistivity. The cumulative effect of this axial flow will terminate Pease-Braginskii collapse earlier than the one-dimensional limit based on the plasma becoming optically thick.

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There were many others involved in bringing me back to Livermore. Lyle Fox deserves special credit for his memory of an earlier application and his awareness of current events. During the course of this work, I was supported by the U. S. Air Force and the John and Fannie Hertz Foundation. My sincerest appreciations to both organizations. In the Air Force, I must thank Generals Lew Allen, Jr. and Robert C. Mathis for inspiration and encouragement and Colonels Larry Epplay, Dick White, and Tony Johnson, my military supervisors during my tenure in Livermore.

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- $\kappa_{\perp}$ transverse magnetic flux diffusion coefficient, resistivity
- $\kappa_{\parallel}$ parallel diffusion coefficient
- $\kappa_{\perp T}$ transverse thermoelectric diffusion coefficient
- $\kappa_{\perp T}$ thermal diffusion coefficient (perpendicular to field and temperature gradient)
- $\kappa_{\parallel T}$ parallel electron energy
- $\kappa_{\perp T}$ transverse thermal conductivity
- $\kappa_{\parallel T}$ thermal conductivity transverse to both the field and the temperature gradient
- $L_{\parallel}$ axial length
- $L_{\parallel e}$ electron-ion coupling coefficient
- $L_{\parallel r}$ electron-radiation coupling coefficient
- $L_{\parallel r, br}$ electron-radiation coupling due to bremsstrahlung radiation
- $L_{\parallel r, fb}$ electron-radiation coupling due to free-bound processes
- $L_{\parallel r, cs}$ electron-radiation coupling due to Compton scattering
- m plasma density
- ω_{pe} electron plasma frequency
- τ_e electron-ion collision time
- τ_i ion-ion collision time
- ω_{ce} electron cyclotron frequency
- ω_{ci} ion cyclotron frequency
- ω_{UH} lower hybrid frequency

Chapter 1

INTRODUCTION

1.1. Problem Statement

The purpose of this study is to computationally investigate the limits to the one-dimensional radiation enhanced collapse of a *z-pinch*. A *z-pinch* is defined simply as a hollow or solid, current carrying cylinder of plasma confined by its self-induced azimuthal magnetic field. In this sense, the "Z" of Z-pinch refers to the direction of current flow in cylindrical coordinates, (see Figure 1.1). The actual implosion or collapse of the plasma column is radial.

For the purposes of this report, radiation enhanced collapse also has a special meaning. Obviously, any radiation, however small, will aid the radial implosion since radiation will cool the plasma, reducing the thermal energy and, consequently, the kinetic pressure of the column. In this report, however, radiation enhanced collapse refers to the specific case where radiative energy losses exceed ohmic heating--where the current exceeds the Pease-Braginskii current.¹⁻³

Figure 1.1 Z. Pinch Geometry. The interaction of the z directed current and its induced azimuthal field create a $J \times B$ force that is directed radially inward.

neglecting, for the moment, other heat sources and macroscopic instabilities, this condition would lead to a catastrophic collapse of the plasma column -- the plasma would get denser and cooler until the increasingly degenerate electron gas approached its Fermi energy limit.

Despite many years of research, no experiments have indicated such a dense, cool plasma and it is obvious, therefore, that other physical phenomena limit the collapse. On the other hand, a radiation enhanced Pease-Braginskii collapse may play an important role in the dynamic evolution of a Z-pinch. In particular, Shearer⁴⁻⁵ has suggested that such a collapse might be responsible for the formation of minute, hot spots repeatedly observed in Z-pinch experiments.

Since the Pease-Braginskii current is essentially a balance between radiative cooling and ohmic heating, the limits to radiation enhanced collapse may involve these phenomena -- through increased radiative opacity which diminishes the energy flow out of the plasma or through enhanced resistivity which increases the energy flow into the plasma. Enhanced resistivity is of particular interest since there are several known kinetic instabilities that may lead to increased resistivity. The effects of two of these instabilities -- the ion-acoustic instability and the lower hybrid instability -- are examined.

This study investigates one-dimensional radiation enhanced collapse through the use of a sophisticated Lagrangian magneto-hydrodynamic code (see Appendices A and B for a more complete description) which includes flux limited single and multi-group radiation transport models, and zonal, self-consistent phenomenological

models for both the ion-acoustic instability and the lower hybrid instability. Through simulation, the dynamic evolution of a radiating Z-pinch is observed and the limits to a one-dimensional radiation enhanced collapse are demonstrated.

1.2 History of the Z-Pinch

Z-pinchs have been studied for many years.⁶ One of the earliest recorded observations of a Z-pinch was made by R. W. Wood in 1897.⁷ Although several of Wood's conclusions were erroneous, he correctly noticed the pulsed nature of the pinch's x-ray fluences as well as its remarkable intensity. Wood's Z-pinch was a type now known as a vacuum spark, formed by a high current discharge between electrodes in a vacuum. Wood suggested a use for his vacuum spark Z-pinch but interest in it quickly waned due to the development of other x-ray sources and the vacuum spark's poor reproducibility.

Following World War I, the vacuum spark Z-pinch found a new and interesting application. R. A. Millikan and various collaborators began using it as an internal light source for spectroscopic studies.^{8,9} These researchers were motivated by their desire to extend spectral observations to increasingly shorter wavelengths, but the more exciting discovery was that the resulting spectra indicated that electrode atoms had been ionized four or five times. These high ionization states quickly established the vacuum spark as the primary source for vacuum ultraviolet and x-ray spectroscopy.¹⁰ This line of research has continued to the present and has led to the accumulation of spectra for highly ionized species of high Z elements. More recent

publications, all of which concentrate on spectra demonstrating helium-like and hydrogenic ions of elements such as titanium, copper, and iron, include Cohen et al. (1968),¹¹ Lie and Elton (1971),¹² Cilliers, Datla, and Griem (1975),¹³ and Negus and Peacock (1979).¹⁴

A related but separate line of Z-pinch research was based on the self-pinchng phenomenon described by W. H. Bennett in 1934.¹⁵ Bennett discovered a pressure equilibrium relationship between the current carried by a cylindrical plasma and the product of the plasma line density and kinetic temperature. If the critical or Bennett current was exceeded, the magnetic pressures would exceed kinetic pressures and the plasma would contract radially until sufficient PdV heating brought the plasma into a new pressure equilibrium. This simple concept excited researchers trying to develop a controlled thermonuclear reactor. It appeared that simply passing a sufficiently large current through a deuterium-tritium plasma would satisfy all of the requirements for a fusion reactor (primarily satisfying the Lawson criteria, $n\tau \geq 10^{14}$ sec/cm³ and attaining an ignition temperature ≥ 10 keV). In addition, such a device would be easily cycled and could operate at power densities containable by conventional structural materials.

Unfortunately, experimentalists and theorists quickly determined that the Z-pinch was subject to several macroscopic instabilities. Two of the most troublesome are among the easiest to visualize -- the sausage instability and the kink instability. In the sausage instability (see Figure 1.2a), localized inhomogeneities result in the

Figure 1.2a The Sausage ($m=0$) instability produces localized constrictions in the plasma.

Figure 1.2b The Kink ($m=1$) instability moves the plasma off axis and often results in a helical structure.

plasma necking off or pinching down at one or more points along the column resulting in a plasma similar in appearance to a link sausage -- several plasma sausages joined by filamentary links of plasma. In the kink instability (see Figure 1.2b), the plasma itself moves off axis, locally bending. Often this manifests itself in a helical, spring-like structure. These instabilities were predicted theoretically by Kruskal and Schwarzschild using mode analysis and a skin current,¹⁶ by Low and using the energy principle and a distributed current,¹⁷ and by Kadomtsev and used both a skin current and a distributed current.¹⁸

Researchers then attempted to avoid the consequences of these instabilities either by making the pinches faster or by devising stabilizing mechanisms. Faster pinches did succeed in producing hotter, denser plasmas whose stability exceeded the bounce time of the plasma and, in some cases, the instability growth time. To their credit, these faster pinches produced even more highly ionized plasmas but, unfortunately, confinement times have remained too low for appreciable thermonuclear yield. Stabilized pinches, on the other hand, have greatly increased confinement time, but have done so at the expense of density and temperature. Stabilization of a Z-pinch by an axial magnetic field is one of the simplest schemes and provided the first evidence of the stabilizing properties of various plasma field configurations.¹⁹ Increasingly sophisticated stabilization configurations, with names such as the reversed axial field pinch, the screw dynamic pinch, the inverse pinch, and Triax, all failed as potential fusion devices. Although these devices demonstrated increased stability, new instability modes eventually disrupted the

pinches despite theoretical predictions. These predictions were generally made assuming infinite conductivity. Later studies showed that the introduction of finite conductivity into the system of equations results in new instabilities.²⁰ Despite these disappointments in Z-pinch fusion programs, the research advances made in these efforts provided a rich legacy for the currently more promising toroidal pinch experiments.

Current interest in Z-pinchs rests primarily on their use in spectroscopic work and as x-ray sources although the more exotic possibility of controlled fusion is still the focus of some research. There are four methods used to produce Z-pinch plasmas: vacuum sparks, plasma focus machines, exploding wires, and gas puffs. Each method has its own advantages and disadvantages.

The first of these methods is the vacuum spark produced Z-pinch. This was the method used by early workers in the field such as Wood¹ and Millikan^{6,9} and involves a high current discharge between two electrodes in a vacuum. Generally a third electrode is used to create a plasma of electrode material just prior to the main discharge. Although this method produces some of the hottest, most ionized plasmas, it is inherently and infamously nonreproducible. The experimenter has little control over the initial plasma density (produced by blow-off of electrode material), and the timing between the initial and main discharges may vary. Still, the vacuum spark Z-pinch has been a valuable research tool^{11-14,21} and it continues to be the simplest, most energy efficient Z-pinch device.

The plasma focus attempted to overcome problems in scaling the vacuum spark to higher input energies. It does this by delaying the time at which the plasma stagnates on itself through the use of a two-dimensional geometry. The washer-like ring of plasma is electro-magnetically accelerated down a cylindrical channel between electrodes before it ultimately reaches the end of the inner electrode forming a more conventional Z-pinch. Temperatures are generally cooler than in vacuum sparks and, unfortunately, only a small fraction of the gas reaches the focused Z-pinch stage. The rest is left behind often wreaking havoc on the experiment via flashovers or restrikes that short circuit the pinch. The plasma focus program at Livermore has been essentially curtailed, but work on focus machines continues at many laboratories around the world including, most prominently, the U.S.S.R.

Exploding wire experiments involve stretching one or more wires of conducting material between two electrodes (in a similar vein, but with increased symmetry, some researchers have used a cylindrical foil instead of an array of wires). Then a capacitor bank or a pulse line is connected across the electrodes. The wire or wires resistively heat until the vaporization temperature is reached. The vapor is initially a poor conductor and, therefore, the current drops and the vapor explodes radially. As the vapor's density falls, breakdown occurs and the vapor becomes a highly conductive plasma. The current rises quickly and the plasma begins to implode radially as a conventional Z-pinch. While usually cooler than vacuum spark plasmas, exploding wire devices generally involve more massive plasmas coupled to more energetic sources. Consequently, they produce more x-rays than vacuum

sparks although the intensity of the fluence may be lower. One limitation to exploding wire plasmas is that all experiments must start with solid density. Exploding wire and foil plasmas are currently being investigated at many sites including Physics International, Maxwell Laboratories, the Naval Research Laboratory, and the Air Force Weapons Laboratory.²⁵

The most common method for creating Z-pinch plasmas is the gas puff method developed by J. Fisher in 1973.²⁶ In the gas puff, a jet of gas is introduced into the vacuum chamber prior to the capacitor discharge. The jet may be a solid cylinder (akin to the vacuum spark or a single exploding wire) or a hollow cylinder (similar to an exploding foil). In either event, the researcher has much better control over the initial density of the plasma and can choose from a wide variety of gaseous materials. In his Ph.D. dissertation, J. H. Shilton demonstrated the reproducibility of the gas puff Z-pinch as well as the increased energy conversion efficiency of the hollow jet over the solid jet.²⁷ The use of gas puff Z-pinch has made it more practical to examine finite Larmor radius (FLR) stabilized Z-pinch as suggested by J. W. Marshall.²⁸ In particular, K. W. Struve, using a gas puff Z-pinch, successfully demonstrated the scaling of instabilities with electron line density which, in turn, is related to the relative importance of FLR effects.²⁹ Other work with gas puff Z-pinch is continuing, including a promising atomic physics experiment at Livermore by Price, Stewart, Dietrich, and Fortner.³⁰

and self-consistent simulations, in particular, advanced during the 1980's. Hameiri and Killeen^{35,36} used tensor thermal conductivity and alternating direction implicit numerical techniques. Later, separate researchers included resistivity due to the low-frequency ion instability in their theta-pinch simulations. Hameiri and Killeen³⁴ used a phenomenological model for the frequency-dependent thermal conductivity. Hamasaki and Enli³⁷ and Hameiri and Killeen³⁸ used a hybrid multi-particle model where the electrons are treated as a fluid and the ions are treated as particles.

Even in the case of axisymmetrized θ -pinch, however, computational time is still prohibitive, especially when radiative effects are included. In fact, intense radiation is accompanied by the emission of high-energy neutrons whose radii are so small in relation to the size of the plasma core that any strictly numerical code would be required to follow them closely. On the other hand, two-dimensional or three-dimensional codes would involve extreme spatial discretization which would ultimately affect the simulation's accuracy and possibly require a very small timestep. Even one-dimensional codes may have to be careful to avoid the problem of restrictive timestep inside the imploding plasma. There is also the nontrivial problem of modeling electron transport, ionization, equation of state for extreme regimes, and the effects of important microinstabilities.

Despite these difficulties, the intense x-ray production of nonstabilized z-pinchs has continued to tantalize researchers into the arduous task of continual development of new and better simulation codes. At the Lawrence Livermore National Laboratory, an early one-dimensional code, MAGPIC⁴¹, was used by Kidder⁴² to simulate tube compression experiments. Later, a modified version of MAGPIC was used by another group, Cheng, Cooper, Eddleman, and Munger⁴³ to study the behavior of the jet of the gas-embedded Z-pinch. Still later, Sneider² and others supported the hypothesis that the minute hot spots in the jet were due to localized Pease-Braginskii collapse. In these experiments, it was assumed that electrons and ions have the same temperature and that the plasma is optically thin.

Two-dimensional simulations of the dense plasma focus were conducted by Ginzburg and Iosadnick⁴⁴ and by Potter⁴⁵. Chase, et al.,⁴⁶ who have developed a two-dimensional Eulerian code⁴⁶ which, although not specifically designed for the focus, was used extensively for two-dimensional studies. Using this code, Maxon and Eddleman⁴⁷ conducted a study which improved upon Potter's earlier work and closely matched available experimental data. Meanwhile, in the Soviet Union, Viskrey and others⁴⁸ conducted studies⁴⁸⁻⁵⁰ that studied the effects of radiation and ionization on the final phases of a deuterium plasma focus.

A whole circle of researchers has been involved with the simulation of exploding wires and liners. The two-dimensional Eulerian code SPEDM⁵¹ has been used extensively at the Air Force Weapons Laboratory for exploding wire⁵² and imploding foil⁵³⁻⁵⁷ simulations. At the Naval Research Laboratory, Mosher et al.^{53,58}

used a one-dimensional, single fluid Eulerian code to simulate exploded wires driven by dielectric transmission line pulse generators. This code successfully predicted the electrical characteristics of the discharge as well as the x-ray characteristics of high atomic number exploding wire plasmas. More recently, Colombant, Lampe, and Bloomer⁴⁹ have developed WHYRAC, a one-dimensional exploding wire code that includes an artificial enhanced resistivity in the low density region while Boris⁵⁰ has developed ADINC, an implicit Lagrangian code. At Livermore, Lindemuth expanded upon his Ph.D. thesis to produce WHIDAT⁵¹, an extremely versatile two-dimensional Eulerian code that has been used for liner implosion studies⁵². Mason⁵³ has determined that, on similar Z-pinch problems, ANIMAL, the Lawrence Livermore Laboratory's BRAHMA, and the Chase, LeBlond and Wilson code⁵⁴ agree with only minor differences -- a reassuring test of these Eulerian codes.

This dissertation addresses a problem suggested by Shearer⁴ and becomes relevant by the advent of the gas puff Z-pinch conceived by Fisher⁵. Shearer resurrected the idea of the Pease-Braginskii collapse and invoked it as a plausible explanation for the intense radiation emitting hot spots observed in Z-pinchs. His simulations with MCODE suggested the importance of phenomena not included in that code, such as anomalous resistivity and opacity, in the dynamic evolution of a collapsing pinch. Shilon²⁷ demonstrated that Fisher's gas puff geometry created a reproducible Z-pinch. As a result, the fast RC electrical circuits originally designed for exploding wires and

foils could be used to drive plasmas with less than solid initial densities. Two obvious questions arose: What are the limits to radiation collapse of these lower density plasmas and is there a radiation optimizing combination of initial plasma density and rate of current rise? These are the questions addressed in this dissertation.

1.1 Organization of the Report

Following the Introduction, Chapter 2 describes several analytical models important to the concept of the Z-pinch and to an understanding of the subsequent simulations. Chapter 3 reports the results of a series of simulations, none of which includes enhanced resistivity, which demonstrate a regime of Pease-Braginskii collapse for a tungsten plasma. Chapter 4 examines the effects of increased resistivity, either artificial or microinstability induced, on a plasma in radiation enhanced collapse. It also considers the potential impact of two-dimensional effects. Chapter 5 summarizes the results of this study and suggests further work in this field. Appendix A describes the simulation code, WICK, in general terms while Appendix B details the code's geometric field subroutines.

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Chapter 2

ANALYTICAL MODELS

2.1 The Bennett Current

In his 1934 paper, "Magnetically Self-Focussing Streams,"¹ W. H. Bennett presented a relationship that tantalized controlled thermonuclear researchers of the 1940's and 1950's. Bennett's original paper developed the relationship relativistically, but this, in general, is not necessary for the electron drift velocities present in Z pinch devices. The Bennett relationship is essentially a pressure balance between magnetic and kinetic pressures. Using the one-dimensional, one fluid magnetohydrodynamics (MHD) equations and assuming only radial dependencies, the steady state momentum transfer equation is

$$\frac{dp}{dr} = - \frac{J_z B_\theta}{c} \quad (2.1)$$

Using the MHD limit of Ampere's law,

$$J_z = \frac{c}{4\pi r} \frac{d}{dr} (r B_\theta) \quad (2.2)$$

and substituting into equation 2.1 results in

$$\frac{dp}{dr} = - \frac{1}{8\pi r^2} \frac{d}{dr} (rB_\theta)^2 \quad (2.3)$$

This equation and the boundary condition that $p(R) = 0$, where R represents the plasma radius, are all that is needed to derive the expression for the Bennett current.

The average pressure in the plasma can be found from

$$\langle p(r) \rangle = \frac{\int_0^R p(r) 2\pi r dr}{\int_0^R 2\pi r dr} \quad (2.4)$$

$$= \frac{2}{R^2} \int_0^R p(r) r dr \quad (2.5)$$

Integration by parts and imposing the boundary condition that $p(R) = 0$ results in

$$\langle p(r) \rangle = - \frac{2}{R^2} \int_0^R \frac{r^2}{2} \frac{dp(r)}{dr} dr \quad (2.5)$$

Now, substitute for dp/dr using equation 2.3 to obtain

$$\begin{aligned} \langle p(r) \rangle &= \frac{1}{8\pi R^2} \int_0^R \frac{d(rB_\theta)^2}{dr} dr \\ &= \frac{B_\theta(R)^2}{8\pi} \end{aligned} \quad (2.6)$$

which simply states that, at equilibrium, the average kinetic pressure is equal to the magnetic pressure at the plasma radius. But in c.g.s. units, $B_\theta(R) = 2I/R$ and $\langle p(r) \rangle = \frac{1}{2} n_u k T_u$ where u represents the

various species present. For a single ion species in a quasi neutral plasma, these substitutions transform equation 2.6 into

$$n_e k \left(T_e + \frac{I^2}{Z^2} \right) = \frac{I^2}{2 \pi c^2 R^2} \quad (2.7)$$

or, reorganizing and assuming that $T = T_i = T_e$, the Bennett relation

$$I^2 = 2 N_e c^2 \left(1 + \frac{1}{Z} \right) kT \quad (2.8)$$

where $n_e = n_e \omega r^2$, the line density. Since current is more often measured in the m.k.s. units of amperes, divide by $(c/10)^2$ to obtain

$$I^2 = 200 N_e \left(1 + \frac{1}{Z} \right) kT \quad (2.9)$$

where I is expressed in amperes, N_e in cm^{-1} and kT in ergs.

This is the Bennett current, I_B . It is important to note that the Bennett current is derived without any knowledge of the specific radial distribution of the current or pressure within the plasma. At the Bennett current, the plasma is in a pressure equilibrium and the magnetic pressure, $B_\theta^2/8\pi$, is just balanced by the kinetic pressure, $n_i (1 + 1/Z) kT$. At higher currents, the magnetic pressure is greater than the kinetic pressure and the plasma contracts, heating the plasma until the Bennett current equals the circuit current. On the other hand, if the current is lower than the Bennett current, kinetic pressures outbalance magnetic pressures and the plasma expands, cooling until the Bennett current reaches the circuit current value.

For an isothermal plasma with a radially independent electron drift velocity, v_d , the momentum equation can be more directly integrated, avoiding the loss of information in the pressure averaging. For this

constant drift velocity model, an equation for density in terms of radius results

$$\frac{d}{dr} \left(\frac{r}{n} \frac{dn}{dr} \right) + \frac{4\pi e^2 v_d^2}{c^2 kT(1 + 1/2)} nr = 0 \quad (2.10)$$

Imposing the natural boundary condition that $dn/dr = 0$ when $r = 0$ and by assuming that $n(0) = n_0$, the solution of this equation can be found to be

$$n(r) = \frac{n_0}{(1 + n_0 b r^2)^2} \quad (2.11)$$

where $b = \frac{4\pi e^2 v_d^2}{c^2 kT(1 + 1/2)}$

This solution is known as the Bennett distribution (see Figure 2.1). Actual Z-pinches are generally not in pressure equilibrium, are rarely isothermal, and have electron drift velocities that are very radially dependent. Consequently, density distributions are often radically different from the Bennett distribution. The primary benefits of Bennett's model rest in his derivation of the Bennett current with its simple explanation of magnetically driven implosions and in the flurry of work it inspired which resulted in more realistic analytical models.

2.2 The Pease-Draginskii Current

While the Bennett current provides a simple explanation for the radial contraction or expansion of a plasma column due to current induced magnetic pressures, it neglects many phenomena that are

Figure 2.1 The Bennett distribution is the solution to the momentum equation assuming a constant electron drift velocity.

important processes are: two such processes are the bremsstrahlung radiation of energy away from the plasma and the resistive heating of the plasma by the (separatrix) current¹⁻⁴ couples the effects of radiative heating and resistive cooling with the concept of a Bennett equilibrium, while assuming that the plasma is optically thin.

Let us first calculate the eddy work per unit length performed on the plasma⁵

$$W = \frac{1}{2} \int_0^2 \frac{d}{dt} \left(\frac{1}{2} \right) dt = 0 \quad (2.1)$$

then we have

$$W = \frac{1}{2} \int_0^2 \frac{d}{dt} \left(\frac{1}{2} \right) dt = 0 \quad (2.2)$$

where the terms are radiative cooling are considered to be the same as in the case of a plasma with a uniform temperature, then we have⁶

$$W = \frac{1}{2} \int_0^2 \frac{d}{dt} \left(\frac{1}{2} \right) dt = 0 \quad (2.3)$$

Let us first calculate the eddy work per unit length for a fully ionized hydrogen plasma, equation (2.1) becomes⁷ $W = 4k_B T \ln 2$, substituting this expression for W into equation (2.3) produces

$$W = \frac{1}{2} \int_0^2 \frac{d}{dt} \left(\frac{1}{2} \right) dt = 0$$

$$\frac{d}{dt} = \frac{1}{\tau_R} \left(\frac{\rho}{\rho_R} - 1 \right) \quad (2.15)$$

where $\tau_R = 4\pi\epsilon_0 kT_e / e^2$, the radiation time constant.

Now, let us return to equation 2.15, digress for a moment to the plasma current, I_{p0} , itself. If the plasma is in equilibrium, then the Bennett relationship must hold.

Let us first express the current, and then express both the current and the density, n_p , as integrals of current density and the density, respectively, to obtain

$$I_{p0} = \frac{1}{2} \int_0^R (2\pi r) dr \int_0^\pi n_p(r, \theta) d\theta \quad (2.16)$$

and the density, n_p , in equation 2.15, to obtain the radiation loss rate per unit length when a pressure equilibrium has been achieved, i.e., when heating equals radiative cooling. The pressure equilibrium condition, through the use of the Bennett relation, the equilibrium temperature, and length is

$$n_p = \frac{4\pi}{3} \frac{e^2}{kT_e} \int_0^R r^2 dr \quad (2.17)$$

where $\frac{4\pi}{3} \frac{e^2}{kT_e}$ is the β or resistivity, constant for the plasma temperature, the magnetic field^b ($\beta_p = \mu_0 I_{p0}^2 / 4\pi n_p kT_e$ in eqs. 2.16, 2.17) is the Coulomb logarithm, and Z is the ionization state. Similarly, the radiation energy loss rate per unit length is

$$P_{\text{rad}} = 4\pi\epsilon_0 q_{\text{eff}} (kT_e)^{1/2} \int_0^R n_p^2(r) r dr \quad (2.18)$$

where σ_{eff} is the effective bremsstrahlung radiation constant¹⁰

$\sigma_{\text{eff}} = \sigma_{\text{eff}}^0 (1 + 1.2 \log_{10} n_e)^{-1}$ ($\sigma_{\text{eff}}^0 = 1.4 \times 10^{-25} \text{ sec}^{-1}$), n_e is the electron density,

and σ_{eff}^0 is a correction factor for including the effects of bound-free absorption, $\sigma_{\text{eff}}^0 = 1.4 \times 10^{-25} \text{ sec}^{-1}$.

Substituting $\sigma_{\text{eff}} = \sigma_{\text{eff}}^0 (1 + 1.2 \log_{10} n_e)^{-1}$ into equation (1) and solving for kI results in

$$kI = \frac{\sigma_{\text{eff}}^0}{\sigma_{\text{eff}}} \left(\frac{\int_0^R n_e^2 dr}{\int_0^R n_e dr} \right)^{-1} \left(\frac{\int_0^R n_e dr}{\int_0^R n_e^2 dr} \right) \quad (2.1)$$

where $\sigma_{\text{eff}} = \sigma_{\text{eff}}^0 (1 + 1.2 \log_{10} n_e)^{-1}$ in equation (2.1) is a field

$$kI = \begin{bmatrix} \frac{1}{2} \frac{\int_0^R n_e^2 dr}{\int_0^R n_e dr} \\ \frac{1}{2} \frac{\int_0^R n_e dr}{\int_0^R n_e^2 dr} \end{bmatrix} \left(\frac{\int_0^R n_e^2 dr}{\int_0^R n_e dr} \right)^{-1} = \frac{\int_0^R n_e^2 dr}{\int_0^R n_e dr} \quad (2.2)$$

which is a function of n_e only. Letting $n_e = n_e(r)$ and substituting equation (2.2) into equation (1) results in

$$kI = \left(\frac{\int_0^R n_e^2 dr}{\int_0^R n_e dr} \right)^{-1} \frac{\int_0^R n_e^2 dr}{\int_0^R n_e dr} \frac{\int_0^R n_e dr}{\int_0^R n_e^2 dr} \quad (2.3)$$

where the quantities in the right-hand side are in cgs units and $\int_0^R n_e dr$ is the electron column current in megamperes, $\int_0^R n_e^2 dr$

is a parameter that is a function of current and density distributions and $\int_0^R n_e^2 dr$ is the bremsstrahlung current. The results show that the bremsstrahlung current is very sensitive to impurities (or higher Z).

Probably, σ_{eff}^0 would decay much quicker than the $(1 + 1.2 \log_{10} n_e)^{-1}$

indicate. This strong inverse dependency is due to g_{ff} which increases dramatically with increasing Z .

The full significance of the Pease-Braginskii current, however, rests in its relationship to P_{in}/P_R in equation 2.15. Algebraic manipulation will show that, in fact,

$$\frac{\dot{r}}{r_R} = \left(\frac{1/P_B}{1} \right)^2 \quad (2.22)$$

which can be substituted into equation 2.15 into

$$\frac{\dot{R}}{R} = \frac{3}{4} \frac{1}{1/R} \left(\left(\frac{1/P_B}{1} \right)^2 - 1 \right) \quad (2.23)$$

This relationship shows that the plasma will be in pressure and energy equilibrium only at the Pease-Braginskii current. At lower currents, Joule heating is larger than radiative losses and the increasing internal energy of the plasma will cause it to radially expand. When the current is larger than the Pease-Braginskii current, radiative losses dominate Joule heating, cooling the plasma and the plasma contracts. The model contains no mechanism for halting this convergent implosion and, in fact, indicates a catastrophic collapse to $r = 0$. Obviously, additional phenomena, not included in the model, such as shock propagation, anomalous resistivity and opacity, stop the collapse before $r = 0$.

2.3 The Snowplow Model

The Bennett current and the Pease-Braginskii current are useful equilibrium values that indicate when a cylindrical plasma should expand

or contract. Neither predicts the behavior of the plasma during the expansion or contraction. The snowplow (or infinite conductivity) model was developed independently by Leontovich and Osovet's⁸ and Rosenbluth, Garwin, and Rosenbluth⁹ to predict the dynamic behavior of rapidly imploding Z-pinches. Despite the simplicity of the model, it has survived a quarter century as a basic indicator of pinch time scaling for fast discharges.

The snowplow model begins by assuming infinite conductivity. Therefore, all of the current is considered to flow in a thin skin or sheath at the outer radius of the plasma. The model neglects kinetic pressure, as well as radiation, ionization and shock waves. Since the conductivity is infinite, no ohmic heating occurs and, since kinetic pressure is ignored also, temperature does not enter the equations developed. The theory is based solely on the effects of the magnetic pressure and the plasma inertia which may be expressed as:

$$\rho \frac{d}{dt} \left(\frac{r}{\sigma} \right) = - \frac{d}{dt} M \frac{dr}{dt} \quad (2.24)$$

where M is the mass per unit length of the plasma being pushed by the magnetic sheath. The magnetic field, B , is simply related to the current, i , by $B = i/r$. The mass per unit length can be expressed as $M = \rho_0 r_0^2 = r^2 \rho$ where r_0 is the initial plasma radius and ρ is the initial plasma density. Substitution of these terms into equation 2.24 results in

$$\frac{1}{\rho_0 r} \frac{dr}{dt} = - \frac{d}{dt} (r_0^2 - r^2) \frac{dr}{dt} \quad (2.25)$$

assuming that the current rises linearly in time, $i = (di/dt)t$.

(Struve¹⁰ discusses the case where the current rises linearly to a given value and then remains constant. Shiloh¹¹ lets the current rise sinusoidally. Each of these current models may more accurately reflect specific experiments. For this report, the linearly rising current model is sufficient, although the Shiloh sinusoidal current would be slightly more accurate.) Transforming to dimensionless variables, the equation reduces to

$$\frac{1}{x^2} \frac{dx}{dt} = -\frac{dI}{dt} (1 - x^2) \frac{dx}{dt} \quad (2.26)$$

where $x = \frac{r}{r_0} = \frac{\text{radius}}{\text{initial radius}}$

$$t = \frac{t}{t_c} = \frac{\text{time}}{\text{characteristic time}}$$

with $t_c = \left(\frac{c^2 \pi \mu r_0^4}{(dI/dt)^2} \right)^{1/4}$

Numerical integration of equation 2.26 with initial conditions of $x(0) = 1$ and $(dx/dt)|_{t=0} = 0$ has shown that the radius of the pinch equals zero when $t = 1.5, t_c$. The pinch time, t_p , therefore is

$$t_p = 1.5 t_c = 1.5 \left(\frac{c^2 \pi \mu r_0^4}{(dI/dt)^2} \right)^{1/4} \quad (2.27)$$

A refinement of the basic theory involves the inclusion of a kinetic pressure term which precludes the nonphysical result that the radius of the plasma would ever equal zero. Assuming an adiabatic equation of state, this requires the addition of a term $\alpha x^{-7/3}$, to the right-hand side of equation 2.26. The coefficient α is a dimensionless variable related to the initial pressure ($\alpha = 2 p_0 t_c^2 / \mu r_0^2$).

In addition to preventing the total collapse of the plasma, the inclusion of pressure slightly increases the pinch time. Figure 2.2 plots the radius versus time for the snowplow model and for the snowplow model with kinetic pressure.

The snowplow model was the first theory to recognize the importance of inertial forces in the dynamic development of a Z-pinch.¹³ At the pinch time, the internal temperature and pressure of the pinch are higher than that predicted from the equilibrium based Bennett model. This is caused by the sudden thermalization of the kinetic energy represented by the imploding plasma. In turn, this elevated pressure causes the plasma to "bounce" and again, inertial forces bring the plasma part an equilibrium value. The implosion is then repeated in a series of damped bounces until instabilities destroy the plasma column.

2.2.2 Shock Model

Shock waves have generally indicated the propagation of shock waves inside the plasma column. This contradicts the adiabatic compression equation in the snowplow model. An alternate model based on magnetically driven shock waves was presented by J. L. Allen.¹⁴

Allen's model consists of a one-dimensional sheet pinch with infinite conductivity, an instantaneous constant current, and shock waves that are thin relative to the discharge chamber's dimensions. The constant current produces a constant force which drives the current sheath towards the axis. At $t = 0$, a shock wave leaves the magnetically driven current sheath and races ahead of it toward the axis. As the

Figure 2.2 Snowplow radius versus time in normalized units. Kinetic pressure is ignored in the model represented by the solid line, while the dashed lines are based on models which include an adiabatic equation of state with the indicated coefficient, a .

shock wave passes through the Z-pinch, it heats the plasma and accelerates it. Ultimately the plasma is accelerated to the speed of the imploding current sheath.

Eventually, the forward racing shock wave reaches the axis and is reflected. The reflected shock further heats the plasma through thermalization of kinetic energy as it speeds back toward the advancing current sheath. When the reflected shock reaches the sheath, the pressures behind the shock are greater than the magnetic pressures, the implosion stops, and an expansion begins.

In a conventional Z-pinch, the current sheath is cylindrical instead of planar and the current is not constant but rises until near the pinch time. Both of these differences result in a constantly accelerating current sheath and increasing magnetic pressure as opposed to Allen's model of a constant velocity sheath with constant magnetic pressure. As a result, the effects of shock waves are greatly dependent on the initial density of the plasma and the rate of current rise. This will be demonstrated in Chapter 3.

3.5 MHD Instabilities

The once bright prospect of a controlled thermonuclear reactor based on the Z-pinch was tarnished irreparably by the effects of MHD instabilities. For significant thermonuclear yield from a D-T plasma, the Lawson criteria required that the product of the confinement time and the particle density be greater than a given derived number, $n\tau \geq 10^{14} \text{ cm}^{-3} \text{ sec}$, while plasma temperatures exceeded 10 keV.¹⁶ For the highest densities observed in early Z-pinch ($n = 10^{20} \text{ cm}^{-3}$), this

meant a confinement time of microseconds or longer. Unfortunately, it was quickly observed that MHD instabilities disrupted the pinches before the required confinement times were achieved. Much of the ensuing history of fusion research has focused on the search for stabilizing geometries or regimes that would increase the confinement time.

One of the earliest theoretical analyses of MHD instabilities affecting Z-pinches was by Kruskal and Schwarzschild in 1954.¹⁷ Using a surface current and a perturbation of the form $\exp(i\omega + ikz + \omega t)$, they demonstrated that the simple Z-pinch was unconditionally unstable. For each wave vector k ($k \neq 0$), there was a real positive solution for ω' in the dispersion equation indicating instability growth.

R. O. Taylor¹⁸ generalized the Kruskal-Schwarzschild perturbation to a form $\exp(im\phi + ikz + \omega t)$. He obtained the dispersion relation

$$\frac{1 + \lambda_0 \omega_0'}{\omega_0' + \omega_0'} \frac{1}{[1 + \lambda_0 \omega_0']^{1/2}} = \frac{1}{1 + \lambda_0 \omega_0'} \frac{1 + (\lambda_0^2 + \omega_0'^2)^{1/2}}{1 + (\lambda_0^2 + \omega_0'^2)^{1/2}} = \lambda_0 + m^2 \frac{K_{|m|}(\lambda_0)}{K'_{|m|}(\lambda_0)} \quad (2.28)$$

where $\lambda_0 = |k|r_0$ and $\omega_0' = \nu_0 \omega^2 r_0^2 / \mu_0$. This equation reduces to the Kruskal-Schwarzschild equation if $m = 1$. This relationship shows that for $m = 0$ and $m = 1$, there is instability for all wavelengths while for sufficiently short wavelengths there is instability for all m . In the short wavelength limit (which is thought to be the most unstable) the actual growth rate is

$$\omega = (2|k| \mu_0 / \nu_0 r_0)^{1/2} \quad (2.29)$$

which is often approximated as c_s/r_0 where c_s is the sound speed and r_0 is the plasma radius. This means that the e-fold growth time is approximately the time it takes a sound wave to travel across the plasma radius.

The pinch may be stabilized against the $m = 0$ instability and against small wavelength $m = 1$ instabilities by the inclusion of an axial field within the plasma. Tayler¹⁹ demonstrated this theoretically and showed the stabilizing effect of conducting walls on the longer wavelength $m = 1$ instabilities. Figure 2.3 summarizes this data.

Plasma stability based on the energy principle was first analyzed by Bernstein et al.²⁰ for the infinite conductivity case. Their findings are in agreement with Tayler.

The next step in complexity is the analysis of the diffuse pinch. Here the current is no longer confined to the plasma surface. Tayler²¹ used the energy principle to obtain a necessary condition for stability of a diffuse pinch. It is modest:

$$\frac{1}{2} \left(\frac{B^2}{\mu_0} + \frac{1}{2} \frac{E^2}{\mu_0} \right) \geq 0 \quad (2.30)$$

where $\mu_0 = 4\pi \times 10^{-7}$ and primes indicate derivatives with respect to r . Newbold²² demonstrated that the Suydam criterion was not sufficient and went on to construct necessary and sufficient conditions for stability of a diffuse pinch. His results include the special case of a sharp pinch core (with only a skin current). Separate analyses of diffuse pinches are given by Tayler¹⁸ and Kadomtsev²³ among

Figure 2.3 MHD Stability Regime. This graph demonstrates the stabilizing influence of axial fields and conducting walls. The cross-hatched area is a region of stability against both the $m = 0$ and $m = 1$ instabilities.

(from Glasstone and Lovberg¹⁵)

others,²⁴ while Furth, Killeen and Rosenbluth²⁵ more thoroughly discuss the effects of finite resistivity which introduces new instabilities such as the tearing and rippling modes.

Just as MHD instabilities had important effects on Z-pinch devices, they may also affect the dynamic collapse of a Z-pinch in a Pease-Braginskii contraction. The quasi-stationary, sausage like hot spots repeatedly observed in experimental Z-pinch devices may be a result of $m = 0$ instability. If so, the effect is hardly detrimental since the hot spots are the most intense radiation sources. Furthermore, these spots have demonstrated stability 10-100 times beyond the $m = 0$ instability e-folding time.⁷

2.6 Anomalous Resistivity

One phenomenon that could limit the Pease-Braginskii collapse of a Z-pinch would be the onset of microinstability enhanced resistivity.⁷ Resistivities higher than classically predicted values have been deduced in many Z-pinch experiments.²⁶ In these cases, it is easy to show that the electron drift velocities are beyond the requisite thresholds for several kinetic instabilities. Theta-pinch researchers have reported agreement between experimentally observed transport coefficients and theoretical models based upon the effects of the lower hybrid drift instability.²⁷⁻³⁴ Computer simulations have verified this agreement by using fluid codes with phenomenological models^{35,36} and by using hybrid fluid-particle codes.^{37,38} Vikhrev and Korzhavin^{2b} have examined the effects of the lower hybrid drift instability on a deuterium plasma focus Z-pinch using a one-dimensional fluid code

including a phenomenological model for the collision frequency and incorporating axial plasma flow.

Two kinetic instabilities are included in the simulations of this report - the lower hybrid drift instability and the ion acoustic instability. The threshold drift velocity for the lower hybrid drift instability is smaller than the ion thermal velocity. The effective collision frequency is given by Vikhrev and Korzhavin²⁶ as

$$\nu_{eff} = \frac{v}{V_{Th}} \frac{1}{2} \omega_{UH}^2 \quad (2.31)$$

where v is the drift velocity, V_{Th} , the ion thermal velocity, and ω_{UH} , the lower hybrid frequency.

The ion acoustic instability has a much higher threshold velocity. Although the actual condition is rather complicated, the threshold drift velocity is roughly equal to the sound speed if $\omega_{UH} \gg \omega_{pi}^{1/2}$. The effective collision frequency was determined by Sagdeev and Galeev⁴⁰⁻⁴¹ to be

$$\nu_{eff} = 10^{-4} \frac{1}{T_i} \frac{v}{c_s} \omega_{pi}^2 \quad (2.32)$$

where c_s is the sound speed and ω_{pi} is the ion plasma frequency. Gary and Paul^{42,43} arrive at the same value for the ion acoustic limit of a more general electrostatic turbulence derivation.

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Chapter 3

SECTION 1. ONE-DIMENSIONAL Z-PINCH

1.1. The analytical models

The analytical models presented in Chapter 2 are useful for gaining a general perspective of the z-pinch and for understanding the structure of certain key phenomena. However, each model is very limited and cannot predict exhaustively the phenomenologically rich behavior of the z-pinch. For example, the Bennett and the snowplow models are basically equilibrium models concerned with the effects of small perturbations. Neither model can predict the transient dynamic behavior of the pinch. The snowplow model is useful for rough scaling of macroscopic pinch behavior but disregards at least three important phenomena -- shock heating, radiation, and magnetic field diffusion. The inadequacy of analytical models naturally leads one to computational simulation.

In one dimension, a Lagrangian code is clearly preferred to an Eulerian one since the plasma compresses strongly and a large

low density region is required to correctly model the plasma's inductance. The code must include coupled equations for hydrodynamic motion, magnetic induction, ion energy transport, electron energy transport, radiation transport, and a driving electrical circuit. Furthermore, since a broad range of densities and temperatures are involved in a pinch, the code must have access to detailed equation of state data to determine the correct pressure, internal energy and specific heat of the materials; full ionization cannot be assumed and the ionization state of the ionization state.

One such code, although not specifically designed for 2-pinch fusion, is LASNEX, developed for this task. LASNEX is a two-dimensional code that is primarily destined for laser fusion studies. As such, the code is more comprehensive than is required for the present 1D pinch simulation. For example, models for laser interaction with matter and thermonuclear burn physics are not included in the code; the simulation code used is a reduced version of the code in this report, as well as dimensionality.

Since the quadrilateral zones. Therefore, although the quadrilateral zones in this chapter are one-dimensional in the radial direction, there is a fixed z . One-dimensionality is insured by requiring the components at $z = 0$ to advance according to their related dependence at $z = z_0$. The only quantities actually defined at the despoth are position and velocity. All of the others, such as temperature, pressure, density, and magnetic field strength, are zonal quantities, defined at the zone center.

In one dimension, the equations being solved reduce to the following set:

1. Hydrodynamic Motion

$$\frac{dv_r}{dt} = \frac{1}{\gamma} \frac{d^2 r}{dr^2} = \frac{1}{\rho c} \left(\frac{B_0}{4\pi r} \frac{\partial}{\partial r} (r B_0) \right)$$

2. Magnetic Induction

$$\frac{dB_0}{dt} = -\frac{1}{c} \frac{d}{dr} (v_r B_0) + \frac{c^2}{4\pi e^2 n_e^2} \left(\frac{1}{r} \frac{\partial}{\partial r} (r B_0) + \frac{ckp_\perp}{en_e} \frac{\partial i_e}{\partial r} \right)$$

3. Ion Energy Transport

$$r \frac{d}{dt} \left(\frac{1}{r} \right) = \frac{1}{r} (I_e - I_i) + \frac{\sigma}{dr} \left(\kappa_\perp \frac{\partial T_i}{\partial r} - (P_i + Q) \frac{\partial v_r}{\partial r} \right)$$

4. Electron Energy Transport

$$\frac{dT_e}{dr} \frac{dr}{dt} = \frac{1}{r} (I_e - I_i) + \frac{\sigma}{dr} \left(\kappa_\perp^e \frac{\partial T_e}{\partial r} - p_e \frac{\partial v_r}{\partial r} \right)$$

$$+ \frac{1}{c} \frac{1}{en_e} \frac{d}{dr} \left(\frac{ckp_\perp}{r} \right) + \frac{p_\perp}{en_e} \frac{d}{dr} \left(\frac{\partial i_e}{\partial r} \right) \\ + p_e \frac{\partial v_r}{\partial r} \frac{\partial v_r}{\partial r} + \frac{1}{r} (I_e - I_i)$$

5. Radiation Transport

$$4a \frac{dr}{dt} \frac{dr}{dr} = \frac{1}{r} (I_e - I_i) - \frac{\sigma}{dr} q_r - p_r \frac{\partial v_r}{\partial r} - a \frac{dr}{dr} \frac{\partial v_r}{\partial r}$$

In the listed equations, the following conventions are used (terms are further defined in the List of Symbols, page xii):

$P = P_i + P_e + P_r$ total pressure

$P_i = n_i k T_i$ ion pressure

$P_e = P_e(x, y)$ electron pressure from equation of state tables

$P_r = a T^4$ radiation pressure

$\nu = \nu_{\text{coll}} + \nu_{\text{artificial}} = \begin{cases} 0 & \text{if } \Delta v \leq 0 \\ \Delta v & \text{if } \Delta v > 0 \end{cases}$ artificial viscosity

$\kappa_{\text{eff}} = \kappa_{\text{eff}}(x, y, z)$ transport coefficients from Braginskii

$\gamma_{ei} = \frac{4\pi e^2 n_i}{m_e m_i} k$ electron-ion coupling coefficient

$\gamma_{er} = \frac{\gamma_{\text{brems}}}{4\pi} + \frac{\gamma_{\text{free}}}{4\pi} + \frac{\gamma_{\text{compton}}}{4\pi}$ electron-radiation coupling due to Bremsstrahlung, free-bound, and compton processes

$q_r = -\frac{1}{4\pi} \frac{\partial}{\partial x} \left(\frac{1}{\mu_0} \frac{\partial B^2}{\partial x} \right) - \frac{1}{4\pi} \frac{\partial}{\partial y} \left(\frac{1}{\mu_0} \frac{\partial B^2}{\partial y} \right) - \frac{1}{4\pi} \frac{\partial}{\partial z} \left(\frac{1}{\mu_0} \frac{\partial B^2}{\partial z} \right)$ radiation heat flux

$\lambda_e = \lambda_e(x, y, \text{composition})$ Rosseland mean free path from equation of state tables

$\epsilon_e = \epsilon_e(x, y)$ internal electron energy from equation of state tables

A more complete description of LASNEX is given in Appendix A and a detailed description of the magnetic field subroutines is given in

Appendix B. Two points made in these appendices should be highlighted.

First, the most obvious differencing of the magnetic pressure term in the hydrodynamic motion equation,

$$-\frac{J_z B_\theta}{c} = -\frac{1}{4\pi r} \frac{B_{\theta J+1} + B_{\theta J}}{2} \frac{(rB_\theta)_{J+1} - (rB_\theta)_J}{\Delta r}$$

is computationally inefficient for a Lagrangian code. This differencing requires extremely small zones in the low density region outside the plasma to achieve proper acceleration for the plasma column. The error is caused by the coefficient B_θ , a zone centered quantity that must be averaged to determine the meshpoint acceleration. All averaged values of B_θ will be smaller than the actual value of B_θ at the plasma's outer edge, since this last plasma zone will always carry some of the current. For large low density zones (desirable due to Alfvén velocity timestep restrictions), the effect may be significant. Consequently, instead of this formula, CRASH uses

$$-\frac{J_z B_\theta}{c} = -\frac{1}{8\pi r^2} \frac{(rB_\theta)^2_{J+1} - (rB_\theta)^2_J}{\Delta r}$$

where no averaging is required and large zoning in the low density region has no effect on the meshpoint Lorentz force.

Second, large zoning in the low density region enhanced the

efficiency of the simulations, but initially resulted in poor energy conservation. The error was found to be in the computation of the check quantities, not in the simulation algorithms. The magnetic energy in a given zone is determined by multiplying the zone's volume by the magnetic energy density:

$$\begin{aligned} L_{\text{mag}} &= (4\pi r_{\text{avg}}^2 \Delta z) (B_\theta^2/8\pi) \\ &= 4\pi r_{\text{avg}}^2 \Delta z (41^2/8\pi r_{\text{avg}}^2) \\ &= 41^2 \Delta z \Delta r/r_{\text{avg}} \end{aligned}$$

This is a good approximation for thin zones, but not for thick zones. In the low density zones, no current is allowed to flow (all of the current is in the plasma column) and, hence, the actual magnetic energy in these zones is

$$\begin{aligned} L_{\text{mag}} &= \int_{r_1}^{r_0} \frac{B_\theta^2}{2} 2\pi r dr dz \\ &= \int_{r_1}^{r_0} \frac{1}{r} dr dz \\ &= \Delta z \ln(r_0/r_1) \\ &= \Delta z \ln(1 + r \frac{dr}{r_1}) \end{aligned}$$

where r_{avg} is the zone's average radius, r_0 is the zone's largest (outside) radius, and r_1 is the zone's smallest (inside) radius. These two formulas approach each other as Δr approaches zero, but the

actual energy, as represented by the logarithmic formula, will always be larger, significantly so for thick zones.

5.2 Description of the Simulation Series

To examine the limits to a process as complex as Pease-Braginskii collapse, it is important to isolate the effects of various independent variables. This was the goal of the first simulation series. The parameters of interest were the initial density of the plasma and the initial current rise rate. To minimize the effects of other dependencies, the current inductance was kept constant at 5 nanohenries; the energy of the capacitor bank was kept constant at 100,000 joules; and the unperturbed dc circuit's maximum current was kept constant at 6.4 megamperes. With these constraints, the desired initial current rise rates were obtained by varying the capacitance and the charge on the capacitor. Table 2.1 summarizes the circuit parameters.

The element used for this study was tungsten, one of the highest Z elements existing in vacuum spark machines. A high Z element was chosen to enhance radiation losses which would lower the Pease-Braginskii current. At each current rise rate, tungsten plasmas with initial ion number densities of 10^{15} cm^{-3} , 10^{16} cm^{-3} , 10^{17} cm^{-3} , and 10^{18} cm^{-3} were studied. The plasma column was 1 cm long, 1 mm in radius, and was uniformly heated to 10 eV.

The simulations were performed by the CASHEX code¹. The three temperature model was chosen for radiation transport. Due to the large

(Circuit Parameters

Current Rise Rate	amp/sec	6×10^{11}	6×10^{12}	6×10^{13}	6×10^{14}
Circuit Capacitance	uf	22500	225	2.25	.0225
Circuit Charge	coulomb	67.5	6.75	.675	.0675
Circuit Voltage	kV	3	30	300	3000
Circuit Inductance	nh	5	5	5	5
Maximum Current	MA	6.4	6.4	6.4	6.4
Bank Energy	kJ	100	100	100	100
Quarter Cycle Rise Time	ns	16700	1670	167	16.7

Table 3.1. Parameters for circuits that drive the magnetic implosion. Inductance, maximum current and bank energy are held constant while capacitance and charge are varied to yield the desired current rise rates.

densities expected for the collapsing plasma, local thermodynamic equilibrium was assumed. The resistivity model is from Braginskii.² It includes the effects of magnetic fields on resistivities transverse to these fields, but does not include the resistivities enhancing effects of the ion acoustic or lower hybrid drift instabilities.

3.3 Energy Flow in a Z Pinch

The initial density of the plasma and the current rise rate play an important role in the plasma's evolving distribution of energy. As will be shown later, small changes in the relative distribution can greatly perturb the dynamic behavior of the pinch and, consequently, the amount of energy radiated. Nevertheless, there is a general pattern of energy flow that is worth noting prior to examining the differences.

Initially, almost all of the energy in the problem is stored in the capacitor bank. The exception is the small amount of internal ion and electron energy due to initializing the plasma at a temperature of 10 eV. For a given temperature and volume of plasma, this internal energy scales with density. However, even for the simulated plasmas with the highest starting densities, the initial internal energy is only 0.1% of the bank energy.

As the capacitor discharges through the plasma, some of its energy is stored inductively in the circuit. The remaining energy is delivered to the plasma itself (see Figure 3.1). This delivered energy is invested in the establishment of a strong magnetic field. Most of this field energy is located in the vacuum outside of the plasma column.

Figure 1.1 Energy flow from the Capacitor tank for the simulation with $n_1 = 10^{15} \text{ cm}^{-3}$ and $I = 1 \times 10^{12} \text{ amperes/sec}$.

Initially, the current is confined to a thin sheath at the outer radius of the plasma and, therefore, there is little magnetic field energy in the plasma. At this point, the plasma satisfies most of the assumptions inherent in the snowplow model and the model is useful in explaining the plasma's subsequent behavior.

The magnetic field energy, in turn, is dissipated in several ways (see Figure 3.1). First, small amounts of energy go into the ohmic heating associated with field diffusion. Second, through the Nernst terms in the magnetic induction and electron energy equations, the plasma weakly attempts to oppose the effects of resistive diffusion. Third, the magnetic pressure performs PdV work on the ions and electrons, including shock heating. Since shock heating is attributed solely to the ions, the ions may be appreciably hotter than the electrons. On the other hand, since there are Z times as many electrons as ions and since the electrons can be directly heated by magnetic field diffusion, the internal electron energy will generally be greater than the internal ion energy. Finally, the magnetic pressure sets the plasma into ordered motion creating the kinetic energy associated with the radial implosion.

As the pinch progresses, the mass of plasma affected by the magnetic field grows, densities increase and other forms of energy coupling become important. Of course, as described above, the magnetic field energy is coupled to the plasma primarily through PdV work and resistive diffusion which manifest themselves in the plasma as kinetic energy and internal energy. In addition, energy in the ions may be coupled into the electrons. This depends on the difference in temperature of the

Figure 3. Distribution of delivered energy for the simulation with $n_1 = 10^{15} \text{ cm}^{-3}$ and $I = 0.5 \times 10^{10} \text{ amperes sec.}$

two species and the electron density of the plasma. In low density plasmas, large temperature differences may exist for long periods. Similarly, electron energy can be converted into radiation energy through bremsstrahlung, free-bound, or line radiation processes. Again this is species dependent.

As the external amplification approaches the axis, the plasma begins to stagnate (see Fig. 3.3). Kinetic energy is converted into internal energy (see the temperature rise in the last five nanoseconds of Figure 3.3). At the moment of the highest density phase of the pinch, electron-ionization and electron-ion-radiation coupling can be important and the kinetic energy may be radiated away. The residual energy is then in the form of kinetic pressure that overcomes the magnetic pressure and causes expansion of the plasma column, ending the pinch.

The stagnation and thermalization is similar in effect to the reflected shock wave in Allen's planar model.³ In the simulated case, the divergence of the initial shock wave, the cylindrical geometry of the reflected shock, the increasing field pressure due to the increasing and decreasing plasma radius, and the plasma's ability to radiate energy away modify Allen's simple model. During this process it should also be noted that the plasma is never in a true ion-ion or Debye-Braginskii equilibrium. Dynamic effects dominate and the importance of shock heating violates the assumptions in these models as well as in the snowplow model. Nonetheless, the importance of radiation as an energy loss mechanism enhances the significance of the Debye-Braginskii model while the snowplow model is in rough agreement with the simulation until the moment of stagnation.

The speed with which the plasma thermalizes kinetic energy and possibly radiates away internal energy is an important attribute of the Z-pinch. Electrical energy is delivered to the pinch during the entire duration of the implosion. It is stored in kinetic and internal energy and through thermalization and coupling is partially released in a quick burst of radiation. The power of this burst can be many times higher than the peak power of the bank discharge. Therefore, the Z-pinch is not only a device for the conversion of electrical energy into radiation; it is also an important power multiplier.

The large differences that can result from variations in starting density and current rise rate can be examined by a comparison of two runs from the simulation series -- a low density run and high density run. The low density run begins with an ion density of 10^{15} cm^{-3} and a current rise rate of 6×10^{12} amperes/sec. The high density run starts with an ion density of 10^{17} cm^{-3} and a current rise rate of 6×10^{13} amperes/sec. These two runs were chosen for several reasons. First, their theoretical pinch times, t_p , are the same ($t_p \propto n^{1/4} i^{-1/2}$) and the simulated pinch times are virtually the same (89.8 μs versus 89.2 ns). Second, the amount of energy per ion delivered to the plasmas was nearly the same. Empirically, this delivered energy per ion scaled approximately as $n^{-1/2} i$. Third, the results, in terms of radiation and maximum density, are greatly different.

To demonstrate the similarities and differences of these low density and high density cases, a series of overlay plots was made. Energies for the high density simulation were scaled down by a factor

of 100, the ratio of densities. Therefore, all energies in the plots are actually energies per 3.14×10^{15} ions (the plasma's initial volume is 3.14 cm^3). As can be seen in Figures 3.4-3.15, the energy distribution is very similar in the two pinches until each plasma approaches stagnation. This is especially true of the energy delivered to the pinch (Figure 3.4), the magnetic field energy (Figure 3.5), the kinetic energy until 85 ns (Figure 3.6), the internal electron energy until 85 ns (Figure 3.7), PdV work done on the electrons until very near the maximum contraction (Figure 3.8), and the overall PdV work performed by the magnetic field (Figure 3.9). On the other hand, the internal ion energy (Figure 3.10), the PdV work done on the ions (Figure 3.11) and the energy removed from the plasma by radiation (Figure 3.12) are very distinct. The high density plasma is a relatively efficient radiator, releasing 12 kJ of energy, and undergoes a radiation enhanced collapse, radiating about five times the energy developed through ohmic heating. The low density plasma, despite the same available energy per ion, is an inefficient radiator, releasing only 3 J of energy, and does not enter a radiation enhanced collapse, radiating only 20% of the ohmic energy.

The key difference in the two runs rests in their respective abilities to couple ion energy due to shock heating into the electrons and, subsequently, to radiate this electron energy away. In the high density case, the ions are able to share the energy they receive from shock heating with the electrons. This moderates ion temperatures. In addition, at this higher density, the electrons begin to couple radiatively which releases energy from the plasma, reducing electron

Figure 3.4 Energy delivered to the plasma.

Figure 3.5 Energy in the magnetic field.

Figure 3.6 Kinetic energy.

Figure 3.7 Internal electron energy.

Figure 3.8 PdV work done on the electrons.

Figure 3.9 PdV work done by the field.

Figure 3.10 Internal ion energy.

Figure 3.11 PdV work done on the ions.

Figure 3.12 Radiated energy.

Figure 3.13 Electron-ion coupling.

Figure 3.14 Electron-radiation coupling.

Figure 3.15 Ohmic heating.

temperatures. At these cooler electron and ion temperatures, less PdV work is done on the ions and electrons and more of the field energy is converted to kinetic energy. The plasma enters a feedback loop where lower temperatures enable it to reach higher densities where radiation coupling is even more efficient. At stagnation, the increased kinetic energy is also available for efficient radiation coupling when the plasma thermalizes since the density is very high.

In the low density case, however, the ions are not able to couple their energy with the electrons. The hotter ions are on a higher adiabat and more field energy is required to compress them. This continues to raise their temperature and less field energy is available for kinetic energy. This results in a softer, "mushier" pinch where the densities, even at stagnation, are too low for efficient coupling. (See Figure 3.1 for a comparison of the two plasmas' overall energy distributions.) The pinch radius of maximum contraction is much larger than in the high density case (331 microns versus 4 microns) and virtually no energy is radiated.

This would seem to indicate that a higher initial density pinch would always be preferable to a lower density one, but the overall dependency is more complex than this. It must be remembered that the shock heating that effects the pinch is related to the current rise rate and the current's magnetic pressure. At a given rate of current rise, an initial density that is too high may involve too much inertia or may become optically thick too soon for an efficiently radiating collapse.

Figure 3.16 Relative distribution of energy for the low ($n_1 = 10^{15} \text{ cm}^{-3}$, $I = 6 \times 10^{12} \text{ amps/sec}$) and high ($n_1 = 10^{17} \text{ cm}^{-3}$, $I = 6 \times 10^{12} \text{ amps/sec}$) density simulations. B represents the magnetic field energy; K, kinetic energy; E, internal electron energy; I, internal ion energy; and R, radiation energy.

3.4 Limits to Pease-Braginskii Collapse

As stated in Section 3.1, a series of simulations was performed to investigate Pease-Braginskii collapse. Initial densities for the tungsten plasma were varied from 10^{15} cm^{-3} to 10^{18} cm^{-3} in powers of ten. Current rise rates were varied from 6×10^{11} amperes/sec to 6×10^{14} amperes/sec, also in powers of ten. Radiated energy exceeded ohmic heating in many of the sixteen simulations -- none at a starting density of 10^{15} cm^{-3} , all at 10^{16} cm^{-3} and 10^{17} cm^{-3} , and one at 10^{18} cm^{-3} . Each of these runs exemplified a Pease-Braginskii radiative collapse. The dependency on initial plasma density and rate of current rise demonstrated interesting limits to radiation production by Z-pinchs.

According to the snowplow model, the pinch time, t_p , scales roughly with $\rho^{1/4} i^{-1/2}$. This is confirmed by the simulation series (see Table 3.3). Consequently, the average velocity of the imploding plasma, v , scales as $\rho^{-1/4} i^{1/2}$. It would appear then that lower starting densities and higher rates of current rise would result in higher specific kinetic energies and, ultimately, in more efficient radiation production. In fact, energy delivered to the plasma by the circuit scales approximately as $\rho^{1/2} i$ (see Table 3.5). What is less obvious, but important, is how this delivered energy manifests itself.

For a given initial density, greater rates of current rise result in increasing amounts of shock heating. Conversely, for a given rate of current rise, lower densities are accompanied by increased shock heating. As demonstrated in the low density case of Section 3.2, this shock heating places the ions on a higher adiabat which requires more PdV

work for compression and reduces or delays ion-electron and electron-radiation coupling. The ions stay hot and radiation is reduced. This effect was confirmed by the entire simulation series and is shown in Figure 3.6, Radiated Energy, and Table 3.8, the ratio of radiated energy to ohmic energy. Thus, while the average implosion velocity scales as $\propto I^{1/4}/I^{1/2}c$ and the average specific kinetic energy scales as $\propto I^{-1/2}$, the maximum velocity and maximum specific kinetic energy, v_{max} and E_{max} , are limited by the possible effects of ion-ion shock heating. For example, see Figure 3.6. Snowplow-like scaling would require the same specific kinetic energies for these pinches. In fact, the average and very nearly the same, yet the maximum specific kinetic energy of the high-density run is more than twice that of the low-density run. Consequently, for a given rate of current rise, there is a density limit to resist-Braginskii collapse due to ion-ion shock heating. For a given density, there is an upper current rise rate limit to resist-Braginskii collapse.

At sufficiently high, sufficiently high density, a self-heating regime appears, less shock heating and better ion-electron and electron-radiation coupling. This study pointed out two cases for this. First, while the pinch time, t_p , scales as $\propto I^{1/4}/I^{1/2}c$, the quarter cycle time of the current scales as $I^{-1/2}$. Therefore, the ratio of t_p to the quarter cycle time scales as $\propto I^{1/4}/I^{1/2}c$. At high densities and fast current rise rates, the pinch time may approach or go beyond the quarter cycle time. Such a pinch is "soft" and bounces due to the declining current and magnetic field as well as the desired stagnation and

Summary of Series Results

Ion Density	Ion Density (cm^{-3})			
	10^{15}	10^{16}	10^{17}	10^{18}
0.5×10^{-4}	.220	.085	.153	.543
0.5×10^{-5}	.351	.005	.026	.095
0.5×10^{-6}	.585	.136	.004	.025
0.5×10^{-7}	.699	.576	.00.	--

Table 3.2 Minimum Pinch Radius (mm)

Ion Density	Ion Density (cm^{-3})			
	10^{15}	10^{16}	10^{17}	10^{18}
0.5×10^{-4}	290.	533.	1000.	1830.
0.5×10^{-5}	89.9	159.	284.	513.
0.5×10^{-6}	28.6	50.3	89.2	163.
0.5×10^{-7}	9.2	16.7	33.0	--

Table 3.3 Pinch Time (ns)

Summary of Series Results (continued)

Series	Ion Density (cm^{-3})			
	10^{15}	10^{16}	10^{17}	10^{18}
Series 1	15.	11.	6.	3.
Series 2	37.	33.	21.	11.
Series 3	95.	73.	64.	39.
Series 4	290.	140.	90.	--

Table 3.4 Maximum Implosion Velocity ($\text{cm}/\mu\text{s}$)

Series	Ion Density (cm^{-3})			
	10^{15}	10^{16}	10^{17}	10^{18}
Series 1	29	100	370	1100
Series 2	270	950	2900	9000
Series 3	2600	8400	27000	67000
Series 4	23000	60000	75000	--

Table 3.5 Delivered Source Energy at Pinch (Joules)

Summary of Series Results (continued)

$\dot{I}$ (A/sec)	Ion Density (cm^{-3})			
	10^{15}	10^{16}	10^{17}	10^{18}
6×10^{11}	5	65	190	440
6×10^{12}	8	660	1300	1700
6×10^{13}	5	3000	13000	12000
6×10^{14}	2	770	45000	--

Table 3.6 Radiated Energy (Joules)

$\dot{I}$ (A/sec)	Ion Density (cm^{-3})			
	10^{15}	10^{16}	10^{17}	10^{18}
6×10^{11}	6	39	150	500
6×10^{12}	19	250	690	1900
6×10^{13}	36	420	2800	5800
6×10^{14}	56	360	4100	--

Table 3.7 Ohmic Heating (Joules)

Summary of Series Results (continued)

I (A/sec)	Ion Density (cm ⁻³)			
	10 ¹⁵	10 ¹⁶	10 ¹⁷	10 ¹⁸
6×10^{11}	.74	1.67	1.27	.88
6×10^{12}	.43	2.64	1.88	.89
6×10^{13}	.14	7.14	4.64	2.07
6×10^{14}	.03	2.14	10.98	--

Table 3.8 Radiated Energy/Ohmic Heating

I (A/sec)	Ion Density (cm ⁻³)			
	10 ¹⁵	10 ¹⁶	10 ¹⁷	10 ¹⁸
6×10^{11}	.16	.65	.51	.40
6×10^{12}	.03	.69	.45	.19
6×10^{13}	<.01	.36	.48	.18
6×10^{14}	<.01	.01	.60	--

Table 3.9 Radiated Energy/Delivered Energy

thermalization. This effect made data from the simulation with $n_i = 10^{19} \text{ cm}^{-3}$ and $i = 6 \times 10^{14} \text{ amperes/sec}$ completely meaningless (the theoretical pinch time greatly exceeded a quarter cycle time) and partially perturbed the results of the runs with $n_i = 10^{18} \text{ cm}^{-3}$, $i = 6 \times 10^{13} \text{ amperes/sec}$ and $n_i = 10^{17} \text{ cm}^{-3}$, $i = 6 \times 10^{14} \text{ amperes/sec}$. Evidence of this perturbation appears in Table 3.5, where the delivered energy fails to scale as $\rho^{1/2} i$. Second, the increased plasma mass, resulting in slower velocities due to inertia, and the increased density itself, resulting in an earlier propensity for the plasma to become optically thick, reduce the possibility for prodigious radiative cooling of the plasma.

The real interest of the simulation series, however, lies in the intermediate range--where Pease-Braginskii collapse does in fact occur computationally. From Table 3.8, it can be seen that there is a valley in the density-current rise rate regime where radiation collapse becomes very important. The most efficient radiator occurs at a starting density of 10^{16} cm^{-3} and an initial current rise rate of $6 \times 10^{14} \text{ amperes/sec}$. Here, the conversion efficiency of delivered source energy into radiative energy is 69%. (This is still less than 1% of the total bank energy.) Other simulations, while not as efficient in converting delivered energy, were more prodigious radiators and were more efficient vis a vis the available bank energy. These astounding radiation efficiencies are accompanied by other physical parameters that are not experimentally verified -- for example, average plasma electron densities of 10^{24} cm^{-3} . In several of the simulations the plasma radius at maximum contraction is on the order of several microns versus

a starting radius of 1 cm. In fact, it is the optical thickness of the plasma brought on by these extreme densities that finally limits the collapse in these computational simulations.

Initially, this may seem like a catastrophic failure of the code. It is important, however, to understand the physical analogue of the simulation. The simulation, by its one dimensional nature, precludes the development of important hydrodynamic phenomena such as the sawtooth instability. Such an instability would introduce axial flow into the system somewhat reducing the densities at the most constricted sites. Therefore, one interpretation of the present simulation is that it represents the optimal energy radiated from the densest parts of the bead-like structure repeatedly observed in Z-pinches. The actual energy radiated would be lower for several reasons: first, lower densities due to axial flow will reduce energy coupling; second, the densest parts of the plasma will account for only a small fraction of the pinches total length introducing a significant geometrical factor; and, finally, non-linear phenomena, such as enhanced resistivity, may play an important part in pinch dynamics and energy flow.

The next phase of the study was directed at examining these nonlinear phenomena.

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Chapter 4

EFFECTS OF ENHANCED RESISTIVITY

4.1 Resistivity Multiplication

To study the effects of enhanced resistivity, one of the more extreme collapses from Chapter 3 was chosen as a base -- the simulation with a starting density of 10^{17} cm^{-3} and a current rise rate of 6×10^{13} amperes/sec. Enhanced resistivity was first investigated by simply multiplying the code's existing resistivity which is based on Spitzer's formula¹ as modified by Braginskii² for the effects of magnetic fields. This had a global, not zonal, effect. In other words, the resistivity of each zone was multiplied by the same number. There was no differentiation of zones near the center of the plasma from those in the current carrying sheath. The overall impact of this multiplication was to increase the importance of ohmic heating relative to other energy gain and loss mechanisms.

The early time behavior of several of these simulations can be seen in Figures 4.1-4.6. As expected, increased resistivity allows the

magnetic field to penetrate further into the plasma (Figure 4.1). The peak magnetic fields decrease with increasing resistivity for two reasons -- first, extra resistivity slows the current rise slightly and, second, increased field diffusion and ohmic heating broaden the pressure profile and, consequently, slow down the radial contraction. Since $B = 1/r$ in the code's units, both of these effects reduce B .

At this early time in the simulations, shock and PdV heating are generally less important than ohmic heating, electron thermal conduction and electron-ion coupling. The electron temperatures (Figure 4.3) demonstrate the increasing importance of ohmic heating inside the plasma for higher resistivities as well as the increase of temperature at the surface. For the normal resistivity model, electron temperature falls off somewhat at the surface due to some PdV cooling. At the enhanced resistivities, the ohmic heating becomes increasingly explosive and PdV expansion is inadequate to cool the outer zones.

Ion temperatures (figure 4.4) are cooler, but generally follow the electron temperature profile. This confirms the relative insignificance of shock and PdV heating. In fact, at this point energy audits show that less than 1 J of energy has gone into shock and PdV heating in each of these simulations while 11-34 J have gone into ohmic heating. For the 1x, 2x, and 5x simulations, ion temperatures fall off more rapidly near the surface of the plasma. This is due to lower densities which decrease electron-ion coupling. For the 10x simulation, ion temperatures in the second and third zones from the surface are greatly increased (.35 keV and .14 keV, respectively) by

Figure 4.1 Magnetic field at $t = 20$ ns.

Figure 4.2 Density profile at $t = 20$ ns.

Figure 4.3 Electron temperature at $t = 20$ ns.

Figure 4.4 Ion temperature at $t = 20$ ns.

shock heating (see the extra Q spike for 1-x Spitzer resistivity in Figure 4.5). This shock heating is not due to accelerations brought on by magnetic pressures, but rather is due to the explosive electron heating in the surface plasma zone.

In the primary Q spike (Figure 4.5), the decreasing amplitude of Q with increasing resistivity is a result of magnetic field diffusion and electron conduction of ohmic heating at the surface. Both of these phenomena work to minimize zonal velocity differences by giving inner zones an earlier, "softer" push. This view is confirmed by the pressure plots (Figure 4.6) where increased resistivity results in broader pressure spikes. Again, the perturbation on the 10x Spitzer resistivity curve is a result of shock heating near the surface.

The cumulative effect of field diffusion and the various heating mechanisms can be seen in the density plots (Figure 4.2). Early in time, as expected, increased resistivity results in broader, smaller density spikes. As field diffusion thickens the current sheath and the snowplow model becomes less applicable,

later in time, the temporal development of these simulations is complicated by an instability caused by this global resistivity multiplication. Since resistivity is inversely proportional to v_p^2 , better zones are more conductive. Consequently, increased heating in the surface zones eventually concentrates current there.³ At 20 ns, the 10x simulation is just beginning to fall into the instability. By 60 ns, the current is concentrating near the surface and overall field diffusion is less than in the 1x simulation. The 5x simulation also

Figure 4.5 Q profile at $t = 20$ ns.

Figure 4.6 Pressure profile at $t = 20$ ns.

demonstrates this instability although it develops slightly slower. In effect, by globally multiplying resistivity, ohmic heating overcomes energy loss mechanisms locally and eventually results in less overall resistivity and less resistive dissipation of energy. A time plot of ohmic heating (Figure 4.7) shows that this has little impact on energy flow until near the points of maximum contraction. For the 5x and 10x simulations, current is concentrated in exterior regions of the plasma where densities are low and electron temperatures are relatively unperturbed by the contraction. On the other hand, in the 1x and 2x cases, the current is more diffuse. In the interior zones of the plasma, maximum contraction is accompanied by high densities and good electron-radiation coupling. Electron temperatures drop, increasing resistivity precipitously.

Cumulatively, these simulations demonstrate an important point. Although ohmic heating accounts for only a small portion of the total energy delivered to the plasma, the plasma is extremely sensitive to increases in resistivity. This demonstrated sensitivity is encouraging from the viewpoint of ion acoustic or lower hybrid enhanced resistivity. Table 4.1 summarizes several key parameters for this series of simulations. As expected, minimum pinch radius and pinch time increase with increasing resistivity while maximum implosion velocity, radiated energy and radiated energy as a fraction of delivered energy decrease. Total delivered energy remains relatively constant while ohmic energy defies intuition due to the instability. This renders the radiated energy as a fraction of ohmic energy meaningless.

Figure 4.7 Ohmic heating versus time for the enhanced resistivity simulations that globally multiply the classical resistivity.

Simulation Description	Minimum Radius (mm)	Pinch Time (ns)	Maximum Velocity (cm/us)	Delivered Energy (J)	Radiated Energy (J)	Ohmic Energy (J)	Rad E Ohm E	Rad E Del. E
1x Spitzer	.004	89.2	-64.	27000	13000	2800	4.6	.48
2x Spitzer	.005	89.5	-64.	27000	1400	4200	3.3	.52
5x Spitzer	.057	89.8	-43.	25000	9000	700	12.9	.36
10x Spitzer	.113	90.2	-41.	25000	7000	500	14.0	.28

For each simulation:

Initial ion Density = 10^{17} cm^{-3}

Initial Current Rise Rate = $6 \times 10^{13} \text{ amperes/sec}$

Table 4.1. Summary of resistivity multiplication simulations.

4.2 Ion-Acoustic Resistivity

The ion-acoustic instability has been studied as a potential cause of anomalously high plasma resistivity for many years and from several different approaches. Kadomtsev⁴ was the first to present a weak turbulence theory. Building upon this work, Sagdeev and Galeev⁵ arrived at what is now called the Sagdeev resistivity formula. These theories are based on nonlinear ion Landau damping involving mode coupling. A separate theoretical line began with Dupree⁶ who used a statistical set of exact particle orbits instead of the unperturbed orbits of linear theory. Using these exact orbits in a perturbation solution to the Vlasov equation, Dupree predicted a broadening of the resonance regime for particle-wave interactions. This ion resonance broadening theory was developed further by Sleeper, Weinstock, and Bezzerides,⁷ Wesson and Sykes,⁸ and Caponi and Davidson⁹ who demonstrated nonlinear saturation mechanisms for the instability. These two theoretical avenues -- mode coupling and resonance broadening -- were brought together by the work of Choi and Horton,¹⁰ and Horton, Choi and Koch.¹¹ A more inclusive review of theoretical developments involving the ion acoustic instability is given by Papadopoulos.¹²

Before modifying the simulation code to include the effects of ion-acoustic resistivity, data from previous simulations were analyzed to ensure that the instability's threshold condition would be met and that the ensuing resistivity would be significant. A simple post-processing code was written to analyze the voluminous data at hand.

Results indicated that the threshold condition was met only during a short interval near the pinch time and that resistivities of 30-50x classical values were possible. The short temporal range can be explained through the auxiliary threshold condition that $Z T_e$ must be greater than T_i for ion-acoustic turbulence. In the imploding Z-pinch, shock heating can result in ion temperatures that are much

The results of several simulations were modest. Although it was clear that the threshold condition was met and that the Sagdeev formula did result in enhanced resistivity, the effect was too small and occurred for too short a time to affect the pinch dynamics in any meaningful way. Even when the Sagdeev resistivity was multiplied by 10, the differences from simulations without ion-acoustic resistivity were inconsequential. The collapse was stopped not by additional resistivity but by shock wave propagation when the plasma became optically thick.

4.3 Lower Hybrid Instability

As mentioned in Section 2.6, the lower hybrid instability has been successfully invoked for explaining previously anomalous behavior in theta pinches.¹³ These devices operate in regimes with much lower line densities and magnetic fields than are commonly present in Z-pinches. Struve¹⁴ has speculated that the lower hybrid instability was responsible for anomalous heating observed in his experiment investigating finite Larmor radius effects in Z-pinches. Again, this effect was important only at low line densities. Vikhrev and Korzhavin,¹⁵ on the other hand, have computationally demonstrated the significance of this instability in a high density, high current plasma focus simulation. While essentially a one-dimensional model, their code allows for axial flow depending entirely on the particle pressure gradient of the plasma and its initial pressure. This axial flow results in declining line densities throughout the simulation. Vikhrev

and Korzhavin cite the development of resistivities 1,000-10,000 times larger than classical Spitzer values.

Using the post-processor code initially developed for analyzing ion-acoustic resistivity, data were reanalyzed for the possible importance of the lower hybrid instability. Despite the absence of axial flow, this analysis indicated resistivities of 1000x Spitzer for some current carrying zones during periods of the pinch.

The zonal implementation of lower hybrid instability induced resistivity followed the work of Vikhrev and Korzhavin. The effective collision frequency was

$$\nu_{\text{eff}} = \left(\frac{V_d}{V_{Ti}} \right)^2 \omega_{LH} \quad (4.3)$$

where V_d is the drift velocity, V_{Ti} is the ion thermal velocity, and ω_{LH} is the lower hybrid frequency. This instability results in ion heating as well as electron heating. The heating model used by Vikhrev and Korzhavin is actually based on the relative heating due to the electron-acoustic instability. This model smoothly switches resistive heating based on the significance of lower hybrid resistivity. When nonlinear resistivity is insignificant, all of the heating is in the electrons as expected. When nonlinear resistivity dominates, 87% of the heating occurs in the ions.

In the simulations, this instability resulted in a different distribution of current and temperature, but, again, most plasma parameters such as average electron density, radiated energy, and

minimum pinch radius were modified only slightly. Again, the primary mechanism for halting collapse was the developing optical thickness of the plasma which inhibits energy loss through radiation. Apparently, the lower hybrid instability significantly affects resistivity in the early sheath-like current layer. In response to this enhanced resistivity, the magnetic field diffuses into the plasma. But in this constant line density model, the current eventually distributes itself into a configuration where the instability induced resistivity is insignificant. This is due to high ion thermal velocities and relatively low drift velocities. Low drift velocities result from line densities that are too high and currents that are too low for onset of instability induced resistivity in the column as a whole since

$$v_d = \frac{j}{en_e} = \frac{I}{eN_e} \quad (4.4)$$

Obviously, therefore, any axial flow away from constricted regions will magnify the effects of nonlinear resistivities that depend on the electron drift velocity since line densities will decline.

4.4 Preliminary Two-Dimensional Simulations

Two-dimensional simulations are required to realistically simulate axial flow. In addition, two-dimensional simulations can represent the actual dynamic development of Z-pinches more accurately. One of the

principal characteristics of the fast, nonstabilized Z-pinchs used as x-ray sources is the evolution of a bead-like structure of hot spots along the plasma column. These hot spots are the primary sources of radiation and are, perhaps, the only part of the plasma in Pease-Braginskii collapse. Two-dimensional simulations can reproduce this experimentally observed structure while one-dimensional simulations can only appeal to intuitive geometrical arguments to normalize their results to experiment. Using two-dimensional Eulerian codes, Hussey, Roderick and various collaborators have begun to study Z-pinch instability growth¹⁶⁻¹⁸.

In a comparison of one-dimensional Eulerian and Lagrangian codes, Maxon and Nielsen have found a flaw in the zoning criterion used in many Eulerian Z-pinch simulations¹⁹⁻²⁰. In plasma focus simulations, zone size was chosen to insure accurate representation of shock wave position. Maxon and Nielsen have shown that, in fact, much smaller Eulerian zones are required to represent the amplitude of the shock accurately as well as the position and, furthermore, that the amplitude of the propagating shock greatly affects the final state of a pinch simulation. Lagrangian simulations require far fewer zones for accurate shock representation. This is true not only for the obvious case of the initial zoning, but also for zoning distribution during shock propagation. As a result, many two-dimensional Eulerian simulations must be viewed with some reservation and the need for two-dimensional Lagrangian simulations, however difficult, has become more obvious.

Figures 4.8-4.12 contrast one- and two-dimensional collapse for a preliminary, simplified Lagrangian simulation in which radiation transport is neglected. This simulation is for an argon plasma in a gas puff geometry. The plasma is 2 cm long and is located from 1.7 to 2.2 cm radially. The initial ion density is $6 \times 10^{16} \text{ cm}^{-3}$ and the circuit parameters include a capacitance of 0.8 microfarads, an inductance of 28 nanohenries, and a potential difference of 30,000 volts. The left side of each figure represents a plasma with no initial perturbation. This plasma is essentially one-dimensional and agrees with one-dimensional simulations. The right side of each figure represents a plasma whose outer radius is initially perturbed according to

$$r_0 = 2.200 + 0.001 \cos 2\pi x \text{ cm.}$$

Figures 4.8, 4.9, and 4.11 show the mesh or grid at $t = 0$, 550, and 600 ns respectively, while Figures 4.10 and 4.12 show the vectored velocities of the meshpoints at $t = 550$ and 600 ns. The pinch time is 600 ns.

The growth of the initial perturbation and the consequent development of axial flow are clearly demonstrated. This axial flow will affect both the energy coupling and instability induced resistivity. Coupling will be reduced and resistivity increased. Both of these changes are in the right direction to place an earlier limit on Pease-Braginskii collapse than the presently indicated one-dimensional limit based on the plasma becoming optically thick.

Figure 4.8 Two-dimensional zoning at $t = 0$
 for a) no initial radial perturbation and
 b) a sinusoidal perturbation for the outer
 plasma radius ($r = 2.2 + 0.001 \cos 2\pi x$ cm).

Figure 4.9 Two-dimensional zoning at $t=550$ ns
for a) no initial radial perturbation and
b) a sinusoidal perturbation for the outer
plasma radius ($r = 2.2 + 0.001 \cos 2\pi x$ cm).

Figure 4.10 Velocity vectors at $t = 550$ ns
for a) no initial radial perturbation and
b) a sinusoidal perturbation for the outer
plasma radius ($r = 2.2 + 0.001 \cos 2\pi x$ cm).

Figure 4.11 Two-dimensional zoning at pinch time for a) no initial radial perturbation and b) a sinusoidal perturbation for the outer plasma radius ($r = 2.2 + 0.001 \cos 2x$ cm).

Figure 4.12 Velocity vectors at pinch time for a) no initial radial perturbation and b) a sinusoidal perturbation for the outer plasma radius ($r = 2.2 + 0.001 \cos 2\pi x$ cm).

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Chapter 5

SUMMARY AND RECOMMENDATIONS

5.1 Summary

This report describes a one dimensional study of the limits to Pease-Braginskii collapse of a Z-pinch. Initially, LASNEX, a sophisticated simulation code used in laser fusion studies, was modified to efficiently simulate a Z-pinch. These modifications primarily involved the magnetic field subroutines which are more thoroughly discussed in Appendix B. Later, the code was further modified to allow for enhanced resistivity effects due to the ion-acoustic and lower hybrid instabilities.

Following these modifications, a series of Z-pinchs was examined varying initial density and rate of current rise. The effects of these two parameters on pinch dynamics and energy flow are discussed in Chapter 3. Several of these simulations demonstrated Pease-Braginskii or radiation enhanced collapse. For these pinches, Chapter 4 studied the effects of enhanced resistivity. This was done first through the

use of a completely artificial, but simple, global resistivity multiplication which demonstrated the sensitivity of the simulations to enhanced resistivity. Later, simulations were performed using a zonally determined resistivity due to the ion-acoustic and lower hybrid instability.

Several conclusions can be drawn from this study:

1. There is a low density, high current rise rate limit to Pease-Braginskii collapse. This is due to strong shock heating and poor electron-ion and electron-radiation coupling. The pinch implosion is terminated by quickly rising particle pressures due to thermalization of kinetic energy. This is analogous to the effects of the reflected shock wave in Allen's planar model.
2. There is a high density, high current rise rate limit due to the relative scaling of the pinch time and the capacitor bank's quarter cycle time. The ratio of pinch time to quarter cycle time scales as $\rho^{1/4} i^{1/2}$. Therefore, at high densities with fast energy banks, it is possible for the natural pinch time to exceed the quarter cycle time. This "soft" pinch is ended partly by the effects of the declining current and magnetic field.

3. There is a high density, low current rise rate limit. In this case, the inertia of the plasma limits the kinetic energy available for eventual thermalization and radiation. In addition, the high density and lower energy available per particle result in an earlier propensity for the plasma to become optically thick, reducing the possibility for prodigious radiative cooling.
4. Despite these limits, there is a regime where the one-dimensional fluid equations lead to a Pease-Braginskii collapse in which significant densities are reached and prodigious radiation fluences are produced. The collapse is ultimately terminated by the plasma becoming optically thick as density increases and temperatures fall.
5. Apparent discrepancies between these one-dimensional simulations and experiments can be explained by dimensional and geometric arguments. Due to inhomogeneities in real plasmas, sausage-like instabilities limit the maximum contractions simulated in this study to only a small fraction of the plasma column's total length. Furthermore, axial flow permitted in higher dimensions will reduce densities and energy coupling.

6. Resistivities only 5-10 times greater than classical values can significantly affect pinch dynamics, reducing pinch densities and radiation fluences.
7. In one dimension, neither the ion acoustic instability nor the lower hybrid instability significantly alter pinch behavior although their thresholds are exceeded and enhanced resistivity enters the simulation.
8. Reduced line densities possible in two-dimensional simulations via axial flow from high density regions will undoubtedly increase the effects of instability induced resistivity.

5.2 Recommendations for Further Work

5.2.1. An obvious area for further work is in two-dimensional Lagrangian simulation of Z-pinches. Two-dimensional simulations performed for this dissertation included from one-half to ten wavelengths of a sinusoidal perturbation and completely ignored radiation. Future two-dimensional simulations should include radiation transport since this dissertation demonstrates the importance of radiational effects in one dimension. Also, some future two-dimensional simulations should concentrate on the gas puff geometries that actually result from experimental nozzles. In general,

the puffed gas does not form a simple cylinder of plasma, but more of a diverging or converging cone.

5.2.2. These two-dimensional Lagrangian simulations should be carefully compared with previous Eulerian simulations. The one-dimensional comparisons by Maxon and Nielsen^{1,2} raise suspicions of existing Eulerian simulations. While these Lagrangian simulations are being conducted, the Eulerian simulations could be redone with much smaller zoning to test the convergence of the previous solutions.

Of course, no code is sacrosanct. As other Lagrangian codes are developed, comparisons with LASNEX in one and two dimensions would be useful. These comparisons can point out errors and inadequacies in the codes that might otherwise be indiscernible. For example, during the course of this research, such code comparisons identified an error in the formula for averaging the thermal conductivity of two adjacent zones in LASNEX. Conversely, LASNEX highlighted several errors in other codes.

5.2.3. The low density, fast current rise limit to Pease-Braginskii collapse indicates a possibly fruitful avenue of research. Presently, research and development efforts are directed at developing driving circuits with increasingly fast current rise times. Since shock heating can keep the ions, electrons, and radiation decoupled, it might be beneficial to investigate ways of minimizing this heating. Instead of only developing faster circuits, some attention should be given to tailoring the current rise. Existing

circuits have current rise rates that are nearly constant or that decline with time while a rise rate that accelerates for some period of time may be more advantageous.

5.2.4. Another suggestion is the use of two or more gas puffs of different materials to tailor the response of the imploding plasma. This would be somewhat similar to existing imploding liner systems³ but the density regime would be more flexible. Furthermore, some of the ambiguity concerning the initial vaporization of the liner would be avoided and a wide variety of materials could be used in the various shells.

5.2.5. A final suggestion would be the use of two-dimensional simulations to investigate the effects of nonlinear resistivity and radiation on the growth of the $m = 0$ MHD instability. Perhaps heating due to nonlinear resistivity is responsible for the anomalously long stability of the bead-like structure in Z-pinches.

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Appendix A

DESCRIPTION OF THE LASNEX CODE

A.1 General Description

LASNEX is a two-dimensional magnetohydrodynamic (MHD) code that has been developed over a number of years primarily for laser fusion studies.¹ LASNEX is basically a Lagrangian code although some semi-Eulerian options are included by means of rezoning routines. Axial symmetry is assumed. Therefore, the two coordinates are generally interpreted as the cylindrical coordinates r and z . The difference mesh is chosen to form quadrilateral zones and, in general, follows the development of Schulz.²

LASNEX is a comprehensive code that includes theoretical and phenomenological models for all of the physical processes presently envisioned in laser fusion. As such, it simulates the effects and mutual interactions of laser light, photons, thermal ions, thermal electrons, suprathermal electrons, magnetic fields and thermonuclear burn products. Additionally, a wide variety of source terms may be

included: lasers, electric circuits, ion beams, suprathermal electron spectrums, as well as more generic sources such as pressure, velocity, temperature or internal energy sources, specific energy deposition rates or frequency dependent radiation sources. The code is written with numerous user accessible multipliers. It is through the judicious use of these multipliers that specific models are chosen, others discarded; and various effects are enhanced or reduced.

For the study of radiation enhanced collapse in Z-pinch plasmas, a reduced version of LASNEX was used. Significantly, there was no need to access the subroutines involving laser transport and energy deposition, suprathermal electron generation and transport, and thermonuclear burn product physics. Also, the only source required is an external electric circuit. Despite these simplifications, a complex interactive system remains involving the electric circuit, hydrodynamic motion, thermal ion and electron transport, magnetic field physics, and atomic physics effects such as ionization and radiation absorption and emission. Braginskii transport coefficients³ are used in ion and electron transport and the magnetic induction subroutines to account for the significant effects of large magnetic fields.

A.2 Equation Summary

The actual model can be summarized in five equations, describing hydrodynamic motion, magnetic induction, ion energy transport, electron energy transport, and radiation transport. Using the convention that d/dt is the convective derivative, these equations are:

Hydrodynamic Motion

Hydrodynamic motion includes the Lorentz force as well as an artificial viscosity. The scalar pressure includes pressure contributions from the three components of the plasma: ions, electrons, and photons.

$$\frac{\partial \tilde{v}}{\partial t} = -\frac{1}{\rho} \nabla(P+Q) + \frac{1}{\rho c} (\tilde{\nabla} \times \tilde{B})$$

where

$$\tilde{v} = v + v_{\theta} \hat{e}_{\theta} + v_{\phi} \hat{e}_{\phi}$$

$$\tilde{\nabla} = \nabla + \hat{e}_{\theta} \frac{1}{r}$$

$$P_{\text{ion}} = P_{\text{ion}}(x, y, z) \text{ from detailed equation of state tables}$$

$$P_{\text{ph}} = aT_{\text{ph}}^4/2 \text{ for three temperature models}$$

$$\frac{\partial}{\partial t} = \nabla \cdot \tilde{B}$$

$$Q = c(\text{ion})^2 (\Delta v / \Delta r)^2 \text{ if } \Delta v < 0; 0 \text{ otherwise}$$

The hydrodynamic equation is solved explicitly, with subcycling allowed. Velocity is time-centered. Because the equation is explicitly solved, both the sound speed and the Alfvén velocity may limit the timestep. In the simulation of a Z-pinch, the Alfvén velocity can become especially restrictive in the relatively low density region outside of the plasma column. Here, the density may be one percent or less of the plasma column's starting density while the magnetic fields

can approach megagauss. As a result, it is computationally practical to make these zones as large as possible. LASNEX has been modified to allow this low density, insulating region to be represented by only one zone. This solves the time step problem but introduces a new problem in determining the hydrodynamic acceleration due to the magnetic pressure. In many Lagrangian codes the magnetic field is a zone centered quantity and the Lorentz term is solved as

$$\frac{d^2x}{dt^2} = -\frac{1}{4\pi c} \cdot \nabla(rB),$$

but this equation is inaccurate since the differenced quantity, rB , is constant in a given zone, and, therefore, in the differencing there is no concept over where rB is defined. On the other hand, the coefficient must be point centered and must be formed by some averaging of the zone centered B 's defined in the code. If one zone is large, as remedied by the time step remedy above, a simple average will give the wrong value for the coefficient. In the particular case of the copper, the acceleration of the plasma due to magnetic forces will always be too small. This error greatly effects the simulation especially in terms of the minimum pinch radius, maximum pinch density and radiation. The pinch time is not significantly affected. To correct this error due to an occasional larger zone, LASNEX has been modified to solve the Lorentz term as

$$\frac{\tilde{J} \times \tilde{B}}{c} = - \frac{1}{8\pi r^2} v(rB)^2.$$

In this formulation, no averaging of B is required and the code will determine the proper force to apply at the zonal interface regardless of zone sizes.

Magnetic Induction

The magnetic induction equation includes the Hall term, the $\nabla_{\perp}^2 \times \nabla_{\perp}^2$ term and the Braginskii thermal terms.

$$\frac{\partial \tilde{B}}{\partial t} = - \nabla_{\perp}^2 \times \nabla_{\perp}^2 \tilde{B} + \frac{c}{4\pi} \nabla_{\perp}^2 (\nabla \cdot \tilde{P}_{\perp} - \frac{\tilde{J} \times \tilde{B}}{c} - \tilde{R})$$

where

$$\nabla_{\perp}^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad \nabla \cdot \tilde{P}_{\perp} = \tilde{p}_{\perp} \nabla \cdot \tilde{V} = \tilde{p}_{\perp} \tilde{b} \times \nabla \cdot \tilde{v}_{\perp}$$

$$\tilde{R} = - \frac{1}{4\pi} \nabla_{\perp}^2 (\tilde{v}_{\perp} \cdot \nabla_{\perp}^2 \tilde{v}_{\perp}) \quad \text{Braginskii's transport coefficients}^2$$

The equation is solved implicitly, using splitting. One limitation of the code is that only a 0 directed magnetic field is allowed. Thus, coupled with the assumed azimuthal symmetry of the problem, explains the absence of Braginskii's $\alpha_{||}$ and $\beta_{||}$ transport terms.

Ion Energy Transport

The ion energy transport equation is also solved implicitly. An incomplete Cholesky Conjugate Gradient (ICCG) package is used to solve the diffusive term.⁴ Presently, the code ignores Braginskii's $\kappa_{\perp}^{-1}$ term which may not always be small. (This is discussed further in Appendix A.)

$$\rho_i \frac{dT_i}{dt} = \chi_{ei}(T_e - T_i) - \nabla \cdot \tilde{q}_i - (P_i + Q) \nabla \cdot \tilde{v}$$

where

$$\chi_{ei} = \frac{3m_e}{m_i} \frac{n_e k}{T_e} \quad \text{electron-ion coupling coefficient}$$

$$\tilde{q}_i = -\kappa_{\perp}^i \nabla T_i \quad \text{ion heat flux}$$

$$\kappa_{\perp}^i = \kappa_{\perp}^i(I_i, h_i, Z_i, \omega_{ci}, T_{ii}) \quad \text{transverse thermal conductivity}^5$$

Electron Energy Transport

As for the ions, the electron energy transport equation is solved implicitly and uses the ICCG package for the diffusive term. In this case, however, all applicable Braginskii terms are included.

$$\begin{aligned} \rho_e \frac{dT_e}{dt} = & \chi_{ei}(T_i - T_e) - \nabla \cdot \tilde{q}_e - P_e \nabla \cdot \tilde{v} + P_e \nabla \cdot \frac{\tilde{J}}{en_e} \\ & + \nabla \cdot \frac{3}{2} \frac{kT_e}{e} \tilde{J} + \frac{1}{en_e} \tilde{J} \cdot \tilde{R} + \mu^2 \frac{d\epsilon}{d\rho} \nabla \cdot \tilde{v} + \sum_{er} (T_r - T_e) \end{aligned}$$

where

$\epsilon_e = \epsilon_e(\rho, T_e)$ internal electron energy from
equation of state tables

$\gamma_{ei} = \frac{3m_e}{m_i} \frac{n_e k}{T_{ei}}$ electron-ion coupling coefficient

$\tilde{q}_e = \mu_{\perp} \tilde{U}_{\perp} + \mu_{\parallel} \tilde{b} \times \tilde{U} - \kappa_{\perp} V T_e - \kappa_{\parallel} \tilde{b} \times V T_e$ electron heat flux

$\tilde{R} = \frac{1}{en_e} (\alpha_{\perp} \tilde{J}_{\perp} + \alpha_{\parallel} \tilde{J} \times \tilde{b}) - \mu_{\perp} \nabla T_e - \mu_{\parallel} \tilde{b} \times \nabla T_e$ momentum transfer term

$\mu, \nu, \kappa = \mu, \nu, \kappa(T_e, Z, \omega_{ce} \tau_{ei})$ Braginskii's transport terms³

$\gamma_{er} = \gamma_{Brem} + \gamma_{F-B} + \gamma_{Comp}$ electron-radiation coupling due
to Bremsstrahlung, free-bound,
and Compton processes

Radiation Transport

Radiation transport can be handled by either of two models: a three temperature model or multigroup photons. Both models are solved implicitly and use the ICG package for the diffusive term. For most of this study, the three temperature model was used.

$$4aI_r^j \frac{dT_r}{dt} = \gamma_{er}(T_e - T_r) = V \cdot \tilde{q}_r - p_r V \cdot \tilde{v} - aI_r V \cdot \tilde{v}$$

where

$\gamma_{er} = \gamma_{Brem} + \gamma_{F-B} + \gamma_{Compton}$ electron-radiation coupling due
to Bremsstrahlung, free-bound,
and Compton processes

$$\tilde{q}_r = \frac{-\frac{c}{3}a V_r^4}{\kappa_K + \frac{2}{3} \frac{|V_r|^4}{l_r^4}} \quad \text{radiation heat flux}$$

$$k_R = k_R(\omega, I_e, \text{Composition}) \quad \text{Rosseland mean free path}$$

$$P_r = \frac{1}{3}a l_r^4 \quad \text{radiation pressure}$$

The second term in the denominator of q_r is simply a radiation heat flux limiter.

In addition to these five equations, a source term representing the external electric circuit is required. The code allows the user to specify the circuit resistance, capacitance and inductance as well as the initial charge on the capacitor, an initial current, and the voltage on the circuit battery. In its most general form, this equation would appear as

$$V_p = V_b + \frac{Q}{C} - IR_c - L_c \frac{dI}{dt}$$

where V_p is the voltage across the plasma, V_b is the circuit battery voltage, Q is the charge on the capacitor, C is the capacitance, I is the current, R_c is the circuit's resistance, and L_c is the circuit's inductance. This equation is solved consistently with the magnetic induction equation which includes the effects of the plasma's resistance and inductance including the time rate of change of inductance.

A.3 Computational Considerations

In the full two-dimensional form, LASNEX uses a nine point differencing scheme for its diffusion equations. On a general Lagrangian mesh where zones can become highly skewed, Kershaw has shown that a nine point differencing scheme is the smallest one that will approximate the diffusion operator to second order and satisfy the consistency condition.⁴ To avoid overly restrictive time step limitations, these equations are solved implicitly using an Incomplete Cholesky Conjugate Gradient solver.⁵ For an orthogonal two-dimensional mesh, the nine point difference operator simplifies to the standard five point operator. Furthermore, for one-dimensional problems, this reduces to the three point difference scheme generally solved by the tridiagonal system described by Richtmyer and Morton.⁶ Splitting is used in the two-dimensional implicit differencing of the magnetic induction equation. In the one-dimensional case, this too reduces to the familiar tridiagonal system.

References For Appendix A

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APPENDIX B

MAGNETIC FIELD SUBROUTINES

The efficient simulation of a Z-pinch required some revision of the magnetic field subroutines in LASNEX. The following pages detail the new magnetic field subroutines. This description is available as a separate report titled "The Treatment of Theta-Directed Magnetic Fields in LASNEX", Lawrence Livermore National Laboratory Report UCRL-53123, February 5, 1981. The report is co-authored with George B. Zimmerman.

Introduction

LASNEX solves the magnetic induction equation with a series of subroutines called by subroutine MAGI. For θ -directed magnetic fields, these subroutines are CIRCUI, MAGS, HALL, MAGD, and HDIFF. These subroutines also solve the parts of the thermal electron temperature equation that exist only in the presence of a magnetic field.

A magnetic field also introduces effects in other, more generally present terms. Both the ion and electron thermal conductivities are perturbed by a magnetic field for directions perpendicular to that field. In the presence of a magnetic field, the ion heat flux, q_i , would include a component perpendicular to both the field and the temperature gradient, a $\kappa_{\perp}^{(i)}$ term in Braginskii's formalism. Although this term is not always small, LASNEX ignores it. The code assumes there are no magnetic effects on the real viscosity and effects on alpha particles. Suprathermal electrons, etc., are introduced only through modification of the respective $\kappa_{\parallel}$ term. Magnetic fields do, of course, introduce the $\mathbf{J} \times \mathbf{B}$ term into the equation of motion.

In performing the finite differencing required by the induction equation and the electron temperature equation, the principles of flux conservation and energy conservation are important. After hydromotion has been considered in a given cycle, the zonal areas are constant. As a result, it is convenient to determine flux changes, rather than magnetic field changes,

$$\frac{\delta \Phi}{\delta t} = \int \frac{\delta \mathbf{B}}{\delta t} \cdot d\mathbf{a}$$

$$\text{but } \frac{\delta \mathbf{B}}{\delta t} = -c \nabla \times \mathbf{E}$$

$$\frac{\delta \Phi}{\delta t} = \int -c \nabla \times \mathbf{E} \cdot d\mathbf{a}$$

$$\oint c \mathbf{E} \cdot d\mathbf{l}$$

In this form, it is easy to see that the flux in a zone changes only when flux lines enter or leave across a zonal interface. When this occurs, one zone's loss is guaranteed to be another zone's gain, ensuring conservation. To implement this equation, the tangential component of $\mathbf{E}$ is needed for each interface. This is readily available for the magnetic diffusion and Nernst terms of the induction equation where $\mathbf{E} \propto \nabla \times \mathbf{B}$ or $\mathbf{E} \propto \mathbf{B} \times \nabla T_e$, since the tangential component naturally arises on the interface. On the other hand, for the two-dimensional terms, the normal component is the one that arises naturally and the contour integral cannot be done simply around the zone. There is some arbitrariness to the solution of this problem. Zone-centered quantities could be averaged to form point-centered quantities, making the tangential $\mathbf{E}$ easier to form. Instead, what LASNEX does is to define a point-centered flux derivative, $\dot{\Phi}$, for two-dimensional terms and then to integrate along the contour defined by the zone centers.

These ideas are more thoroughly discussed in the applicable subroutines. As a guide to the overall solution of the problem, the induction equation and the thermal electron temperature equation appear on the following pages, along with a key to term multipliers, advancing subroutines, etc. Following these equations is an analysis of each subroutine in the magnetic field packages. Finally, there is a brief discussion of other subroutines affected by magnetic fields and a guide to the edit quantities due to magnetic fields.

Equation Summary

The magnetic induction equation

$$\frac{d\mathbf{B}}{dt} = \nabla \times \left[\mathbf{v} \times \mathbf{B} + \frac{c}{en_e} \left(\nabla P_e \frac{\mathbf{J} \times \mathbf{B}}{c} - \frac{\alpha_A \mathbf{J} \times \mathbf{b}}{en_e} - \frac{\alpha_i \mathbf{J}_i}{en_e} + \beta_i \nabla_i T_e + \beta_A \mathbf{b} \times \nabla T_e \right) \right]$$

Advancing subroutine	HYDRG	HAU1	HAU1	MAGD	HAU1	MAGD
Coefficient formed	HYDRG	HAU1	MAGS	MAGS	MAGS	MAGS
Term multiplier	SMGPRM	SMGGPEM	SMGHAU1M	SMGDH1M	SMGGH1M	SMGNERM

The thermal electron temperature equation

$$\rho \frac{dT_e}{dt} - \frac{dT_e}{dt} = \xi_e (T_i - T_e) - \nabla \cdot \left(\frac{\beta_i T_e \mathbf{J}}{en_e} - \frac{\beta_A T_e \mathbf{b} \times \mathbf{J}}{en_e} - \kappa_i \nabla T_e - \kappa_A \mathbf{b} \times \nabla T_e \right)$$

Advancing subroutine	TRDLP	HD01	HD01	COND	HD01
Coefficient formed	TRDLP	MAGS_HAU1	MAGS_MAGD	COND	MAGS_HAU1
Term multiplier	TRDPM	SMGGH1M	SMGNERM	CONDH1M	SMGHRM
Zonal energy transfer	TRDPLZ	SMGCLZ	SMGCLZ	CONDZ	SMGCLZ
Cumulative energy transfer	TRDPL	SMGGPEL	SMGNERM		

$$P_e \frac{d\mathbf{v}}{dt} + P_e \nabla \cdot \frac{\mathbf{J}}{en_e} + \nabla \cdot \frac{3}{2} \frac{k T_e}{e} \mathbf{J}$$

Advancing subroutine	THERMO	HD01	HD01
Coefficient formed	THERMO	HAU1	MAGS_HAU1
Term multiplier		SMGGPEM	SMGHRM
Zonal energy transfer	PDW	SMGCLZ	SMGCLZ
Cumulative energy transfer	THDPL	SMGGPEL	

$$+ \frac{1}{en_e} \mathbf{J} \cdot \left(\frac{\alpha_i \mathbf{J}}{en_e} - \beta_i \nabla_i T_e - \beta_A \mathbf{b} \times \nabla T_e \right)$$

Advancing subroutine	MAGD	HD01	HD01
Coefficient formed	MAGS_MAGD	MAGS_HAU1	MAGS_MAGD
Term multiplier	SMGDH1M	SMGGH1M	SMGNERM
Zonal energy transfer	SMGCLZ	SMGCLZ	SMGCLZ
Cumulative energy transfer	SMGDH1	SMGGPEL	SMGNERM

$$+ \rho \frac{\delta T_e}{\delta \rho} \frac{d\mathbf{v}}{dt} + \text{Suprathermal coupling} + \text{Radiation coupling}$$

Subroutines

MAG

Subroutine MAG is the entrance to all time-dependent magnetic field physics. MAG is called by subroutine CYCLE whenever one of the flags MAGF or MAGZ is set to 1. This occurs when a properly executed MAGF card is included in the generator deck. Specifically, the flags are set according to the following:

MAGF card	MAGF flag	MAGZ flag
MAGF	1	0
MAGF ZSYM	1	0
MAGF Z-only	0	1

Subroutine MAG, in turn, calls subordinate subroutines based on the dimensionality of the problem and the values of three flags: ICIRC, MAGF and MAGZ. As stated above, the values of MAGF and MAGZ are determined by the MAGF card in the generator deck. Similarly, ICIRC is set to 1 if the generator deck includes an external circuit source. This source must also indicate key parameters of the circuit and must include time-dependent information on subsequent TP cards. These cards should appear as

S	CIRC	Region	CIRI	CIRC	CIRQ	ICROW	CIRI
TP	time 1		CIRV1	CIRR1			
TP	time 2		CIRV2	CIRR2			
.							
.							
.							
TP	time n		CIRVn	CIRRn			

where the variables are defined and expressed in units according to the following table:

Variable	Definition	Units
Region	Region to which circuit is applied	An integer number
CIRI	Initial circuit current	10^9 A
CIRC	Reciprocal of circuit capacitance	$(10^{-7} \text{ F})^{-1}$
CIRQ	Initial capacitor charge	10^1 C
ICROW	Crowbar flag	0 or 1
CIRI	Circuit inductance	10^{-9} H
time1, ..., time n	Time	Shakes (10^{-8} s)
CIRV1, ..., CIRVn	Circuit voltage	10^8 V
CIRR1, ..., CIRRn	Circuit resistance	10^{-1} Ω

Operationally, subroutine **MAG** first checks the **ICIRC** flag. If this flag indicates that an external circuit is included in the problem, subroutine **MAG** calls subordinate subroutine **CIRCUIT**, which updates several circuit parameters. Since subroutine **MAG** is the only subroutine that calls **CIRCUIT**, subroutine **CIRCUIT** can only be called when a properly executed **MAGF** card is in the generator deck. In other words, to include the effects of an external circuit, both a **MAGF** card and an **SCIRC** card must be included in the generator deck.

Next, subroutine **MAG** checks for a **Z-only** directed magnetic field via the value of the **MAGZ** flag. If such a field is indicated, then subroutine **ZMAG** is called.

Finally, subroutine **MAG** checks for a **θ -directed** magnetic field via the value of the **MAG θ** flag. If a **θ -directed** field is indicated, subroutine **MAG** calls a series of four subroutines: **MAGS**, **HALL**, **MAGD**, and **JDOTE**. **HALL** is not actually called unless the problem is two-dimensional since it includes only two-dimensional effects.

Following these calls, subroutine **MAG** returns control to its calling subroutine, **CYCLE**.

Subroutine MAG

CIRCUIT

This subroutine is called by **MAG** whenever an **S CIRC** card is included in the generator deck. **CIRCUIT** determines the capacitor charge, voltage and resistance for the external circuit. Additionally, depending on the flag **ICROW**, the subroutine will crowbar the circuit when the charge passes through 0.

Presently, the code requires that the external inductance be constant. Consequently, the values for the reciprocal capacitance, **CIRC**, and the circuit inductance, **CIRL**, are the user specified values from the generator deck's **S CIRC** card.

CIRCUIT computes the value of the charge, **CIRQ**, according to $d\text{CIRQ}/dt = -\text{CIRL}$, which is differenced simply as $\text{CIRQ}^{n+1/2} = \text{CIRQ}^{n-1/2} - \text{CIRL} \Delta t$. The initial value of the charge, **CIRQ**, is specified on the **S CIRC** card. The initial value of the current, **CIRI**, is also on the **S CIRC** card while subsequent values are computed in subroutine **MAGD**.

Values for the external voltage, **CIRV**, and circuit resistance, **CIRR**, are both linearly interpolated from values given on the time history cards (TP cards). The actual interpolation is done by subroutine **SORCINT**. **CIRCUIT** simply calls the values from a stored array.

Finally, if desired, **CIRCUIT** will crowbar the circuit when the charge reaches 0. If the **S CIRC** card includes a nonzero value for **ICROW**, the **ICROW** flag is set to 1. If **ICROW**'s flag is set to 1, **CIRCUIT** checks **CIRQ** in the following manner:

1. Prior to computing $\text{CIRQ}^{n+1/2}$, **CIRCUIT** stores the value of $\text{CIRQ}^{n-1/2}$ in **QMFM**.
2. If **ICROW** equals 1, **CIRCUIT** multiplies **QMFM** by $\text{CIRQ}^{n-1/2}$.
3. If this product is less than or equal to 0, $\text{CIRQ}^{n+1/2}$ and **CIRC**, the capacitance, are set to 0 which effectively crowbars the circuit.

This test allows the capacitor to be charged initially with either a positive or a negative charge, permitting current to flow in either direction.

Subroutine CIRCUIT

MAGS

Subroutine MAGS computes various quantities used by subsequent subroutines. MAGS itself does not advance the magnetic field or any other principal dependent variable. The quantities that MAGS computes fall into three major categories: geometrical quantities, electron-ion collision times, and Braginskii transport coefficients.

The geometrical quantities are the first to be computed. Among these quantities are R, the average zonal radius; AOR, the zonal area divided by the average radius; CIRAOR, the integral of zonal area divided by the radius for circuit zones; CIREM, a magnetic energy multiplier that corrects for the effect of the integrated magnetic field energy in circuit zones; PSYOLD, the zonal value of rB_θ at the beginning of this cycle; and PSYNEW, the zonal value of rB_θ after the hydro portion of this cycle.

In computing these values, the subroutine uses several previously defined variables. Specifically, for each zone, the subroutine begins by computing four of the above mentioned quantities: R, AOR, PSYOLD, and PSYNEW. They are computed according to the following formulas:

$$R = \frac{VI(J)}{2\pi (2 \cdot FJAC(J))}$$

where VI(J) is the zone's volume and FJAC(J) is half of the zone's area.

$$AOR(J) = \frac{2 \cdot FJAC(J)}{R}$$

$$PSYOLD(J) = HFD(J) \cdot R$$

$$PSYNEW(I) = HNEW(J) \cdot R$$

where HFD(J) is the zonal magnetic field at the beginning of this time step and HNEW(J) is the zonal magnetic field after this step's hydrodynamics contributions. (Note: The terms for

R and AOR(J) above, as well as all subsequent terms in LASNEX that could result in a "divide by zero" actually are incremented by a term $FUZ = 10^{-99}$ to avoid this error. For simplicity and clarity, FUZ is omitted from all equations in this report.)

These quantities are altered in vacuum regions and in circuit regions. In vacuum regions, AOR(J) is arbitrarily set to $BIG = 10^{99}$ and PSYOLD(J) and PSYNEW(J) are set to XMGRBND, a changeable parameter defaulted to 0.

In circuit regions, the alterations are more complex. First, a zonal flag is set, designating the zone as a nonphysical zone (either a circuit zone or a vacuum zone): ZZOM(J) = 1. Then, for circuit zones, three quantities are calculated: CIRAOR, TAOR, and PHE. CIRAOR is the sum of the areal integral of $1/r$ for all circuit zones.

$$CIRAOR = \sum_{\text{Circuit zones}} \iint \frac{dr dz}{r}$$

TAOR is a similar sum but weighted by the ratio of a zone's integrated area/ r to its simply computed area $2*FJAC(J)/R$.

$$TAOR = \sum_{\text{Circuit zones}} \left(\iint \frac{dr dz}{r} \frac{\iint \frac{dr dz}{r}}{\frac{2 * FJAC(J)}{R}} \right)$$

For noncircuit zones, $AOR(J) = 2*FJAC(J)/R$, as computed previously. For circuit zones, however, $AOR(J) = \iint dr dz / r$ and $AORZ(J) = 2*FJAC(J)/R$. For zones narrow in the r direction, the two formulas for $AOR(J)$ approach each other, and TAOR approaches CIRAOR. However, for many problems, the circuit region is a region of low density. To avoid unnecessarily restrictive time steps, it is advantageous to make this region's zones as large as possible. Consequently, the differences between the two $AOR(J)$'s and between TAOR and CIRAOR can become appreciable. Finally, PHE, the total magnetic flux in the circuit region, is the sum of the product of rB_θ , PSYNEW(J), and A/r , AORZ.

$$PHE = \sum_{\text{circuit zones}} PSYNEW(J) \cdot AOR(J)$$

Since the calculation of the magnetic energy in the circuit region will also involve a correction for the difference in the two formulas for AOR(J), a magnetic energy correction multiplier is formed.

$$CIREM = CIRAOR \cdot AOR$$

In the circuit region, no current is allowed to flow. Consequently, all of the current flows around the circuit region and hence,

$$HROLD = 2 \cdot CIRI$$

The value of HRNEW, rB# in the circuit region after hydro, is a constant for all zones and is simply the total flux, PHE, divided by the sum of the areal integral of 1/r. CIRAOR.

$$HRNEW = PHE / CIRAOR$$

Finally, for all circuit zones, PSYOLD(J) and PSYNEW(J) are corrected.

$$PSYOLD(J) = HROLD$$

$$PSYNEW(J) = HRNEW$$

After completing this geometric section, MAGS computes the electron-ion collision time, TAUE, and optionally corrects it for several effects: metallic resistivity, ion acoustic resistivity and Bohm diffusion. It does this by first determining TAUE including the correction term for temperatures above 0.0362 keV. MAGS computes ZB, the average ionic charge, and then calculates X2, the square of the Coulomb logarithm argument.

$$X2 = \left(\frac{3}{2} \frac{(kT_e)^{3/2}}{\pi n_e e^2 T} \left(\frac{1}{1 + \frac{T_e}{Z^2 \cdot 0.0362}} \right)^{1/2} \right)^2$$

$$= \frac{9}{4} \frac{k^3 T_e^3}{\pi n_e e^6} \frac{1}{Z^2 + 27.64 \cdot T_e}$$

Using the code's units, $k = 1.6 \times 10^{-25}$ jerks/keV and $e = 4.8 \times 10^{-18} \times 10^8$ statcoulombs

$$X2 = \frac{2.405E+29 * TE(J) * * 3 * VI(J)}{TOTNE(J) * (ZB * * 2 + 27.64 * TE(J))}$$

The code gives the user the option of specifying two quantities that affect TAUE. First, ACLGD is a decrement to the Coulomb logarithm. ACLGD is a changeable parameter defaulted to 0. If it is nonzero, it is subtracted from the Coulomb logarithm. Second, ACLGMN is a minimum value for the Coulomb logarithm. ACLGMN is defaulted to 1. With this in mind, and remembering that X2 is the square of the argument,

$$FIAM = AMAX1(ACLGMIN, .5 * ALOG(X2) - ACLGD)$$

This value is then used immediately to compute TAUE(J),

$$TAUE(J) = \frac{3}{4} \frac{m_e}{2\pi} \cdot \frac{k}{c^2} \frac{I_e}{n} \frac{1}{\langle Z^2 \rangle \ln \Lambda}$$

$$FI(J) * SQR1(1E(J)) * VI(J)$$

$$9.19E-19 * SIG4(J) * FIAM$$

where $SIG4(J)/VI(J) = n_e Z^2$. The formula actually includes a correction term when the number of ions, perhaps of zero charge, exceeds the number of electrons. These electron-neutral collisions can be important.

Three corrections can be made to TAUE(J). First, if the region card specified "METAL," then the TAUE(J) computed above is smoothly fitted with a metallic resistivity formula which was chosen to give the correct resistivity for copper at room temperature. A user-specified multiplier, EITAUM, is available to force the formula to fit other metals' resistivities. Therefore, for regions specifying METAL, TAUE is corrected according to

$$TAUE(J) = TAUE(J) + \frac{5.12E+12 * EITAUM}{n_e I_e}$$

$$= TAUE(J) + \frac{5.12E+12 * EITAUM * VI(J)}{TOTNE(J) * TE(J)}$$

A second correction can be made for anomalous resistivity due to the ion acoustic instability. This correction is based on Sagdeev's formula. This correction is implemented whenever parameter ESAGM is set to a value other than its default of 0. The algorithm goes through several steps to determine a quantity related to the square of the current in each zone. This instability only occurs when the drift velocity, VDRIF, exceeds a critical velocity, VCRIT. The actual formula for VCRIT is

$$VCRIT = \frac{(ZkT_e M)^{1/2}}{(1 + k^2 \lambda_D^2)^{1/2}} \left[1 + Z \left(\frac{T_e}{T_i} \right)^{1/2} \left(\frac{M}{m} \right)^{1/2} \exp \left\{ \frac{3}{2} - \frac{1}{2} \frac{Z T_e / T_i}{1 + k^2 \lambda_D^2} \right\} \right]$$

For $T_e \gg T_i$, this reduces to $VCRIT \approx C_s$, the sound velocity. In the LASNEX implementation of this formula, $1 + k^2 \lambda_D^2$ is set to 1.5 and called DFAC. Then

$$\begin{aligned} VCRIT = & 4.0029E-13 \cdot \left[\frac{TOTNE(J) \cdot TE(J)}{DFAC \cdot YMT(J)} \right]^{1/2} \left\{ 1 + 3.3132E+13 \right. \\ & \cdot \frac{TOTNE(J) \cdot TE(J)}{TOTNP(J) \cdot TI(J)} \left[\frac{TE(J) \cdot YMT(J)}{TI(J) \cdot TOTNP(J)} \right]^{1/2} \\ & \left. \cdot \exp \left(1.5 - 0.5 \cdot \frac{TOTNE(J) \cdot TE(J)}{TOTNP(J) \cdot TI(J) \cdot DFAC} \right) \right\} \end{aligned}$$

Now, the drift velocity is determined.

$$\begin{aligned} VDRIF = & \left| \frac{\mathbf{J}}{en_c} \right| = \frac{I}{en_c} \frac{c}{4\pi} \left| \nabla \times \mathbf{B} \right| \\ = & \frac{1}{c} \frac{2\pi r dr dz}{TOTNE(J)} \frac{c}{4\pi} \left| \frac{\delta B_\theta}{\delta r} r + \frac{1}{r} \frac{\delta}{\delta r} r B_\theta z \right| \\ = & \frac{c}{2c \cdot TOTNE(J)} \text{SQRT} \left\{ r' dr' dz' \left(\frac{\delta B_\theta}{\delta r} \right)' + dr' dz' \left(\frac{\delta}{\delta r} r B_\theta \right)' \right\} \\ = & \frac{3.1215E+19 \cdot \text{SQRT}(CUR(J))}{TOTNE(J)} \end{aligned}$$

Finally, TAUSAG, the effective electron-ion collision time due to the ion acoustic instability is calculated.

From Sagdeev, we know that

$$\nu_{\text{eff}} = 10^{-2} \frac{Z I_e}{I_i} \frac{V_d - V_e}{V_{\text{the}}} \omega_{pe}$$

Therefore,

$$\begin{aligned} \tau_{\text{eff}} &= 10^2 \frac{I_i}{Z I_e} \frac{V_{\text{the}}}{V_d - V_e} \frac{1}{\omega_{pe}} \\ &= 10^2 \frac{I_i}{Z I_e} \frac{(k I_e m)^{-1}}{V_d - V_e} \frac{1}{(4\pi n_e e^2 m)^{-1}} \\ &= \frac{10^2 k}{2\pi^2 e} \frac{I_i}{Z I_e} \frac{I_e}{V_d - V_e} \frac{1}{n_e} \\ &= 2.3511\text{E}+6 \cdot \frac{\Pi(J) \cdot \text{TOTNP}(J)}{\text{TE}(J) \cdot \text{TOTNE}(J)} \cdot \text{SQRT} \left(\frac{V1(J) \cdot \text{TE}(J)}{\text{TOTNE}(J)} \right) \cdot \frac{1}{\text{VDRIF} - \text{VCRIT}} \\ &= 2.3511\text{E}+6 \cdot \frac{\Pi(J) \cdot \text{TOTNP}(J)}{\text{TOTNE}(J)} \cdot \text{SQRT} \left(\frac{V1(J)}{\text{TE}(J) \cdot \text{TOTNE}(J)} \right) \cdot \frac{1}{\text{VDRIF} - \text{VCRIT}} \end{aligned}$$

LASNEX limits this quantity to always being greater than or equal to zero (equivalent to turning the instability on only when $\text{VDRIF} \geq \text{VCRIT}$) and allows ESAGM to be used as a collision frequency multiplier in addition to a flag for the ion acoustic correction. Therefore,

$$\begin{aligned} \text{TAUSAG}(J) &= 2.3511\text{E}+6 \cdot \Pi(J) \cdot \text{TOTNP}(J) \cdot \text{SQRT} \left(\frac{V1(J)}{\text{TE}(J) \cdot \text{TOTNE}(J)} \right) \\ &\quad / \text{AMAX1} \left(\text{FUZ}, \text{TOTNE}(J) \cdot \text{ESAGM} \cdot (\text{VDRIF} - \text{VCRIT}) \right) + \text{FUZ} \end{aligned}$$

To complete the correction, TAUE is smoothly fitted with TAUSAG so that the shortest collision time dominates.

$$\text{TAUE}(J) = \frac{\text{TAUE}(J) \cdot \text{TAUSAG}(J)}{\text{TAUE}(J) + \text{TAUSAG}(J)}$$

The third correction to the electron-ion collision time is for Bohm diffusion. This option is selected if parameter EBOHMM is not equal to 0. If the Bohm correction is desired, LASNEX shortens the collision time TAUE according to

$$TAUE(J) = \frac{TAUE(J)}{1 + \frac{1}{16} \frac{eB}{mc} TAUE(J) \cdot EBOHMM}$$

which, for large values of $\omega_{ce}\tau_{ei}$, becomes

$$TAUE(J) = \frac{16 mc}{eB \cdot EBOHMM}$$

The final section of MAGS computes the Braginskii transport coefficients. The ion transport coefficients are separately calculated in subroutine ICOND. Braginskii divides the momentum loss term, R , and the heat-flux term, q , as follows:

$$R = R_u + R_I$$

$$q_e = q_u^e + q_i^e$$

where $R_u = \alpha u + \alpha_\perp u_\perp + \alpha_\Lambda b \times u$ frictional forces

$$R_I = \beta^{-1} \nabla \cdot T_e + \beta \nabla \cdot T_e + \beta_\Lambda b \times \nabla T_e \text{ thermal forces}$$

$$q_u^e = \beta^{-1} u + \beta^{-1} u_\perp + \beta_\Lambda^{lu} b \times u$$

$$q_i^e = \kappa^{-1} \nabla \cdot T_e + \kappa^{-1} \nabla \cdot T_e + \kappa_\Lambda^e b \times \nabla T_e$$

For θ -directed magnetic fields, the parallel ($\parallel$) direction is the θ direction. By the assumption of azimuthal symmetry, these terms vanish.

MAGS now forms several preliminary quantities. RZB is the reciprocal of the average ionic charge, ZB; X2 is $(\omega_{ce}\tau_{ei})^2$, a measure of how important magnetic effects are *vis a vis* collisional effects; RDEL is the reciprocal of the Braginskii variable Δ , so that

$$RDI1 = \frac{1}{\Delta} = \frac{1}{(X2)^2 + \delta_1 X2 + \delta_0}$$

$$\text{where } \delta_1 = \left(\frac{1.35}{Z} + 5.958 \right) \frac{1}{Z} + 7.482$$

$$\delta_0 = \left(\left(\frac{1.3008}{Z} + 1.5956 \right) \frac{1}{Z} + 0.7778 \right) \frac{1}{Z} + 0.0961$$

Since the LASNEX transport coefficients include the effects of the coefficients of the Braginskii transport terms, it is instructive to examine the magnetic induction equation *in toto*.

$$\frac{\delta \mathbf{B}}{\delta t} = -\nabla \times \left[\mathbf{V} \times \mathbf{B} + \frac{c}{en_c} \left(\nabla p_e - \frac{\mathbf{J} \times \mathbf{B}}{c} - \mathbf{R} \right) \right]$$

Now, disregard the $\mathbf{V} \times \mathbf{B}$ term and the ∇p_e term but expand $\bar{\mathbf{R}}$

$$\begin{aligned} \frac{\delta \mathbf{B}}{\delta t} = -\nabla \times \frac{c}{en_c} & \left(-\frac{\mathbf{J} \times \mathbf{B}}{c} - \frac{1}{en_c} (\alpha \cdot \mathbf{J} + \alpha \wedge \mathbf{J} \times \mathbf{b}) + \beta \cdot \nabla \cdot \mathbf{I}_e + \beta \wedge \mathbf{b} \times \nabla \cdot \mathbf{I}_e \right) \\ & + \left[-\frac{c\alpha \cdot \mathbf{J}}{en_c} - \frac{1}{en_c} \left(1 + \frac{c\alpha \wedge}{en_c B} \right) \mathbf{J} \times \mathbf{B} + \frac{c\beta \cdot \nabla \cdot \mathbf{I}_e}{en_c} + \frac{c\beta \wedge}{en_c B} \mathbf{B} \times \nabla \cdot \mathbf{I}_e \right] \end{aligned}$$

but

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} \text{ and } \mathbf{I}_e \text{ in formula is really } k \mathbf{I}_e$$

$$\begin{aligned} \frac{\delta \mathbf{B}}{\delta t} = & \left[-\frac{c^2 \alpha \cdot \nabla \times \mathbf{B}}{4\pi en_c} - \frac{c}{4\pi en_c} \left(1 + \frac{c\alpha \wedge}{en_c B} \right) (\nabla \times \mathbf{B}) \times \mathbf{B} \right. \\ & \left. + \frac{ck\beta \cdot \nabla \cdot \mathbf{I}_e}{en_c} + \frac{ck\beta \wedge}{en_c B} \mathbf{B} \times \nabla \cdot \mathbf{I}_e \right] \end{aligned}$$

Now, let

$$APERP = \frac{c^2 n_e}{4\pi e^2 n_c^2} \cdot AHA1 = \frac{c}{4\pi en_c} \left(1 + \frac{c\alpha \wedge}{en_c B} \right)$$

$$BPERP = \frac{ck\beta \cdot \nabla \cdot \mathbf{I}_e}{en_c} \text{ and BHAT} = \frac{ck\beta \wedge}{en_c B}$$

Then,

$$\frac{\delta \mathbf{B}}{\delta t} = \nabla \times \left[\mathbf{A} \mathbf{P} \mathbf{E} \mathbf{R} \mathbf{P} \nabla \times \mathbf{B} - \mathbf{A} \mathbf{H} \mathbf{A} \mathbf{T} (\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{B} \mathbf{P} \mathbf{E} \mathbf{R} \mathbf{P} \nabla_{\perp} \mathbf{T}_e + \mathbf{B} \mathbf{H} \mathbf{A} \mathbf{T} \mathbf{B} \times \nabla \mathbf{T}_e \right]$$

$$\begin{aligned} \text{where } \mathbf{A} \mathbf{P} \mathbf{E} \mathbf{R} \mathbf{P} &= -\frac{c^2 \alpha_1}{4 \pi e^3 n_e^3} = \frac{c^2}{4 \pi e^3 n_e^3} \frac{m_e n_e}{r_e} \left(1 - \frac{\alpha_1 X_2 + \alpha_0}{\Delta} \right) \\ &= \frac{m_e c^2}{4 \pi e^3} \frac{1}{n_e r_e} \left(1 - \frac{\alpha_1 X_2 + \alpha_0}{\Delta} \right) \\ &= \frac{2.824 E + 11 \cdot V I(J)}{I A T E(J) \cdot T O T N E(J)} \cdot \left(1 - R D E L \cdot (A 1 \cdot X_2 + A 0) \right) \end{aligned}$$

$$\text{with } A 1 = \frac{1.786}{Z} + 4.63$$

$$A 0 = \left(\left(\frac{0.3008}{Z} + 0.976 \right) \frac{1}{Z} + 0.4924 \right) \frac{1}{Z} + 0.0678$$

Let $S I G(J) = \frac{1}{\mathbf{A} \mathbf{P} \mathbf{E} \mathbf{R} \mathbf{P}}$ conductivity,

$$\begin{aligned} \mathbf{A} \mathbf{H} \mathbf{A} \mathbf{T}(J) &= \frac{c}{4 \pi n_e} \left(1 + \frac{c n_A}{e n_e B} \right) = \frac{c}{4 \pi e n_e} \left(1 + \frac{c}{e n_e B} \frac{m_e n_e}{r_e} \omega_{ce} r_{ei} \frac{\alpha_1 X_2 + \alpha_0}{\Delta} \right) \\ &= \frac{c}{4 \pi e} \frac{1}{n_e} \left(1 + \frac{\alpha_1 X_2 + \alpha_0}{\Delta} \right) \\ &= \frac{4.967 E + 18 \cdot V I(J)}{I O T N E(J)} \left(1 + R D E L \cdot (A 1 \cdot X_2 + A 0) \right) \end{aligned}$$

$$\text{with } A 1 = 1.704$$

$$A 0 = \left(\left(\frac{0.3714}{Z} + 0.3142 \right) \frac{1}{Z} + 0.094 \right)$$

$$\begin{aligned} \mathbf{B} \mathbf{H} \mathbf{A} \mathbf{T}(J) &= \frac{c k \beta_A}{e n_e B} = \frac{c k}{e n_e B} \frac{1}{n_e \omega_{ce} r_{ei}} \frac{\beta_1 X_2 + \beta_0}{\Delta} \\ &= \frac{k}{m} \frac{r_{ei}}{r_{ei}} \frac{\beta_1 X_2 + \beta_0}{\Delta} \\ &= 175.9 \cdot I A U E(J) \cdot R D E L \cdot (B 1 \cdot X_2 + B 0) \end{aligned}$$

$$\text{with } B 1 = 1.5$$

$$B 0 = \left(\frac{7215}{Z} + 1.4545 \right) \frac{1}{Z} + 8.77$$

$$\begin{aligned}
 \text{BPERP(J)} &= \frac{ck\beta_i}{en_c} = \frac{ck}{en_c} n_c \frac{\beta_i' X2 + \beta_0'}{\Delta} \\
 &= \frac{ck}{e} \frac{B_i' * X2 + B_0'}{\Delta} \\
 &= 1.E-5 * \text{RDFI} * (B1 * X2 + B0) \\
 \text{with } B1 &= \frac{1.303}{Z} + 3.798 \\
 B0 &= \left(\left(\frac{.3768}{Z} + 1.2998 \right) \frac{1}{Z} + .8583 \right) \frac{1}{Z} + .146
 \end{aligned}$$

One additional transport term is calculated, corresponding to Braginskii's κ_A .

$$\begin{aligned}
 \text{XKHAI(J)} &= \frac{\kappa_A}{B} = \frac{n_e \Gamma_e r_c}{B m_e} \omega_{ce} \tau_{ei} \frac{(\gamma_i'' X2 + \gamma_0'')}{\Delta} \\
 &= \frac{ck^2}{em^2} n_e T_e \tau_{ei}^2 \frac{\gamma_i'' X2 + \gamma_0''}{\Delta} \\
 &= 4.952E-16 * \frac{\text{IOTNE(J)}}{\text{VI(J)}} * \text{TE(J)} * \text{TAUE(J)} * 2 * \text{RDEL} * (C1 * X2 + C0) \\
 \text{with } C1 &= 2.5 \\
 C0 &= \left(\frac{2.31}{Z} + 9.31 \right) \frac{1}{Z} + 10.23
 \end{aligned}$$

The routine wraps up its calculations with corrections to these terms in the vacuum and circuit region. This region is infinitely resistive, so $\text{SIG(J)} = 0$. All of the other transport coefficients are also set to zero: **AHAT**, **BHAT**, **BPERP**, and **XKHAT**.

Following these actions, **MAGS** returns control to its calling subroutine, **MAG**.

Subroutine MAGS

HALL

Subroutine HALL accounts for all the two-dimensional magnetic terms for θ -directed magnetic fields. In the induction equation, these terms are the ∇P_e , $\mathbf{J} \times \mathbf{B}$ and ∇T_e terms. To see this, return to the full induction equation:

$$\frac{\delta \mathbf{B}}{\delta t} = \left[\mathbf{V} \times \mathbf{B} + \frac{c}{en_e} \nabla P_e - \frac{\nabla \times \mathbf{B}}{\text{SIG}(J)} - \text{AHAT}(J) (\nabla \times \mathbf{B}) \times \mathbf{B} \right. \\ \left. + \text{BPERP}(J) \nabla T_e + \text{BHA1}(J) \mathbf{B} \times \nabla T_e \right]$$

The $\mathbf{V} \times \mathbf{B}$ term is the change in $\mathbf{B}$ due to hydro motion. Its effect on $\mathbf{B}$ is determined in the hydro portion of the code. With respect to the other terms, however, we can expand each of them to find out if they are present in one-dimensional problems:

$$\frac{c}{en_e} \nabla P_e = \frac{\delta}{\delta z} \frac{c}{en_e} \frac{\delta P_e}{\delta r} \quad (2-D)$$

$$+ \left(\frac{\nabla \times \mathbf{B}}{\text{SIG}(J)} \right) = \frac{\delta}{\delta z} \frac{1}{\text{SIG}(J)} \frac{\delta B_\theta}{\delta r} + \frac{\delta}{\delta r} \frac{1}{\text{SIG}(J)} \frac{1}{r} \frac{\delta r B_\theta}{\delta r} \quad (1- \text{ and } 2-D)$$

$$+ \left(\text{AHAT}(J) (\nabla \times \mathbf{B}) \times \mathbf{B} \right) = \frac{\delta}{\delta r} \text{AHAT}(J) \frac{B_\theta}{r} \frac{\delta r B_\theta}{\delta r} = \frac{\delta}{\delta r} \text{AHAT}(J) B_\theta \frac{\delta B_\theta}{\delta r} \quad (2-D)$$

$$+ \left(\text{BPERP}(J) \nabla T_e \right) = \frac{\delta}{\delta z} \text{BPERP}(J) \frac{\delta T_e}{\delta r} = \frac{\delta}{\delta r} \text{BPERP}(J) \frac{\delta T_e}{\delta r} \quad (2-D)$$

$$+ \left(\text{BHA1}(J) \mathbf{B} \times \nabla T_e \right) = \frac{\delta}{\delta z} \text{BHA1}(J) B_\theta \frac{\delta T_e}{\delta r} + \frac{\delta}{\delta r} \text{BHA1}(J) B_\theta \frac{\delta T_e}{\delta r} \quad (1- \text{ and } 2-D)$$

In addition to correcting the magnetic field equation for the effects of these three terms, HALL sets up GAIN, which accounts for electron heating due to the ∇P_e and ∇T_e terms and sets up PCHAT which accounts for electron heat flow in the $\mathbf{J}$ and $\mathbf{B} \times \nabla T_e$ directions. HALL accomplishes these actions in four major steps:

First, HALL computes two preliminary quantities: PSYP(J) and PCHAT(J). PSYP(J) is the mesh-point value of rB_z at time n. It is calculated as a weighted average of the zonal values of rB_z for the four adjacent zones, with AOR(J) as the weighting factor. This particular weighting allows us to conserve both magnetic flux and energy:

$$PSYP(J) = \frac{PHI(J) + PHI(J+1) + PHI(J+KMAX) + PHI(J+KMAX+1)}{AOR(J) + AOR(J+1) + AOR(J+KMAX) + AOR(J+KMAX+1)}$$

where $PHI(J) = PSYOLD(J) * AOR(J)$ is the number of flux lines in zone J. Similarly, a mesh-point value of a term involved in the heat flux in the J and $B \times \nabla T_e$ directions is determined. The thermal electron temperature equation terms involved are

$$\begin{aligned} \rho \frac{\delta T_e}{\delta t} \frac{dV}{dt} &= \omega \cdot b + \omega T_e + \omega \left(\frac{3}{2} \frac{ckT_e}{e} \cdot J \right) \\ \rho \frac{\delta T_e}{\delta t} \frac{dV}{dt} &= \omega K HAT(J) \cdot B + \omega T_e + \omega \left(\frac{3}{2} \frac{ckT_e}{4\pi e} \cdot \omega \cdot B \right) \\ &= \left(1 - \omega \right) \cdot \omega K HAT(J) \cdot B + \omega T_e \cdot \left(\left(1 - \frac{3}{2} \frac{ck}{4\pi e} \right) \cdot B \right) \\ &= \left(1 - \omega \right) \cdot \left(\omega K HAT(J) \cdot B + \frac{3}{2} \frac{ck}{e} \cdot \frac{B}{4\pi} \right) \end{aligned}$$

Now let

$$PCHAT(J) = \frac{PHI(J) + PHI(J+1) + PHI(J+KMAX) + PHI(J+KMAX+1)}{XKHAT(J) + XKHAT(J+1) + XKHAT(J+KMAX) + XKHAT(J+KMAX+1)} + \frac{3}{2} \frac{ck}{e4\pi} PSYP(J)$$

where $XKHAT(J) = AOR(J) \cdot KHAT(J)$, $PHI(J) = PSYOLD(J) * AOR(J)$ and PSYP(J) is the point-centered rB_z derived above. Then

$$\begin{aligned} \rho \frac{\delta T_e}{\delta t} \frac{dV}{dt} &= \omega T_e \cdot \left(1 - \frac{PCHAT(J)}{r} \right) \\ &= \left(1 - \omega \right) \cdot \left(\frac{PCHAT(J)}{r} \right) \end{aligned}$$

This value of PCHAT(J) is stored for use in JDOTE.

For vacuum or circuit regions, these two quantities are corrected. For a circuit region, PSYP(J) = 2*I = 2*CIRI and PCHAT(J) = (3/2)(k/e)(c/4π)PSYP(J). For a vacuum region, PSYP(J) = XMGRBND and PCHAT(J) = (3/2)(k/e)(c/4π)PSYP(J), where XMGRBND is a changeable parameter defaulted to 0.

Second, HALL computes the mesh-point magnetic flux change, DPHIP(J), and the zonal change in electron energy TE, GAIN(J), due to the ∇Pe and ∇TE terms. To do this, HALL first calculates the zonal electron density, XNE(J), and the zonal electron pressure including suprathermal pressure, PE(J).

The terms being solved in the induction equation are

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left(\frac{c}{4\pi n} \nabla T_e + \frac{c}{4\pi n} \nabla P_e \right) + \mathbf{B} \text{PERP}(J) \cdot \nabla T_e$$

$$\frac{\delta}{\delta z} \left(\frac{ck}{c - n} \frac{\delta P_e(J)}{\delta r} + \mathbf{B} \text{PERP}(J) \cdot \frac{\delta T_e(J)}{\delta r} \right)$$

$$\frac{\delta}{\delta r} \left(\frac{ck}{c - n} \frac{\delta P_e(J)}{\delta z} + \mathbf{B} \text{PERP}(J) \cdot \frac{\delta T_e(J)}{\delta z} \right)$$

where PE(J) is actually (PS(J) + TOTNE(J)*TE(J))/V1(J), the suprathermal pressure plus the thermal pressure divided by k, the Boltzmann constant.

Actually, however, since the zonal area is constant during this part of the problem, it is simpler to look at the change in magnetic flux, Φ, where Φ = ∫B·da. This approach also ensures conservation of magnetic flux

$$\frac{\delta \Phi}{\delta t} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{a}$$

$$\int \nabla \cdot (\mathbf{E} \times d\mathbf{l})$$

Therefore, for the terms of interest here,

$$\frac{\delta \Phi}{\delta t} = \oint \left(\frac{c}{4\pi n} \nabla P_e + \mathbf{B} \text{PERP}(J) \cdot \nabla T_e \right) \cdot d\mathbf{l}$$

Graphically, this process can be represented on the mesh

where the quantities $PE(J)$, $TE(J)$ are zone-centered, $\overline{T/n_e}$, $\overline{\beta_1}$ are averages of the two zones involved in each difference, and ϕ or $\Delta\phi$ is the mesh-point-defined flux or change in flux.

Now let $CA = (ck,e) \Delta t$ **XMGGPEM** and $CB = \Delta t$ **XMGGTEM** where **XMGGPEM** and **XMGGTEM** are changeable parameters defaulted to 1. Then, using the line integral formula above

$$\begin{aligned}
 \Delta PHIP(J) = & \phi - CA \frac{1}{n_e} \left(PE(J + 1 + KMAX) - PE(J + 1) \right) \\
 & + CB \overline{BPERP} \left(TE(J + 1 + KMAX) - TE(J + 1) \right) \\
 & + CA \frac{1}{n_e} \left(PE(J + KMAX) - PE(J + 1 + KMAX) \right) \\
 & + CB \overline{BPERP} \left(TE(J + KMAX) - TE(J + 1 + KMAX) \right) \\
 & + CA \frac{1}{n_e} \left(PE(J) - PE(J + KMAX) \right) + CB \overline{BPERP} \left(TE(J) - TE(J + KMAX) \right) \\
 & + CA \frac{1}{n_e} \left(PE(J + 1) - PE(J) \right) + CB \overline{BPERP} \left(TE(J + 1) - TE(J) \right)
 \end{aligned}$$

This is the equation used by HALL, although it is implemented in a computationally more economical manner. Designate the lines of the above formula by D_{J+1L} , $-D_{J+KMAXK}$, $-D_{JL}$, and D_{JK} , respectively. Then

$$DPHIP(I) = (D_{I+1L} - D_{IL}) - (D_{I+KMAXK} - D_{IK})$$

where the D 's are centered on the line segments joining adjacent mesh points. Note that

$$DPHIP(I+1) = (D_{I+1} - D_{I+1L}) - (D_{J+1+KMAXK} - D_{I+1K})$$

$$DPHIP(I-KMAX) = (D_{I-KMAX+1L} - D_{J-KMAXL}) - (D_{IK} - D_{I-KMAXK})$$

Therefore, each of the D 's will appear in two equations for $DPHIP$, once with a positive sign and once with a negative sign. This forces conservation of flux for the effects of the ∇P_c and ∇T_c terms. To save computations, HALL does the differencing in only one direction at a time, computing only the single quantity D_{JL} or D_{JK} on each iteration. Then, for the given sweep, HALL increments $DPHIP(J)$ according to

$$\left. \begin{aligned} DPHIP(I+1) &= DPHIP(I+1) + D_{JL} \\ DPHIP(I) &= DPHIP(I) - D_{JL} \end{aligned} \right\} \text{ in K direction}$$

$$\left. \begin{aligned} DPHIP(I) &= DPHIP(I) + D_{JK} \\ DPHIP(I-KMAX) &= DPHIP(I-KMAX) - D_{JK} \end{aligned} \right\} \text{ in L direction}$$

The averaged quantities in the above formulas are found in the following way. The ∇P_c coefficient, $1/n_c$, is a logarithmic average over the zones involved in ∇P_c . For example, if the P_c term is $PE(J+KMAX) - PE(J)$, then

$$\frac{1}{n_c} = \frac{\int \frac{dn_c}{n_c}}{\int \frac{dn_c}{n_c}} = \frac{\ln(n_c(J+KMAX)/n_c(J))}{n_c(J+KMAX) - n_c(J)}$$

This particular form of averaging is used because it gives the exact answer for the potential drop between zone centers for the case where $\nabla T_e = 0$. The potential drop between zone centers is $\int 1/n_e \cdot P_e \cdot dl$. For the case where $\nabla T_e = 0, \nabla P_e = T_e \nabla n_e$. Then

$$\int \frac{1}{n_e} \cdot P_e \cdot dl = \int \frac{1}{n_e} \cdot \frac{dn_e}{n_e} \cdot dl$$

$$= T_e \ln(n_e(J) + KMAX) - n_e(J))$$

$$= T_e \left(\frac{\ln(n_e(J) + KMAX) - n_e(J))}{n_e(J) + KMAX} \right) (n_e(J) + KMAX) - n_e(J))$$

$$= \int \frac{1}{n_e} \cdot P_e \cdot dl$$

The average of BPERP is simply an area-weighted average such that, for the same zones involved in the $1/n_e$ example,

$$\overline{BPERP} = \frac{FJAC(J) + FJAC(J + KMAX)}{FJAC(J) + FJAC(J + KMAX)}$$

$$= \frac{BPERP(J) + BPERP(J + KMAX)}{BPERP(J) + BPERP(J + KMAX)}$$

The zonal energy change, T_e is also computed at the same time. The factoring of T_e out of the expression at this point allows us to difference the equation implicitly in subroutine JDOTF using T_e^{n+1} . The part of the thermal electron temperature equation being solved is

$$\nu \frac{\delta T_e}{\delta t} - \frac{dT_e}{dt} = \left(\frac{\beta_e T_e J_z}{en_e} \right) + \frac{J_z}{en_e} \cdot \left(\beta_e \nabla_z T_e + P_e \nabla_z \frac{J_z}{en_e} \right.$$

$$\left. - \frac{\beta_e J_z}{en_e} \cdot \nabla T_e + T_e \nabla_z \frac{\beta_e J_z}{en_e} - \frac{\beta_e J_z}{en_e} \cdot \nabla T_e + n_e k T_e \nabla_z \cdot \frac{J_z}{en_e} \right.$$

$$\left. + \left(\nabla_z \left[\frac{ck\beta_e}{en_e} \frac{\nabla \times B}{4\pi} \right] + n_e \nabla_z \left[\frac{ck}{en_e} \frac{\nabla \times B}{4\pi} \right] \right)$$

Now multiply by dV and integrate over the given zone's volume.

$$\int \rho \frac{\delta \epsilon_e}{\delta T_e} \frac{dT_e}{dt} dV = \text{DEDT}(J) \frac{dT_e}{dt}$$

and

$$\frac{dT_e}{dt} = \frac{I_e}{\text{DEDT}(J)} \int \nabla \cdot \left[\frac{ck\beta_e}{en_e} \frac{\nabla \times \mathbf{B}}{4\pi} \right] + n_e \nabla \cdot \left[\frac{ck}{en_e} \frac{\nabla \times \mathbf{B}}{4\pi} \right] dV$$

$$\frac{I_e}{\text{DEDT}(J)} \left[\int_S \frac{ck\beta_e}{en_e} \frac{\nabla \times \mathbf{B}}{4\pi} \cdot d\mathbf{a} + n_e \int_S \frac{ck}{en_e} \frac{\nabla \times \mathbf{B}}{4\pi} \cdot d\mathbf{a} \right]$$

$$\Delta T_e(J) = \frac{I_e(J)}{\text{DEDT}(J)} \text{GAIN}(J)$$

HALL does not actually calculate ΔT_e , but only $\text{GAIN}(J)$, where $\text{GAIN}(J)$ includes the effects of Δt and the term multipliers.

As stated earlier, $\text{GAIN}(J)$ is stored and later used in JDOTE, where the equation may be solved implicitly. The actual term for $\text{GAIN}(J)$ is

$$\begin{aligned} \text{GAIN}(J) = & \frac{\text{PSYP}(J-1) - \text{PSYP}(J)}{2} \left(n_e(J) CA \frac{1}{n_e} + CB \overline{\text{BPERP}} \right) \text{Avg } J, J + \text{KMAX} \\ & \frac{\text{PSYP}(J - \text{KMAX} - 1) - \text{PSYP}(J - \text{KMAX})}{2} \left(n_e(J) CA \frac{1}{n_e} + CB \overline{\text{BPERP}} \right) \text{Avg } J, J - \text{KMAX} \\ & + \frac{\text{PSYP}(J) - \text{PSYP}(J - \text{KMAX})}{2} \left(n_e(J) CA \frac{1}{n_e} + CB \overline{\text{BPERP}} \right) \text{Avg } J, J + 1 \\ & \frac{\text{PSYP}(J-1) - \text{PSYP}(J - \text{KMAX} - 1)}{2} \left(n_e(J) CA \frac{1}{n_e} + CB \overline{\text{BPERP}} \right) \text{Avg } J, J - 1 \end{aligned}$$

The averaging is now orthogonal to the differencing. As in the computation of the change in flux, the various components of $\text{GAIN}(J)$ are formed at different iterations and on different sweeps. Computationally, the result is the same, but it is more economical.

Following these calculations, HALL calculates the quantity XMGGPSE , the cumulative magnetic energy coupling with suprathermal electrons through the $\nabla \cdot \mathbf{P}_{\text{suprathermal}}$ term

Third, HALL computes the contribution to flux from the HALL term itself. The term being solved is

$$\frac{\delta \mathbf{B}}{\delta t} = \nabla \times (\mathbf{r} \cdot \mathbf{A} \mathbf{H} \mathbf{A} \mathbf{T}(\mathbf{J}) \times (\nabla \times \mathbf{B}) \times \mathbf{B})$$

Converting again to a change in flux

$$\begin{aligned} \frac{\delta \phi}{\delta t} &= \oint \mathbf{r} \cdot \mathbf{A} \mathbf{H} \mathbf{A} \mathbf{T}(\mathbf{J}) (\nabla \times \mathbf{B}) \times \mathbf{B} \cdot d\mathbf{l} \\ &= \oint \mathbf{r} \cdot \mathbf{A} \mathbf{H} \mathbf{A} \mathbf{T}(\mathbf{J}) \mathbf{B} \times (\nabla \times \mathbf{B}) \cdot d\mathbf{l} \end{aligned}$$

Now, AHAT(J) and HFD(J) are zone-centered quantities, ϕ is again a mesh-point quantity and PSYP is the mesh-point value of rB_θ . Instead of averaging the AHAT(J) terms similar to the averaging of the ∇P_c and ∇T_c coefficient, the flux ϕ is advanced using a different contour where the actual zone-centered values of AHAT(J) are appropriate.

The area of the contour is now twice as large as a zone. Consequently, a factor of 1/2 must be introduced into the equations. Let $CA = (1/4) XMGHALLM \Delta t$, then

$$\begin{aligned}
 AB &= \frac{CA \cdot AHAT(J) \cdot HFD(J) \cdot AOR(J)}{FJAC(J)} \\
 &= \frac{.25 \cdot XMGHALLM \cdot DT \cdot AHAT(J) \cdot HFD(J) \cdot AOR(J)}{FJAC(J)} \\
 &= .50 \cdot XMGHALLM \cdot DT \cdot AHAT(J) \cdot \frac{HFD(J)}{R_{Ag}}
 \end{aligned}$$

Then, using the appropriate AB's, the change in flux becomes

$$\begin{aligned}
 DPHIP(I) &= AB_I \cdot (PSYP(J-KMAX) - PSYP(J-I)) \\
 &\quad + AB_{I+1} \cdot (PSYP(I+1) - PSYP(J-KMAX)) \\
 &\quad + AB_{I-KMAX+1} \cdot (PSYP(J-KMAX) - PSYP(J+1)) \\
 &\quad + AB_{I-KMAX} \cdot (PSYP(J-I) - PSYP(J-KMAX))
 \end{aligned}$$

Again, the code accomplishes this through a computationally more economical method

Considering only one AB per iteration,

$$\begin{aligned}
 DPHIP(I) &= DPHIP(I) + AB \cdot (PSYP(J-I) - PSYP(J-KMAX)) \\
 DPHIP(J-KMAX+1) &= DPHIP(J-KMAX+1) + AB \cdot (PSYP(J-I) - PSYP(J-KMAX)) \\
 DPHIP(I+1) &= DPHIP(I+1) + AB \cdot (PSYP(J) - PSYP(J-KMAX+1)) \\
 DPHIP(J-KMAX) &= DPHIP(J-KMAX) + AB \cdot (PSYP(I) - PSYP(J-KMAX+1))
 \end{aligned}$$

This ensures conservation of flux

HALL corrects DPHIP, the mesh-point flux change, at the boundary points of circuit regions and vacuums. At these points DPHIP(J) is set to zero, with the corresponding actual DPHIP(J) being accumulated in DPHICIR, the change of flux in the circuit. If no circuit is involved, then the value is tallied in SRCME, a scalar that represents the energy input from the boundary.

Fourth, and finally, HALL updates the value of PSYNEW(J) based on the value of the total flux changes at the vertices of each zone. DPHIP(J) is converted from a change in flux to a change in rB , by

$$DPHIP(J) = \frac{DPHIP(J)}{AOR(J) + AOR(J+1) + AOR(J+KMAX) + AOR(J+KMAX+1)}$$

then PSYNEW is updated by averaging

$$PSYNEW(I) = PSYNEW(I) + DPHIP(J) + DPHIP(J-1) + DPHIP(J-KMAX) \\ + DPHIP(J-KMAX-1)$$

since $\phi = PSYNEW \cdot A / r$, this weighted averaging conserves flux.

Following this step, HALL releases several arrays and returns control to MAG.

Subroutine HALL

MAGD

Subroutine **MAGD** handles the one-dimensional magnetic induction terms for θ -directed magnetic fields: the diffusion term and the Nernst term. In addition, **MAGD** couples an external circuit to the problem if that is indicated, computes the magnetic energy coupled to the electrons, updates the circuit current, computes the energy in different circuit elements and determines the magnetic time step limit. **MAGD** represents the last correction to the magnetic field. Following **MAGD**, **HFD(J)** is the magnetic field at the new time step.

First, **MAGD** determines several indices which control whether the subroutine should solve in the **K** or the **L** direction initially and whether the problem is actually one-dimensional. The values of these indices are given in the table below where **JDKLORD=0** indicates solving in the **L** direction first (the default) and **JDKLORD=1** indicates solving in the **K** direction first.

JDKLORD=0				JDKLORD=1			
	2-D	KMAX=2	LMAX=2		2-D	KMAX=2	LMAX=2
ID1	0	0	1	1	1	0	1
ID2	1	0	1	0	0	0	1
IDC	1	1	1	-1	-1	-1	-1

Next, if a circuit is part of the problem, flux quantities due to the circuit are calculated and the magnetic field in the circuit region is incremented to include the effects of the circuit

$$\begin{aligned}
 \text{EXG} \quad & \text{DPHICIR} = (\text{CIRI} * \text{CIRI} + \text{DI} * (\text{CIRV} * \text{CIRQ} * \text{CIRC})) \\
 & \quad \text{CIRAOR} \\
 & \quad \text{DPHICIR} = \text{FI} + \Delta t (\text{V} + \text{Q} * \text{C}) \\
 & \quad \int \frac{d\phi}{r} \quad \text{circuit region}
 \end{aligned}$$

where **DPHICIR** is the change of flux in the circuit region due to two-dimensional effects calculated in **HALL** and division by **CIRAOR** converts the flux quantities into units of **rB₀**. Similarly,

$$EXL = \frac{.5 \cdot (CIRL + D1 \cdot (CIRL + CIRRR))}{CIRAOR}$$

$$= \frac{.5 \cdot (1 + \Delta t (1 + R))}{\int \frac{dr dz}{r} \text{ circuit region}}$$

EXL will represent the fraction of flux lost to the circuit. The factor of 1/2 is introduced since, in the code's units, $I = 1/2 r B_\theta$. Following these calculations, for all circuit zones,

PSYNFW(J) = PSYNFW(J) + EXG
and X1 OS(J) = EXL

Now, MAGD enters the algorithm that actually advances the magnetic field in time. First, the diffusion and Nernst coefficients are determined and combined. Let X2 be the square of the zonal interface distance.

$$X2 = \left[(R1(J) - R1(J+JD1))^2 + (Z1(J) - Z1(J+JD1))^2 \right] \left(1 - \frac{ZZO(J) + ZZO(J+JD1)}{2} \right)$$

where the last factor includes terms ZZO, which are 0 for normal zones and 1 for vacuum zones. If both zones are vacuum zones, X2 = 0, and the diffusion and Nernst coefficients are also 0. Then, in general,

$$BNERN(J) = 4\pi \cdot NMGNERNM \cdot D1 \cdot X2 \cdot \frac{V1(J)}{BHAT(J)} + \frac{V1(J+JD1)}{BHAT(J+JD1)}$$

where volume-weighted averaging is used for BHAT. Of course, the Nernst term also depends on the temperature gradient, so

$$B = BNERN(J) \cdot (TE(J+JD1) - TE(J))$$

Next, the diffusion term is calculated. This term is flux-limited in the code. Ignoring this limit,

$$DC(J) = 4\pi \cdot \text{NMGDIEM} \cdot DT \cdot X2 \cdot \frac{1}{V(I,J) \cdot \text{SIG}(J) + V(I,J+JD) \cdot \text{SIG}(J+JD)}$$

where, again, volume-weighted averaging is used, and $\text{SIG}(J) = 1/\text{APERP}(J)$.

Subroutine MAGD then uses exponential averaging of $B = \text{BNERN}(J) \cdot \Delta TE$ and $DC(J)$ to find the overall diffusion coefficients. In this way, the Nernst and magnetic diffusion terms can effectively cancel each other under the proper conditions. One effect of the averaging is to split $DC(J)$, the diffusion coefficient defined at the interface, into $DCU(J)$ and $DCD(J)$ where $DCU(J)$ is the coefficient of $rB_1^{n+1/2}$ while $DCD(J)$ is the coefficient of rB_1^{n+1} .

As in the other magnetic subroutines, these coefficients are corrected for various effects. No diffusion is allowed between circuit zones and vacuum zones. For these zones, the diffusion coefficients are set to zero. Additionally, the diffusion coefficients are multiplied by the time-dependent NORAD multipliers for the user-specified lines. This is especially useful for eliminating the loss of magnetic flux and energy from zones at the boundary of a problem.

After these corrections are made, the traditional tridiagonal solver is used to implicitly advance the solution. The solver corresponds to a set of simultaneous equations of the form

$$\left[DCU(I) \right] rB_1^{n+1/2} + \left[\frac{A}{r} + \frac{A}{r} \cdot \text{XLOS}(J) + DCD(J) + DCU(J - JD) \right] rB_1^{n+1} \\ + \left[DCD(I+JD) \right] rB_1^{n+1} = \left[\frac{A}{r} \right] rB_1^n$$

where the quantity rB_1^n includes the effect of the circuit, as well as hydro and HALL, that is

$$rB_1^n = rB_1^n + \text{FXG} = rB_1^n + \frac{\text{DPHICIR} \cdot \text{LI}^n + \Delta t (V^{n+1} + Q^{n+1} \cdot C)}{\int \frac{dr dz}{r} \text{ circuit region}}$$

At the same point, notice that circuit effects are included at time $n+1$ by $\text{XLOS}(J)$.

$$\left(1 + \text{XIOS(J)}\right) \tau B_1^{n+1} = \tau B_1^n + \frac{1}{\int \frac{dr dz}{r}} \left(I_1^{n+1} + \Delta t \left(\dot{I}_1^{n+1} + R^{n+1} I_1^{n+1} \right) \right)_{\text{circuit region}}$$

To see the full effect of the circuit, neglect for the moment the diffusion terms and the hydro and HALL contributions. The remaining terms of the equation are

$$\tau B_1^{n+1} = I_1^{n+1} + \Delta t \left(\left(\dot{I}_1^{n+1} + R^{n+1} I_1^{n+1} \right) \right)_{\text{circuit region}} = \tau B_1^n + \frac{1}{\int \frac{dr dz}{r}} \left(I_1^n + \Delta t \left(V^{n+1} + Q^{n+1} \right) \right)_{\text{circuit region}}$$

Rearranging terms

$$\tau B_1^{n+1} - \tau B_1^n = \frac{1}{\int \frac{dr dz}{r}} \left(I_1^{n+1} - I_1^n + \Delta t \left(\dot{I}_1^{n+1} I_1^{n+1} - R^{n+1} I_1^{n+1} + V^{n+1} + Q^{n+1} \right) \right)_{\text{circuit region}}$$

Multiplying through by CIRAOR and letting $I_1^{n+1} - I_1^n = \Delta t \dot{I}_1$

$$\Phi_1^{n+1} - \Phi_1^n = \Delta t \left(1 - I_1^{n+1} - R^{n+1} I_1^{n+1} + V^{n+1} + Q^{n+1} \right)_{\text{circuit region}}$$

$$\frac{\delta \Phi_1^n}{\delta t} = \frac{Q}{C} + V - IR = \frac{\delta I_1}{\delta t}$$

Following the solution of the tridiagonal system, MAGD updates certain energy quantities and reconfigure: the Nernst and magnetic diffusion coefficients for eventual use in the electron temperature equation. At the interface between vacuum or circuit zones and normal zones, SRCMGE, a scalar representing the amount of source energy given up to magnetic energy, is incremented:

$$\begin{aligned} \text{SRCMGE} &= \text{SRCMGE} + 5 * \text{DC(J)} * [\text{PSYNEW(J + JD)} - \text{PSYNEW(J)}] \\ &+ \text{XMGRBND} * [\text{ZZO(J + JD)} - \text{ZZO(J)}] \end{aligned}$$

Next, the Nernst factor is computed

$$\begin{aligned} \text{BNERN}(J) &= \text{BNERN}(J) + (\text{PSYNEW}(J + \text{JD}) - \text{PSYNEW}(J)) \\ &\quad + \text{PSYNEW}(J) + \frac{\text{RHOF}(J) \cdot \text{PSYNEW}(J + \text{JD})}{1 + \text{RHOF}(J)} \end{aligned}$$

The last factor of the modified Nernst term represents the average of rB_z on the interface between mesh points J and J+JD. The factor RHOF(J) was computed while performing the exponential averaging of the Nernst and magnetic diffusion coefficients prior to advancing the magnetic flux in time. RHOF(J) explicitly represents the averaging implicitly performed by the split diffusion coefficients, DCU(J) and DCD(J), in the tridiagonal solver. Energy conservation requires that the same average rB_z be used in the electron temperature equation.

Finally, the magnetic diffusion coefficient DC(J) is reconfigured as well:

$$\text{DC}(J) = \frac{S \cdot \text{DC}(J) \cdot [\text{PSYNEW}(J + \text{JD}) - \text{PSYNEW}(J)] \cdot \cdot 2}{\text{VI}(J) \cdot \text{SIG}(J) + \text{VI}(J + \text{JD}) \cdot \text{SIG}(J + \text{JD})}$$

With these quantities formed, the energy from the magnetic field to the electrons by magnetic diffusion is updated for each zone.

$$\text{XMGEFZ}(J) = \text{XMGEFZ}(J) + \text{VI}(J) \cdot \text{SIG}(J) \cdot [\text{DC}(J - \text{JD}) + \text{DC}(J)]$$

and the value of rB_z is returned to the value suitable for iteration in the other direction, if necessary

$$\text{PSYNEW}(J) = \text{PSYNEW}(J) \cdot (1 + \text{XLOS}(J))$$

This completes the sweep of the one-dimensional terms in one direction. If the problem is actually two-dimensional, MAGD repeats this set of calculations in the other direction.

Once PSYNEW(J), rB_z^{n+1} , has been incremented, taking into account the contribution from each direction, MAGD enters a wrap-up phase of calculations. First, the total energy from the magnetic field to electrons by magnetic diffusion is calculated:

$$XMGDIF = XMGDIF + \sum XMGECZ(J)$$

XMGDIF is a cumulative total for the duration of this problem. Second, the electron temperature is incremented to include the effects of resistive heating

$$TEP(J) = TEP(J) + \frac{XMGECZ(J)}{DEDT(J)}$$

Third, **PSYNEW(J)** is returned to its proper value by dividing out the factor **1+XLOS(J)** by which **PSYNEW(J)** is multiplied at the end of each directional sweep.

Fourth, the current is advanced in time and the energy in each circuit element is computed for problems that involve an external circuit:

$$PSYNEW(\text{Circuit Zones}) = HR \frac{\sum \Phi_{\text{circuit zones}}}{\int \frac{dr dz}{r} \text{ circuit region}} = IB^{n+1}$$

Using **HR**, the average current, **AVEI**; the time difference in current **CIRIR**; and the new current, **CIRI**, are calculated

$$AVEI = \frac{1}{2} (IB^n + IB^{n+1}) = \frac{1}{2} (I^n + I^{n+1})$$

$$CIRIR = \frac{1}{\Delta t} (IB^n - IB^{n+1}) = \frac{I^n - I^{n+1}}{\Delta t}$$

$$CIRI = \frac{IB^{n+1} + IB^n}{2} = I^{n+1/2}$$

Using these values, the energy supplied to the problem due to an external battery, **CIRVE**; external resistance, **CIRRE**; an external charged capacitor, **CIRCE**; external inductance, **CIRIF**, and the sum of these energies, **SCRMGE**, is calculated

$$\begin{aligned}
 CIRV &= CIRV + DT * AVEI * CIRV \\
 CIRRI &= CIRRI - DT * AVEI * AVEI * CIRR \\
 CIRCI &= CIRCI + DT * AVEI * CIRQ * CIRC \\
 CIRL &= CIRL - DT * AVEI * (AVEI * CIRLR + CIRIR * CIRL) \\
 SRUMGE &= SRUMGE + DT * AVEI * (CIRV - AVEI * CIRR \\
 &\quad + CIRQ * CIRC - AVEI * CIRLR - CIRIR * CIRL)
 \end{aligned}$$

The final step in **MAGD** is the formation of the magnetic time step limit. This step also advances **HFD(J)**, the magnetic field. First, the old value of **HFD(J)**, B^n , is stored in **B**. Then, the new value of **HFD(J)**, B^{n+1} , is calculated

$$\begin{aligned}
 HFD(J) &= \frac{PSYNEW(J) * AOR(J)}{2 * IJAC(J)} \\
 &\quad + B^{n+1} \left(\frac{A}{r} \right)^{n+1} \\
 &\quad - 2 * S * A^{n+1} \\
 B^{n+1} &
 \end{aligned}$$

Then, the magnetic time step is calculated

$$DTM = DTMGFC * DT * \left(\frac{ABS(HFD(J)) + DTMGMN}{ABS(B - HFD(J))} \right)$$

where **DTMGFC** is a changeable parameter limiting the relative change in magnetic field per time step, defaulted to 0.1, and **DTMGMN** is a variable parameter designating the minimum magnetic field that will control the timestep, defaulted to 10^{-4} . Assuming that $HFD(J) \gg DTMGMN$

$$DTM = 1 * DT * \left| \frac{B^{n+1} - B^n}{B^n} \right|$$

So

$$DTM = \left| \frac{\Delta B}{B} \right| * DT$$

which limits the next timestep to a value that would have allowed the magnetic field to change by 10% during the last timestep, if that change was *linear in time*.

Following this calculation, MAGD releases numerous arrays and returns control to its calling subroutine, MAG.

Subroutine MAGD

JDOTE

Subroutine JDOTE updates the electron temperature, TEP(J), for effects due to θ -directed magnetic fields. The part of the thermal electron temperature equation being solved is

$$\begin{aligned} n_e \frac{\partial \epsilon_e}{\partial I_e} \frac{dI_e}{dt} &= \left(-\frac{\beta_e I_e \mathbf{J}}{en_e} - \frac{\beta_A I_e \mathbf{b} \times \mathbf{J}}{en_e} - \kappa_A \mathbf{b} \times \nabla I_e \right) \\ &+ P_e \nabla \cdot \frac{\mathbf{J}}{en_e} + \nabla \cdot \left(\frac{3}{2} \frac{k T_e}{e} \mathbf{J} \right) \\ &+ \frac{\mathbf{J}}{en_e} \cdot \left(-\beta_e \nabla I_e - \beta_A \mathbf{b} \times \nabla I_e \right) \end{aligned}$$

Reordering terms,

$$\begin{aligned} n_e \frac{\partial \epsilon_e}{\partial I_e} \frac{dI_e}{dt} &= \nabla \cdot \left(\frac{\beta_e I_e \mathbf{J}}{en_e} \right) + \frac{\mathbf{J}}{en_e} \cdot \left(-\beta_e \nabla I_e \right) + P_e \nabla \cdot \frac{\mathbf{J}}{en_e} \\ &+ \nabla \cdot \left(\frac{\beta_A I_e \mathbf{b} \times \mathbf{J}}{en_e} \right) - \frac{\mathbf{J}}{en_e} \cdot \beta_A \mathbf{b} \times \nabla I_e \\ &+ \nabla \cdot \left(\kappa_A \mathbf{b} \times \nabla I_e \right) + \nabla \cdot \left(\frac{3}{2} \frac{k T_e}{e} \mathbf{J} \right) \end{aligned}$$

By expanding some of the divergences, etc., the equation can be cast in a more recognizable form

$$\begin{aligned} n_e \frac{\partial \epsilon_e}{\partial I_e} \frac{dI_e}{dt} &= I_e \left(\gamma_e - \frac{c k \beta_e}{en_e} \frac{\nabla \times \mathbf{B}}{4\pi} + n_e \nabla \cdot \frac{c k}{en_e} \frac{\nabla \times \mathbf{B}}{4\pi} \right) \\ &+ I_e \nabla \cdot \frac{\beta_A \mathbf{J} \times \mathbf{b}}{en_e} - 2 \nabla \cdot \frac{\beta_A I_e \mathbf{J} \times \mathbf{b}}{en_e} \\ &+ \nabla \cdot \left(I_e \kappa_A \times \kappa_A \mathbf{b} + \frac{3}{2} \frac{k T_e}{e} \mathbf{J} \right) \end{aligned}$$

For the moment, disregard the third and four terms on the righthand side of the equation.

For the remaining terms, multiply by dV and integrate as in HALL:

$$\begin{aligned} \text{DIDIT}(I) \frac{dT}{dt} &= I_c \left(\int_V \frac{ck\beta}{en_c} \frac{\nabla \times \mathbf{B}}{4\pi} dV + n_c \int_V \frac{ck}{en_c} \frac{\nabla \times \mathbf{B}}{4\pi} dV \right) \\ &+ I \int_V \frac{\beta_c \mathbf{J} \cdot \mathbf{b}}{en_c} dV \end{aligned}$$

But for any vector $\mathbf{A}$, $\int_V \nabla \cdot \mathbf{A} dV = \int_S \mathbf{A} \cdot d\mathbf{a}$, so

$$\begin{aligned} \text{DIDIT}(I) \frac{dT}{dt} &= I \left(\int_V \frac{ck\beta}{en_c} \frac{\nabla \times \mathbf{B}}{4\pi} \cdot d\mathbf{a} + n_c \int_S \frac{ck}{en_c} \frac{\nabla \times \mathbf{B}}{4\pi} \cdot d\mathbf{a} \right) \\ &+ I \int_V \frac{\beta_c \mathbf{J} \cdot \mathbf{b}}{en_c} d\mathbf{a} \end{aligned}$$

The term within parentheses is recognizable as $\text{GAIN}(J)$ where $\text{GAIN}(J)$ was defined in HALL. Similarly, the integral of the final term is defined as $\text{GAINN}(J)$, the gain due to the Nernst term

$$\begin{aligned} \text{GAINN}(J) &= \Delta t \int_V \frac{\beta_c \mathbf{J} \cdot \mathbf{b}}{en_c} d\mathbf{a} \\ &= \left(\text{BNERNI}(J - \text{KMAX}) - \text{BNERNI}(J) \right. \\ &\quad \left. + \text{BNERNK}(J - 1) - \text{BNERNK}(J) \right) \end{aligned}$$

where $\text{BNERNI}(J)$ and $\text{BNERNK}(J)$ were defined in MAGD during the respective directional passes for $\text{BNERN}(J)$ and are centered on the zone's interfaces with its neighbors. For these terms, then,

$$\Delta T(I) = \frac{T(I)}{\text{DIDIT}(I)} \left(\text{GAIN}(J) + \text{GAINN}(J) \right)$$

If the total gain from the thermoelectric and Nernst terms is positive, then $\text{TE}(J)$ is advanced explicitly for these terms. If not, then the loss is stored in $\text{XLOS}(J)$ and used in the implicit algorithm that includes the effects of the presently disregarded terms. So, explicitly

$$TEP(J) = TEP(J) + \frac{TE(J)}{D(DEL(J))} \left(GAIN(J) + GAINN(J) \right)$$

while, in preparation for the implicit routine,

$$XILOS(J) = \frac{GAIN(J) + GAINN(J)}{D(DEL(J))}$$

and TEP is not advanced. Remember that in the implicit case, $GAIN(J) + GAINN(J) < 0$, so $XILOS(J) > 0$

The equation solved by the implicit routine includes the second Nernst term, the PCHAT term and, if the total gain was negative, the thermoelectric and Nernst gains. This equation now appears as

$$\begin{aligned} XILOS(J) \cdot TEP(J) &= XILOS(J) \cdot TEP(J) + \frac{\Delta t}{D(DEL(J))} \int_S TE(J) \nabla \times \frac{PCHAT(J)}{r} \cdot da \\ &\quad + \frac{2 \Delta t}{D(DEL(J))} \int_S \frac{\beta_{\Delta}(J)}{en_{\Delta}} \cdot \mathbf{J} \cdot \mathbf{b} \cdot da \\ XILOS(J) \cdot TEP(J) &+ \frac{1}{D(DEL(J))} \int_S \frac{TE(J) \Delta t \Delta XMGERM}{1} \cdot \frac{\delta}{\delta r} PCHAT(J) \cdot r \cdot da \\ &+ \frac{1}{D(DEL(J))} \int_S \frac{TE(J) \Delta t \Delta XMGERM}{1} \cdot \frac{\delta}{\delta r} PCHAT(J) \cdot r \cdot da \\ &+ \frac{1}{D(DEL(J))} \int_S \Delta t \Delta XMGERM \cdot 2 \cdot \frac{TE(J) \beta_{\Delta}}{en_{\Delta}} \cdot \frac{1}{1} \cdot \frac{\delta}{\delta r} r B_{\theta} \cdot r \cdot da \\ &+ \frac{1}{D(DEL(J))} \int_S \Delta t \Delta XMGERM \cdot 2 \cdot \frac{TE(J) \beta_{\Delta}}{en_{\Delta}} \cdot \frac{1}{1} \cdot \frac{\delta}{\delta r} r B_{\theta} \cdot r \cdot da \end{aligned}$$

Using the quantities $BNERNL(J)$ and $BNERNK(J)$ defined in MAGD and letting $CB = 2\pi \Delta XMGERM \Delta t$, this expression simplifies to become

$$\begin{aligned} XILOS(J) \cdot TEP(J) &= XILOS(J) \cdot TEP(J) + \frac{1}{D(DEL(J))} \cdot \left[\delta_1 \left(BNERNL(J) - CB \delta_k PCHAT(J) \cdot TE(J) \right) \right. \\ &\quad \left. + \delta_k \left(BNERNK(J) + CB \delta_l PCHAT(J) \right) \cdot TE(J) \right] \end{aligned}$$

For differencing in either direction, the coefficient of TE in the above equation becomes either DCU(J) or -DCD(J) depending on the sign of the coefficient itself. If the coefficient is positive, then, for example, the δ_1 difference becomes

$$\begin{aligned} \text{DCU}(J) &= \text{BNERNST}(J) - \text{CB} \delta_k \text{PCHAT}(J) \\ \text{DCD}(J) &= 0 \end{aligned}$$

while, if the coefficient is negative, the same term would become

$$\begin{aligned} \text{DCU}(J) &= 0 \\ \text{DCD}(J) &= (\text{BNERNST}(J) - \text{CB} \delta_k \text{PCHAT}(J)) \end{aligned}$$

The effect of this differencing is similar to the upstream-downstream differencing algorithm

Now, the set of implicit equations appears as

$$\begin{aligned} \text{DCU}(J) T_{i+1/2}^{n+1} + \left(\text{DEDI}(J) (1 + \text{XIOS}(J)) + \text{DCD}(J) + \text{DCU}(J - JD) \right) T_i^{n+1} \\ \text{DCD}(J) T_{i+1/2}^{n+1} = \text{DEDI}(J) T_i^n \end{aligned}$$

Note that in each instance, either DCU(J) or DCD(J) is zero. This set of equations is solved using the traditional tridiagonal solver, sweeping in each direction separately.

Several calculations are performed to complete the work in JDOTF. The flux of energy due to the PCHAT and the second Nernst term is calculated,

$$\text{FLX}(J) = \text{TEP}(J) * \text{DCD}(J) - \text{TEP}(J + JD) * \text{DCU}(J)$$

Then the total zonal energy flux is incremented to include these terms

$$\text{XMGECZ}(J) = \text{XMGECZ}(J) + (\text{FLX}(J + JD) - \text{FLX}(J))$$

To compute the energy flux due to the thermoelectric term and the first Nernst term, the proper temperature multiplier must be found—explicit or implicit

$$TB = \frac{1}{2} \left(TE(J) + TEP(J) \right) + \left(TE(J) - TEP(J) \right) \cdot \text{SIGN} \left(5, \text{GAIN}(J) + \text{GAINN}(J) \right)$$

If $\text{GAIN}(J) + \text{GAINN}(J) \geq 0$, $TB = TE(J)$ the correct temperature for explicit differencing.

If $\text{GAIN}(J) + \text{GAINN}(J) < 0$, $TB = TEP(J)$ representing implicit differencing.

Then, the energy flow due to the first Nerst term is accumulated:

$$\text{AMGNERN} = \text{AMGNERN} + \text{GAINN}(J) \cdot TB$$

as is the energy flow due to the ∇P_e and ∇T_e terms

$$\text{AMGGPE} = \text{AMGGPE} + \text{GAIN}(J) \cdot TB$$

and the total zonal energy flux is incremented for these terms

$$\text{AMGTCZ(J)} = \text{AMGTCZ(J)} + TB \cdot \left(\text{GAIN}(J) + \text{GAINN}(J) \right)$$

The final action of JDOTF prior to releasing several files and returning control to MAG is to correct for energy supplied by TE sources. This energy is stored in SRCFH. For zones in which a TE source is applied

$$\text{SRCHF} = \text{SRCHF} + \text{AMGTCZ(J)}$$

Subroutine JDOTE

Other Subroutines Affected by Magnetic Fields

1. HYDRO—Here the magnetic field enters through the $\mathbf{J} \times \mathbf{B}$ force. In HYDRO

$$\begin{aligned} \frac{\mathbf{J} \times \mathbf{B}}{c} &= \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= \frac{1}{4\pi} \left(-\frac{\partial B_\theta}{\partial z} \mathbf{r} + \frac{1}{r} \frac{\partial (r B_\theta)}{\partial t} \mathbf{z} \right) \times \mathbf{B} \\ &= \frac{1}{4\pi} \left(\frac{B_\theta}{r} \nabla B_\theta - \frac{\partial B_\theta}{\partial t} \mathbf{r} + \frac{B_\theta}{r} \frac{\partial (r B_\theta)}{\partial z} \mathbf{z} \right) \\ &= \frac{1}{8\pi} \left(\frac{\partial (B_\theta)^2}{\partial t} \mathbf{r} + \frac{\partial (r B_\theta)^2}{\partial z} \mathbf{z} \right) \end{aligned}$$

2. ICOND—Here the magnetic field enters through its effect on the diffusion coefficient. Only the $\nabla \cdot \kappa \nabla T$ term is included. The $\nabla \cdot \kappa^{ion} \mathbf{b} \times \nabla T_i$ term is neglected. The ratio of the two terms is

$$\frac{\kappa \cdot \mathbf{b} \times \nabla T_i}{\kappa \nabla T_i} = \frac{1}{1 + \frac{1}{2} \omega_i^2 \tau_i^2} \left(\frac{2.5 (\omega_i \tau_i)^2 + 4.65}{2 (\omega_i \tau_i)^2 + 2.645} \right)$$

where $\omega_i \tau_i = Z(m_i/m_e)^{1/2} \omega_e \tau_{ei}$, which even for the most highly ionized high- Z ions results in $\omega_i \tau_i \approx 0.1 \omega_e \tau_{ei}$ and for lesser ionization can easily be of order 0.01 $\omega_e \tau_{ei}$. For highly ionized high- Z materials the neglect of this term could introduce significant error if $\omega_e \tau_{ei}$ is greater than 1.

3. ECOND—Here, as in ICOND, the magnetic field enters through its effect on the diffusion coefficient. Only the $\nabla \cdot \kappa \nabla T_e$ term is solved in ECOND. All of the other terms were included in subroutines MAGS, TAIL, MAGD and JDOTF.

EDITED QUANTITIES

Quantities printed during the various LASNEX edits are helpful in determining the accuracy of the problem and the energy flow within the problem. During the Regional Edit, the code will print out a value for EH, the region's magnetic energy in jerks (10^9J). In general, EH is the sum of the volume times the magnetic field squared divided by 8π for each zone in the region. That is

$$EH(\text{Region}) = \sum_{\substack{\text{all zones} \\ \text{in region}}} \frac{V1(J) * HFD(J) ** 2}{8\pi}$$

For a circuit region, this is modified to yield the analytic answer

$$EH(\text{Circuit Region}) = \left(\sum_{\substack{\text{all zones} \\ \text{in circuit} \\ \text{region}}} \frac{V1(J) * HFD(J) ** 2}{8\pi} \right) * CIREM$$

where CIREM is the circuit region energy multiplier determined in MAGS.

$$CIREM = \frac{CIRAOR}{TAOR} = \frac{\sum \iint \frac{drdz}{r}}{\sum \left(\iint \frac{drdz}{r} \frac{\iint \frac{drdz}{r}}{2 * FJAC(J)} \right)}$$

The overall result is that

$$\begin{aligned} EH(\text{Circuit Region}) &= \iint \frac{B^2}{8\pi} 2\pi r dr dz \\ &= \iint \frac{I^2}{r} dr dz \end{aligned}$$

ENERGY BALANCE EDIT

During this edit, one column is titled **MAGNETIC** and contains information calculated by the magnetic field package.

ENERGY	MAGNETIC
INITIAL	ENINH where $ENINH = \sum_{\text{all zones}} VI(J) HFD(J)/8\pi$ with HFD(J) being the initial magnetic field. For circuit zones, the CIREM correction is included.
COUPLING	No entry
BR+EB+CM	No entry
HYDRO	-XMGPDI computed in HYDRO represents PdV work done by the magnetic field.
MAG	XMGGPFE-XMGNERNE-XMGDIFE-XMGGPSL computed in MAGD and JDOTE representing the various components of the electron temperature equation.
LASER	No entry
PARTDEP	No entry
SOURCE	SRC MGF computed in HALL and MAGD represents the magnetic energy created by a source—either a circuit or a boundary condition
ESCAPED	No entry
PRESENT	ENHFD where $ENHFD = \sum_{\text{region}} EH(\text{Region})$, the quantities determined in the regional edits.
CHECK	The check quantity is the difference between ENHFD, the present magnetic, and the sum of all the other terms in the MAGNETIC column.