Banana Drift Transport in Tokamaks with Ripple

R. Linsker

and

A. H. Boozer

Princeton University, Plasma Physics Laboratory

Princeton, New Jersey 08544

Abstract

Ripple transport in tokamaks is discussed for the "banana drift" collisionality regime, which lies below the ripple plateau regime treated earlier. The physical mechanisms that dominate banana drift transport are found to differ from those considered in previous work on this regime, and the resulting transport coefficients can consequently differ by several orders of magnitude.

DISCLAIMER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

26

I. Introduction

Transport in tokamaks is significantly affected by deviations from axisymmetry that arise owing to the presence of toroidal field coils. This has been studied by several authors.¹⁻⁵ In particular, Boozer⁵ has calculated transport coefficients in the "ripple plateau" regime of principal importance to tokamaks. In this paper we evaluate the transport coefficients in two lower collisionality regimes for comparison with the plateau results and because of the potential importance of these regimes in their own right.

As a trapped particle executes its banana orbit, it experiences a radial drift that oscillates with the ripple phase.⁵ If the collisionality ν is low enough that the particle completes many such orbits, the contributions to the radial drift will average to zero except for the last ripple near each turning point. We shall see that the necessary condition for this is $\nu \ll \nu_*$ $(Nq)^{-2}$ where $\nu_* = \epsilon^{3/2} v_T/qR$, ϵ = inverse aspect ratio, and R = major radius. In this collisionality regime (just below the ripple plateau) the ambipolar ion thermal conductivity will be shown to be [Eq. (26)]

$$\bar{\kappa}_c^A = C n N \langle \delta^2 \rangle_I \rho_\theta^2 \frac{v_T}{R} \frac{\nu_*}{\nu(Nq)^2} \quad (1)$$

where n = density, N = number of coil turns, ρ_θ = poloidal gyroradius, $v_T = (T/m)^{1/2}$ = thermal speed, and $\langle \delta^2 \rangle_I$ denotes a specific weighted average of the square of the ripple amplitude $\delta(\theta)$ [see Eq. (23)]. The value of C is sensitive to the form of the high-energy tail of the distribution function [see Eq. (26)]. For a Maxwellian, $C \approx 16.5$.

When collisionality is decreased further, the effect of toroidal precession (induced, e.g., by an ambipolar radial electric field) on the

ripple phase at the turning point becomes the dominant mechanism. The v^{-1} dependence of the transport coefficients is replaced by a more complicated relation [see Eqs. (33) - (35)], and $D(v)$ is found to decrease with decreasing v . (We have assumed an ordering in which the toroidal precession per bounce is small.)

In the next section we derive the transport coefficients for these two regimes, which we refer to jointly as the banana drift regime. Comparison is made with the ripple plateau results of Boozer.⁵ In Section III other ripple transport effects, and the validity of our precession/bounce ordering assumption, are discussed. In Section IV we discuss earlier work on the banana drift regime,^{3,4} and show why that work requires modification.

II. Banana Drift Regime

The ripple phase at a turning point of a banana orbit changes with time due to collisional scattering, toroidal precession of the orbit, and other effects. We define the banana drift regime to be of sufficiently low collisionality that a particle completes many banana orbits before the ripple phase at a turning point changes substantially. In Part A we impose the additional condition that the change of ripple phase be the result of collisions. The less collisional case in which toroidal precession is the dominant mechanism will be discussed in Part B.

A. Collision-dominant case

We model the magnetic field magnitude by

$$B(\psi, \phi_0, \theta) = B_0(\psi) [1 + 2\epsilon(\psi) \sin^2 \frac{\theta}{2} - \delta(\psi, \theta) \cos(N\phi_0 + Nq\theta)] \quad (2)$$

where the ripple $\delta \ll \epsilon$, $\epsilon = (r/R) \ll 1$ is the inverse aspect ratio, ψ is toroidal flux, θ and ϕ the poloidal and azimuthal angles, and $\phi_0 = \phi - q\theta$. We assume throughout that $Nq \gg 1$. (Also, since we treat banana drift diffusion rather than transport by ripple-trapped particles, we implicitly assume that the ripple-trapped population is unimportant. This is true when $Nq\delta/\epsilon \ll 1$.)

The ripple phase at a banana turning point $\theta = \theta_T$ changes by $\sim 2\pi$ when θ_T changes by $\sim (Nq)^{-1}$. Since, at the midplane, $|v_{\theta}|/v \approx (2\epsilon)^{1/2} \sin(\theta_T/2)$, the required change in v_{θ}/v is $\sim (\epsilon/2)^{1/2} (Nq)^{-1} \cos(\theta_T/2)$. The effective collisionality is therefore

$$\nu_{\text{eff}}(\theta_T) \sim \frac{\nu(Nq)^2}{\epsilon \cos^2(\theta_T/2)}, \quad (3)$$

where ν is the 90° collision frequency. Except for barely-trapped particles, the condition defining the banana drift regime is therefore $\nu \ll \nu_*(Nq)^{-2}$. The next higher collisionality regime is the ripple plateau studied by Boozer.⁵

We estimate the diffusion coefficient as follows. The ripple-induced radial drift velocity is

$$v_r(\phi) \sim N\delta \frac{\rho_{\theta}}{R} v_T \sin N\phi,$$

where $\rho_\theta = mvqR/eB_0 r$ is the poloidal gyroradius (we assume $B_\theta \ll B_0$). The contributions from all ripples, except those at the turning points ($\pm\theta_T$), average to zero. The fraction of time spent traversing the "last" ripple is $\sim (Nq)^{-1/2}$ because of the slowing of the particle near its turning points. Therefore, the orbit-averaged radial drift is

$$v_r \sim N\delta \frac{\rho_\theta}{R} v_T (Nq)^{-1/2}. \quad (4)$$

The fraction of the particles contributing to transport is $\sim \epsilon^{1/2}$. Therefore

$$\begin{aligned} D_c &\sim \epsilon^{1/2} (\Delta r)^2 / \Delta t \sim \epsilon^{1/2} v_r^2 / v_{eff} \\ &\sim N\delta^2 \rho_\theta^2 \frac{v_T^2}{R} \frac{v_*}{v(Nq)^2}. \end{aligned} \quad (5)$$

To calculate the ripple transport, we use the banana kinetic equation⁶

$$-\frac{\partial J}{\partial \phi_0} \frac{\partial f}{\partial \psi} + \frac{\partial J}{\partial \psi} \frac{\partial f}{\partial \phi_0} = \frac{e}{m} \int \frac{d\ell}{v_\parallel} C(f_1) \quad (6)$$

with $J = \oint v_\parallel d\ell$, $f = f_m + f_1$, f_m a Maxwellian distribution, and

$$C(f_1) = v(y) \frac{v_\parallel}{B_0} \frac{\partial}{\partial u} \mu v_\parallel \frac{\partial f_1}{\partial u}, \quad (7)$$

with

$$v(y) = \frac{3}{2} \left(\frac{\pi}{2}\right)^{1/2} v_{By}^{-3} [\text{erf}(y) - \psi(y)]$$

$$y \equiv v(m/2T)^{1/2}$$

$$\Psi(y) \equiv \frac{\text{erf}(y)}{2y^2} - \frac{\pi^{-1/2}}{y} \exp(-y^2) . \quad (8)$$

[An additional term in $C(f_1)$, required for momentum conservation, is omitted here because⁴ it is odd in $v_{\parallel}$ and therefore will not contribute to the right side of Eq. (6).] We approximate $\partial f / \partial \psi$ by $\partial f_m / \partial \psi$, and neglect toroidal precession [the second term in Eq. (6)]. We have

$$\begin{aligned} J(U, \mu, \psi_0, \psi) &\approx 2qR \int_{-\theta_T}^{\theta_T} v_{\parallel} d\theta , \\ v_{\parallel}^2 &= 2[U - \mu B(\psi, \phi_0, \theta) - \frac{e}{m} \phi(\psi)] \\ &= 4\mu B_0 \epsilon [x^2 - \sin^2 \frac{\theta}{2} + \frac{\delta}{2\epsilon} \cos(N\phi_0 + Nq\theta)] , \end{aligned} \quad (9)$$

where $x^2 \equiv (U - \mu B_0 - e\phi/m)/(2\mu B_0 \epsilon)$, U = particle energy/mass, and ϕ = electrostatic potential. Note $x \approx \sin(\theta_T/2)$. Then

$$\begin{aligned} J &\approx 4qR (\mu B_0 \epsilon)^{1/2} A_1 \\ \frac{\partial J}{\partial U} &\approx qR (\mu B_0 \epsilon)^{-1/2} A_2 \\ \frac{\partial J}{\partial \phi_0} &\approx -NqR \epsilon^{-1/2} (\mu B_0)^{1/2} \sin(N\phi_0) A_3 \end{aligned} \quad (10)$$

where

$$\begin{aligned}
 A_1 &\equiv \int_{-\theta_T}^{\theta_T} d\theta (x^2 - \sin^2 \frac{\theta}{2})^{1/2} = 4[E(x) - (1 - x^2) K(x)] , \\
 A_2 &\equiv \int_{-\theta_T}^{\theta_T} d\theta (x^2 - \sin^2 \frac{\theta}{2})^{-1/2} = 4K(x) , \\
 A_3 &\equiv \int_{-\theta_T}^{\theta_T} d\theta \cos(Nq\theta) (x^2 - \sin^2 \frac{\theta}{2})^{-1/2} \delta(\theta) \\
 &\approx 2 \left[\frac{\pi}{Nq x x_1} \right]^{1/2} \cos(Nq\theta_T - \frac{\pi}{4}) \delta(\theta_T) , \\
 x_1 &\equiv (1 - x^2)^{1/2} ,
 \end{aligned} \tag{11}$$

and E and K are complete elliptic integrals. We have assumed $Nq \sin \theta_T \gg 1$; i.e., that the dominant contribution to transport comes from banana orbits whose turning points are many ripples away from $\theta_T = 0$ and $\theta_T = \pi$. We also take $\delta(\theta)$ to be even in θ for simplicity (this is not an essential assumption). Observe that

$$\begin{aligned}
 v \frac{\partial}{\partial \mu} &= - \frac{(U - e\phi/m)}{2B_0 \epsilon \mu} \frac{\partial}{\partial (x^2)} \approx - \frac{1}{2\epsilon} \frac{\partial}{\partial (x^2)} \\
 &= - \frac{\partial}{\epsilon \sin \theta_T \partial \theta_T} ,
 \end{aligned} \tag{12}$$

and that A_3 oscillates rapidly with θ_T (owing to the cosine factor). We may therefore calculate f_1 approximately, by commuting $\partial/\partial \mu$ in Eq. (7) with all factors except the cosine. One finds

$$f_1 \approx - \frac{x^2 x_1^2 N m \epsilon \mu B_0}{e v (Nq)^2} \sin(N\phi_0) \frac{\partial f_m}{\partial \psi} \frac{A_3}{A_1} . \tag{13}$$

The radial drift velocity is

$$\begin{aligned}
 v_r &= - \frac{qm}{eB_0 \tau} \frac{\partial J}{\partial \phi_0} \left(\frac{\partial J}{\partial U} \right)^{-1} \\
 &= \frac{qmN\mu}{e\tau} \sin(N\phi_0) \frac{A_3}{A_2}.
 \end{aligned} \tag{14}$$

[Using $\mu B_0 \approx v^2/2$, and noting that $A_3 \sin(N\phi_0)/A_2 \sim \delta (Nq)^{-1/2}$ times an oscillating term of amplitude $\sim O(1)$, yields the expression used in the heuristic argument above.]

The flux-surface-averaged particle flux for this collisional regime is (for nearly-circular flux surfaces)

$$\Gamma_c \approx \int (4\pi^2)^{-1} d\phi_0 d\theta f_1 v_r d^3v \tag{15}$$

where $|d^3v| = |(8\pi e v^3/v_{\parallel}) \times dx dv|$. Therefore

$$\Gamma_c \equiv \int_0^1 dx \int_0^\infty dy \Gamma_c(x, y) \tag{16}$$

where $\Gamma_c(x, y)$ is the flux due to particles of given $x = \sin(\theta_T/2)$ and $y = v(m/2T)^{1/2}$. We find

$$\begin{aligned}
 \Gamma_c(x, y) &= - \eta N \frac{v_T}{R} \rho_\theta^{-2} \delta(x) I_c(x) \frac{v_* y^6}{(Nq)^2 v(y)} \\
 &\quad (2\pi)^{-1/2} \exp(-y^2) [\eta^{-1} - \tilde{E}_r - \frac{3}{2} + y^2] n'
 \end{aligned} \tag{17}$$

where $\eta = d \ln T / d \ln n$, $\tilde{E}_r \equiv eE_r/T$, the prime denotes d/dr ,

$$I_c(x) \equiv (1 - x^2)^{1/2} [F(x)]^{-1}, \tag{18}$$

$$F(x) \equiv x^{-2} [E(x) - (1 - x^2) K(x)] . \quad (19)$$

The function $I_c(x)$ is plotted in Figure 1. [Note $F(x)$ is a smooth bounded function; $F(0) = \pi/4$ and $F(1) = 1$.] In deriving Eq. (17) we have used

$$f_m = (2\pi v_T^2)^{-3/2} n \exp \left[-\frac{(U - e\phi/m)}{v_T^2} \right] \quad (20)$$

and $d\psi \approx (rB_o/q) dr$.

For the "pure" case in which all trapped particles are in this collision-dominated regime, we may carry out the x and y integrations to obtain

$$r_c = -0.241 \, nN \langle \delta^2 \rangle_I \, \rho_\theta^2 \, \frac{v_T}{R} \frac{v_*}{(Nq)^2 v_d} \left[(\eta^{-1} - \tilde{E}_r - \frac{3}{2}) I_6 + I_8 \right] n' . \quad (21)$$

The flux-surface-averaged heat flu. is

$$Q_c = -0.241 \, nN \langle \delta^2 \rangle_I \, \rho_\theta^2 \, \frac{v_T}{R} \frac{v_*}{(Nq)^2 v_d} \left[(\eta^{-1} - \tilde{E}_r - \frac{3}{2}) I_8 + I_{10} \right] T' . \quad (22)$$

Here, v_d is the deflection collision frequency at $y = 1$ (note $v_d = 1.182 \, v_B$),

$$\langle \delta^2 \rangle_I \equiv \frac{\int_0^1 \delta^2(x) I_c(x) dx}{\int_0^1 I_c(x) dx} \quad (23)$$

and

$$I_k \equiv \int_0^\infty dy \, y^k \exp(-y^2) \frac{y^3}{\operatorname{erf}(y) - \psi(y)} . \quad (24)$$

Numerically, $I_6 \approx 13.7$, $I_8 \approx 66.7$, and $I_{10} \approx 393$.

The evaluation of I_k above assumes a perfect Maxwellian distribution. We can model the effect of a finite energy confinement time by truncating the Maxwellian at $y_{\max}$ where $y_{\max}^2 = E_{\max}/T \sim \ln(\text{energy confinement time/ion collision time})$. For $y_{\max} = \sqrt{5}$, we have $I_6 \approx 8.0$, $I_8 \approx 27$, and $I_{10} \approx 99$.

The large effect of truncation on I_k and therefore on the transport rates is naturally due to the large velocity exponents entering the calculation. To see heuristically why an integral involving $y^{13} \exp(-y^2)$ arises in calculating the heat flux Q_c , note that the radial drift per bounce goes as ρ_0 and the number of bounces per collision goes as v/v_{eff} ; thus D_c goes as $v^4/v_{\text{eff}} \sim v^7$. Since $\partial f_m / \partial \psi$ contributes a term involving $v^2 \exp(-v^2/2v_T^2)$, we have

$$\begin{aligned} Q_c &\sim \int v^2 D_c [v^2 \exp(-v^2/2v_T^2)] d^3v \\ &\sim \int v^{13} \exp(-v^2/2v_T^2) dv \\ &\sim \int E^6 \exp(-E/T) dE. \end{aligned}$$

The above analysis applies equally to electrons and to ions. For the collisionalities of interest here, the ripple transport coefficients scale as $m^{1/2}$. If one accordingly neglects ripple-induced electron transport, ambipolarity implies that the total ripple-induced ion particle flux vanishes. This determines the electric field

$$\tilde{E}_r^A \equiv eE_r^A/T = (I_8/I_6) + n^{-1} - (3/2). \quad (25)$$

Then the ambipolar ion heat flux is

$$Q_c^A = -K_c^A T'$$

where

$$K_c^A = 0.241 \, nN \langle \delta^2 \rangle_I \rho_\theta^2 \frac{v_T}{R} \frac{v_*}{(Nq)^2 v_d} \left[I_{10} - \frac{I_8}{I_6} \right] . \quad (26)$$

The quantity in brackets approximately equals 68 for a Maxwellian distribution, and 8 for one truncated at $y_{\max} = \sqrt{5}$.

B. Toroidal-precession-dominant case

We consider the case in which toroidal precession is due to a radial electric field E_r . The precession rate is $\dot{\phi}_0 \sim E_r / (RB_\theta) \sim qE_r / (rB_0)$. A turning point will precess through one ripple in time $\tau_E \sim (N\dot{\phi}_0)^{-1}$. For a given θ_T , this time is shorter than $\{v_{\text{eff}}(\theta_T)\}^{-1}$ of Part A provided $\cos^2(\theta_T/2) \gtrsim RB_0 Nq v / E_r$. This condition cuts off the divergence that $v_{\text{eff}}(\theta_T)$ [Eq. (3)] would otherwise yield at $\theta_T \rightarrow \pi$; we shall study this in detail later. Heuristically, the diffusion coefficient is therefore

$$\begin{aligned} D_E &\sim \epsilon^{1/2} v_r^2 \tau_E^2 v_{\text{eff}} \\ &\sim N\delta^2 \rho_\theta^2 \frac{v_T}{R} \left(\frac{RB_0}{E_r} \right)^2 v_* v . \end{aligned} \quad (27)$$

To calculate the ripple transport in this regime, we write $f = f_m + f_1 + f_2$,

where f_1 is obtained from the collisionless banana kinetic equation, and f_2 from the collisional equation using $C(f_1)$. We have therefore to solve

$$\begin{aligned} \frac{\partial f_m}{\partial \psi} \frac{\partial J}{\partial \phi_0} &\approx \frac{\partial f_1}{\partial \phi_0} \frac{\partial J}{\partial \psi} \\ \frac{\partial f_2}{\partial \phi_0} \frac{\partial J}{\partial \psi} &\approx \frac{e}{m} \int \frac{dl}{v_{||}} C(f_1) . \end{aligned} \quad (28)$$

The analysis proceeds as in Eqs. (7) - (12). Using in addition

$$\frac{\partial J}{\partial \psi} \approx \frac{eq^2 E_r}{mB_0} (\mu B_0 c)^{-1/2} A_2 , \quad (29)$$

and commuting $\partial/\partial \mu$ with all factors except the rapidly oscillating $\cos [Nq\theta_T - (\pi/4)]$, we obtain

$$f_1 \approx \frac{m\mu r_B^2}{eqE_r} \cos(N\phi_0) \frac{\partial f_m}{\partial \psi} \frac{A_3}{A_2} + \hat{f}_1 \quad (30)$$

with $\partial \hat{f}_1 / \partial \phi_0 = 0$. Then

$$\frac{\partial f_2}{\partial \phi_0} \approx \frac{evJ}{16mB_0 \mu c^2 x^2} \left(\frac{\partial J}{\partial \psi} \right)^{-1} \frac{\partial^2 f_1}{\partial x^2} . \quad (31)$$

[Recall that $x = \sin(\theta_T/2)$.] Note $\partial^2 \hat{f}_1 / \partial x^2$ must be zero; otherwise f_2 would be multivalued in ϕ_0 . Solving for f_2 ,

$$f_2 \approx - \frac{m\mu r_B^2}{e\tilde{E}_r} \frac{cN\mu}{2} \frac{\partial f_m}{\partial \psi} \sin(N\phi_0) \frac{A_1 A_3}{A_2^2 x^2 (1-x^2)} . \quad (32)$$

Since v_r is given by Eq. (14), we note that the ϕ_0 average of $v_r f_1$ is zero. The flux-surface-averaged particle flux for particles of given x and

$y[= v(m/2T)^{1/2}]$ is therefore

$$\begin{aligned} \Gamma_E(x, y) &\approx \int (4\pi^2)^{-1} d\phi_0 d\theta f_2 v_r d^3v / (dx dy) \\ &\approx -nN \frac{v_T}{R} \rho_\theta^2 \delta^2(x) I_E(x) v_* v(y) \\ &\quad \left(\frac{eRB_0}{\tilde{E}_r T'} \right)^2 (2\pi)^{-1/2} y^6 \exp(-y^2) [n^{-1} - \tilde{E}_r - \frac{3}{2} + y^2] n' \end{aligned} \quad (33)$$

where

$$I_E(x) \equiv [K(x)]^{-2} (1 - x^2)^{-3/2} F(x), \quad (34)$$

$F(x)$ as in Eq. (19). The function $I_E(x)$ is plotted in Fig. 1. The heat flux $Q_E(x, y) = Ty^2 \Gamma_E(x, y)$.

C. Mixed case

As noted in Part B, the effective collisionality for barely-trapped particles is always great enough that collisions, rather than toroidal precession, dominate banana drift transport. We may approximate the total (flux-surface-averaged) particle flux by

$$\Gamma \approx \int_0^1 dx \int_0^\infty dy \min[\Gamma_E(x, y), \Gamma_C(x, y)], \quad (35)$$

and similarly for the heat flux Q . The ambipolar electric field is that for which $\Gamma = 0$, as discussed above.

The ambipolar thermal conductivity $\mathcal{R}^A$ (in dimensionless form) is plotted as a function of dimensionless collisionality $\tilde{\nu}_d$ in Figs. 2-4. We define $\mathcal{R}^A$

by

$$Q^A = -nN\delta_m^2 \frac{v_T}{R} \rho_\theta^2 \frac{\tilde{v}_*}{(Nq)^2} \tilde{K}^A(\tilde{v}_d) T, \quad (36)$$

$\tilde{v}_d$ and $\tilde{v}_*$ by

$$\tilde{v}_{d,*} \equiv NqRB_0 \left| \frac{e}{T} \right| v_{d,*}, \quad (37)$$

and model the ripple amplitude as

$$\delta[x = \sin(\theta/2)] = \delta_m (1 + \delta_r \cos\theta). \quad (38)$$

Figure 2 shows the effect of δ_r on $\tilde{K}^A$. We see that δ_r has no effect on $\tilde{K}^A$ (apart from an overall scale factor) in the collision-dominated $1/\tilde{v}_d$ regime, i.e., at large $\tilde{v}_d$. As the collisionality is decreased, the transport becomes toroidal-precession dominated for all banana orbits except the barely trapped ones. In that case, the barely-trapped particles account for much of the precession-dominated transport [note the divergence of $I_E(x)$ in Figure 1] and all of the collision-dominated transport. When δ_r is increased, the "collisional" contribution (which goes as v^{-1}) is decreased and the "precessional" contribution (which goes as v) is relatively increased. This accounts for the steepening of the $\tilde{K}^A$ falloff with increasing δ_r (at low $\tilde{v}_d$).

$\tilde{K}^A$ is seen to be maximal for $\tilde{v}_d \sim 10$. The ripple plateau⁵ would, in the absence of toroidal precession, pass smoothly into the $\tilde{v}_d^{-1}$ regime at a collisionality estimated by setting $K_c^A = K_{\text{plateau}}^A$, where⁵

$$K_{\text{plateau}}^A = 3 \left(\frac{\pi}{2}\right)^{1/2} nN \langle \delta^2(\theta) \rangle \rho_\theta^2 \frac{v_T}{R}. \quad (39)$$

For a Maxwellian distribution and with $\delta_r = 0$, $\kappa_c^A = \kappa_{\text{plateau}}^A$ at

$$\tilde{v}_d \approx 4.4 \frac{T}{\rho_0 T'} \frac{\epsilon}{Nq}^{1/2}.$$

The v^{-1} regime will therefore exist only if this $\tilde{v}_d \gtrsim 10$.

Figure 3 shows the effect of changing n . This parameter affects only the value of the ambipolar electric field; hence it modifies only toroidal-precession-dominated transport.

Figure 4 shows the effect on $\tilde{\kappa}^A$ of truncating the Maxwellian distribution. The effect is large in the v^{-1} regime (as discussed above) and is still significant at lower collisionalities.

III. Competing effects and the validity of the assumed ordering

Throughout this paper we assume that the toroidal precession per bounce is small. The time required to precess one ripple is $\tau_p \sim (Nq_0)^{-1} \sim RB_0/(NE_r)$. The bounce period for a particle of velocity v is $\tau_b \sim qR/(v\epsilon^{1/2})$. The required condition is thus

$$\frac{\tau_b}{\tau_p} \sim \frac{Nq}{\epsilon} \frac{v_T}{v} \frac{\rho_0 T'}{T} - \frac{eE_r}{T'} \ll 1. \quad (40)$$

The dominant contribution to transport comes from particles with $v > v_T$, and typically $eE_r/T' \sim 1$. The condition is thus similar (except for a numerical factor of order unity) to the condition for the existence of a v^{-1} regime derived in the preceding section. It is only marginally satisfied for typical reactor parameters (e.g., $Nq \sim 25$, $\epsilon \sim 1/4$, $\rho_0 T'/T \sim 0.01$).

We turn now to the low- v regime. Apart from toroidal precession induced

by a radial electric field, what other effects can "cut off" the $D \sim v^{-1}$ regime at low v ? (1) Toroidal precession can be induced by magnetic drifts. (2) The radial banana drift could in principle become large enough that the change in $e\phi(\psi)/m$ would result in detrapping. (3) When shear is present, the radial drift can change Nq by an amount of order unity, and thereby change the ripple phase at the banana turning points.⁷ We shall consider each effect in turn.

(1) Magnetic drifts: The precession rate is ⁶

$$\dot{\phi} = \frac{m}{e} \frac{\partial J}{\partial \psi} \left(\frac{\partial J}{\partial U} \right)^{-1} . \quad (41)$$

Ignoring ripple, we find

$$\dot{\phi} \approx - \frac{m \mu B_0'(\psi)}{e} = \dot{\phi}'(\psi) .$$

Thus the condition for the $\vec{E} \times \vec{B}$ drift to dominate magnetic drift as a cause of precession is

$$\left| \frac{e}{m} E_r \right| \gg \left| v \frac{dB_0}{dr} \right| \approx \left| U \frac{d \ln B_0}{dr} \right| .$$

The ambipolar $E_r \sim T/er$, while $d \ln B_0/dr \sim R^{-1}$; for $U \lesssim$ few times T/m the above condition therefore follows from $\epsilon \ll 1$, which is assumed.

(2) We require $|e\Delta\phi/m| \ll U\epsilon$, where $\Delta\phi \sim E_r \Delta r$. For $E_r \sim T/er$ and $U \lesssim$ few times T/m , this is equivalent to $(\Delta r)/r \ll \epsilon$. Now

$$(\Delta r)/r \sim v_r \tau_E / r$$

$$\sim \delta(Nq)^{-1/2}$$

$$\ll \epsilon ,$$

as desired.

(3) Shear is important when $NqQ\Delta r/r \gtrsim 1$, where $Q \equiv d \ln q / d \ln r$. At that v for which the toroidal precession effect "cuts off" the v^{-1} regime, shear is therefore an important competing effect only if

$$\delta Q(Nq)^{1/2} T / (erE_r) \gtrsim 1 .$$

For typical parameters (e.g., $\delta \sim 0.005$, $Nq \sim 25$, $E_r \sim T/er$) one would therefore need $Q \gtrsim 40$.

IV. Comparison with earlier work

Banana drift transport has been discussed by Davidson³ and Tsang.⁴ Davidson treated the v^{-1} regime, obtaining $D \sim \epsilon^{3/2} v^{-1} (\delta N v_T \rho / R)^2$. Our analysis differs in three respects. First, the effective collisionality is not v/ϵ , but $v(Nq)^2/\epsilon$. For successive steps in the diffusion process to be uncorrelated, collisional detrapping is not necessary. One only needs to shift the ripple phase at the turning point θ_T ; therefore, $\Delta \theta_T \sim (Nq)^{-1}$ suffices. Second, the amplitude of $v_r(\phi) \sim v_T \delta N \rho_\theta / R$, not $v_T \delta N \rho / R$ (see Boozer⁵). Third, the only net contribution to v_r comes from the last ripple at each turning point. Since the particle spends $\sim (Nq)^{-1/2}$ of its time in these ripples, one obtains an extra factor of $(Nq)^{-1/2}$ in v_r . Therefore our $D \sim D_{\text{Davidson}} / (N^3 q \epsilon^2)$. For typical parameters, D is therefore 10^{-2} to 10^{-3}

times ν_{Davidson} .

Tsang⁴ obtains essentially the same result as Davidson for the ν^{-1} regime. (Tsang has an extra factor of qQ where $Q \equiv d \ln q / d \ln r$.) The comments of the previous paragraph apply equally to his heuristic result. Concerning his detailed calculation, we note that our $(Nq)^2$ enhancement of ν_{eff} is absent from his result for the following reason. Tsang approximates the action $J \approx J_0 + J_1$, where J_0 is the zero-ripple action and $J_1 \propto Q \delta [E(x) - (1 - x^2) K(x)] \cos N\phi_0$. Thus his $\partial J_1 / \partial \phi_0$ lacks the rapidly-oscillating dependence on x [through $\cos (Nq\theta_T - \pi/4)$] that we have found [Eq. (10)]. It is this oscillation of f_1 that introduces our $(Nq)^2$ factor when the collision term, with its $\partial^2 / \partial \theta_T^2$ operator, is calculated. In essence, the analysis of Ref. 4 lacks the mechanism by which the ripple phase near the turning points determines the net radial drift.

Acknowledgment

This work was supported by the United States Department of Energy Contract No. DE-AC02-76-CH03073.

References

- ¹T. E. Stringer, Nucl. Fusion 12 (1972) 689.
- ²J. W. Connor and R. J. Hastie, Nucl. Fusion 13 (1973) 221.
- ³J. N. Davidson, Nucl. Fusion 16 (1976) 731.
- ⁴K. T. Tsang, Nucl. Fusion 17 (1977) 557.
- ⁵A. H. Boozer, PPPL-1619 (1980).
- ⁶R. J. Hastie, J. B. Taylor, and F. A. Haas, Ann. Phys. (N.Y.) 41 (1967) 302.
- ⁷R. Goldston, private communication.

81T 0079

Fig. 1 The functions $I_C(x)$ [Eq. (18)] and $I_E(x)$ [Eq. (34)], with $x = \sin(\theta_T/2)$, θ_T = banana turning point.

81T0080

Fig. 2 Ambipolar thermal conductivity $\tilde{K}^A$ versus collisionality $\tilde{\nu}_d$ (dimensionless units) for $n = 1$ and several values of the ripple modulation parameter δ_r .

8IT0081

Fig. 3 Dependence of $\tilde{K}^A(\tilde{v}_d)$ upon η , for $\delta_r = 0$.

81T0082

Fig. 4 Effect of truncation of the Maxwellian distribution upon $R^A(C_d)$. Solid curves correspond to Maxwellian; broken curves to a Maxwellian truncated at $y = \sqrt{5}$. For all cases, $\eta = 1$.