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THE IMAGE DISPLACEMENT EFFECT IN LINEAR INDUCTION ACCELERATORS

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THE IMAGE DISPLACEMENT EFFECT IN LINEAR
INDUCTION ACCELERATORS*

V. K. Neil

Abstract

A beam of relativistic particles moving inside a conducting pipe of circular cross-section is displaced from the axis of the pipe. The presence of a discontinuity in the pipe (e.g., an accelerating gap) will cause a redistribution of the image charges and currents, and a net transverse force is experienced by beam particles as they transverse the gap. This force is calculated in the static limit for the particular geometry of 3 cm wide gaps spaced 30 cm apart in a pipe of radius 7.5 cm.

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I. Introduction

In this work we investigate theoretically a transverse force that arises from an accelerating gap or other interruption of a continuous, conducting pipe. The treatment is electrostatic and magnetostatic, thus radiation effects are excluded. The results apply for one geometry only, namely that shown in Figure 1. The method of calculating the electric force as given in Section II is quite general and may be used for similar geometries. However, the method of calculating the magnetic force as given in Section III is probably valid only if the gap width is less than the pipe radius.

A beam of particles moving along the axis of a conducting pipe gives rise to the azimuthally symmetric surface charge and current distributions on the surrounding wall. If the beam is displaced transversely from the axis of the pipe, the charge and current distributions on the wall become asymmetric. The asymmetric surface charge attracts the beam toward the wall while the surface current repels the beam. In a continuous pipe with smooth walls, these two forces cancel to within a factor γ^{-2} , where γ is the particle's total energy in units of the rest energy.

At a discontinuity in the pipe, such as an accelerating gap shown in Fig. 1, the electric self-field will decrease exponentially with r in the region $r > b$ if the gap width is less than the pipe radius. On the other hand, the magnetic self-field will extend much further into the region $r > b$, and decrease in magnitude as r^{-2} . Thus the surface charge and current distributions are displaced from each other, and the cancellation of electric and magnetic forces is lost in the gap region.

For $\gamma \gg 1$ and $I \gtrsim 1$ kA the resulting transverse force becomes an important consideration in the design of the beam transport system.¹

Consider a beam of particles with charge per unit length λ . The average axial speed of the particles is v , so that the beam carries current $I \equiv \lambda v$. If the beam is on axis in a continuous, smooth pipe, the surface charge σ_0 and axial surface current K_0 on the wall are given by

$$\sigma_0 = -\lambda/2\pi b, \quad (1.1)$$

$$K_0 = \sigma_0 v, \quad (1.2)$$

in which b is the radius of the conducting pipe.

If the beam centroid is displaced a distance ξ from the axis of the pipe, there is an additional asymmetric charge distribution σ . To first order in ξ we have

$$\sigma = -\lambda\xi \cos \theta / \pi b^2. \quad (1.3)$$

There is also an asymmetric axial surface current $K \equiv \sigma v$. In Eq. (1.3) the angle θ is measured from the direction of the displacement, which we take to be the x direction.

The electric field arising from the surface charge alone is uniform within the pipe and given by

$$\underline{E} = (2\lambda\xi/b^2)\hat{x}. \quad (1.4)$$

¹V. K. Neil, "Quadrupole vs. Solenoidal Focusing to Suppress the Image Displacement Instability in Linear Induction Electron Accelerators." LLL report UCID 17838, June 20, 1978.

Gaussian units are employed in this work. The magnetic field arising from the surface current alone is also uniform within the pipe and given by

$$\underline{B} = (2\lambda\xi\beta/b^2)\hat{y}, \quad (1.5)$$

in which $\beta = v/c$, with c the speed of light. There are additional fields within the pipe that arise from the charge and current distribution within the beam, but these additional fields can exert no net force on the motion of the beam centroid and need not be considered. The net force F_x in the x direction is given by

$$F_x = e \left[\underline{E}_x - \beta \underline{B}_y \right] = 2e\lambda\xi/b^2\gamma^2, \quad (1.6)$$

in which we have used $\gamma^2 = (1 - \beta^2)^{-1}$. For $\gamma \gg 1$ this force is not appreciable in a continuous pipe.

II. The Electrostatic Force

The system is periodic in the axial coordinate, z , each section has total length L and has a centrally located gap of width w as shown in Fig. 1. We assume that the time variation of the beam displacement ξ and the charge per unit length λ is sufficiently slow to allow a static solution for the electric field (and also the magnetic field, as treated below). The static treatment is valid for frequencies such that $\omega b/c \ll 1$. We further assume that ξ and λ do not vary over the length of one cell. For relativistic particles the variation of v is negligible.

The displaced beam gives rise to a surface charge distribution on the conducting surfaces $r = b$, $w/2 < |z| < L/2$, and $r > b$, $z = \pm w/2$. This distribution may be quite complicated, but the electric field arising from the distribution may be found by super-position. In fact the total electric field $\underline{E}_t$ from the distribution may be written

$$\underline{E}_t = \underline{E} + \underline{E}_1, \quad (2.1)$$

with $\underline{E}$ given by Eq. (1.4). The field configuration $\underline{E}_1$ arises from a surface charge distribution $-\sigma$ on the surface $r = b$, $|z| < w/2$, with σ given by Eq. (1.3). The surface charge σ is negative for $-\pi/2 < \theta < \pi/2$ and gives rise to the electric field $\underline{E}$ in the positive x direction. The electric field $\underline{E}_1$ will have a component in the $-x$ direction and therefore tend to cancel $\underline{E}$. Physically this results from the fact that in the region of the gap, the surface charges are displaced into the region $r > b$, $|z| < w/2$ and exert less force of the beam particles than would be the situation in a smooth pipe.

We define a scalar potential $V(r, z, \theta)$ such that $\underline{E}_1 = -\nabla V$. Because the source of $\underline{E}_1$ has a $\cos \theta$ dependence, it follows that V is of the form

$$V(r, \theta, z) = U(r, z) \cos \theta, \quad (2.2)$$

in which the function U obeys the equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial z^2} - \frac{U}{r^2} = - \frac{4\pi\rho}{\cos \theta}. \quad (2.3)$$

In Eq. (2.3), ρ is the charge density. For the problem we are considering we have

$$\rho = \begin{cases} (\lambda \epsilon / \pi b^2) \delta(r - b) \cos \theta, & |z| < w/2, \\ 0 & |z| > w/2. \end{cases} \quad (2.4)$$

Equation (2.4) cannot be solved analytically, but is readily solved computationally. The boundary conditions are $U = 0$ at $r = 0$ for all z , $U = 0$ on conducting surfaces, and $\partial U / \partial z = 0$ on the planes of symmetry $z = \pm L/2$. We first establish a grid in r and z with a spacing of h cm in r and ℓ cm in z so that $r = ih$ and $z = j\ell$. The finite difference equation to be solved is

$$\frac{U_{i+1}^j}{h^2} \left[1 + \frac{1}{2i} \right] + \frac{(U_i^{j+1} + U_i^{j-1})}{\ell^2} + \frac{U_{i-1}^j}{h^2} \left[1 - \frac{1}{2i} \right] - U_i^j \left[\frac{2}{\ell^2} + \frac{2}{h^2} + \frac{1}{(ih)^2} \right] = \text{RHS}. \quad (2.6)$$

The right hand side of Eq. (2.6) is zero except at the value of i corresponding to $r = b$, and all values of j corresponding to $|z| < w/2$.

At these points we have

$$\text{RHS} = -4\lambda \epsilon / b^2 h.$$

Since the function $U(r, z)$ is even in z , we solve Eq. (2.5) for $z \geq 0$ only. To facilitate computation, an additional conducting boundary is placed at some radius $d > b$ for $z < w/2$. The radius d is chosen large enough so that the presence of this conducting surface has a negligible effect on the solution.

Figure 2 shows a three-dimensional plot of the function U obtained for $b = 7.5$ cm, $w = 3$ cm, and $L = 20$ cm. A value of $d = 2b$ proved sufficient for this geometry. (The value of L has no effect either. If d

and L were increased without limit, the solution in the region of interest would not change.) The peak value of U naturally occurs at $r = b$, $z = 0$. This value is $2.46\lambda\epsilon/b^2$ stat-volts, with λ in esu/cm, ϵ and b in cm. The constant has dimensions of cm.

We wish to calculate the change in transverse momentum, Δp_x , experienced by a particle in the displaced beam as the particle travels from $z = -L/2$ to $z = +L/2$. To evaluate Δp_x we need the quantity I_1 , given by

$$I_1 = \int_{-L/2}^{L/2} E_{1x} dz. \quad (2.7)$$

we have

$$E_{1x} = E_{1r} \cos \theta - E_{1\theta} \sin \theta = -\frac{\partial U}{\partial r} \cos^2 \theta - \frac{U}{r} \sin^2 \theta. \quad (2.8)$$

If, as is true for the geometry of Fig. 1, $U \propto r$ near the axis for all z , then $\partial U / \partial r \equiv U/r$ and $E_x = -\partial U / \partial r$. Then

$$I_1 = - \int_{-L/2}^{L/2} (\partial U / \partial r) dz. \quad (2.9)$$

We have evaluated I_1 numerically found

$$I_1 = -1.06 \lambda \epsilon / b^2 \text{ stat-volts}. \quad (2.10)$$

Again, the constant 1.06 has dimensions of cm. This value is accurate to within 0.1% for $r < 2$ cm.

III. The Magnetostatic Force

Analogous to the method used above for the electric force, we write the total magnetic field $\underline{B}_t$ as

$$\underline{B}_t = \underline{B} + \underline{B}_1, \quad (3.1)$$

with $\underline{B}$ given by Eq. (1.5). The field $\underline{B}_1$ arises from an axial surface current distribution

$$K_1 = \lambda \xi v \cos \theta / \pi b^2 \quad (3.2)$$

on the surface $r = b$, $|z| < w/2$. This surface current is equal and opposite to that which would be present on that surface if no discontinuity were present. The magnetic field $\underline{B}_1$ will have a component in the negative y direction that tends to cancel $\underline{B}$.

It is not convenient to solve for $\underline{B}_1$ computationally. However, an approximate analytic solution may be obtained. In the region $r > b$, $|z| < w/2$ (region II in Fig. 1) we take the field $\underline{B}_1$ to be given by the approximate expression

$$\underline{B}_1(\text{II}) = A(b/r)^2 (\sin \theta \hat{r} - \cos \theta \hat{\theta}), \quad (3.3)$$

in which A is a constant to be evaluated. Equation (3.3) neglects that portion of $\underline{B}_1$ in region II that varies with z . The neglected portion will give a contribution to Δp_x that is much less than the contribution we are considering.

In the region $r < b$ (region I in Fig. 1) we take B_1 to be given by

$$B_1(I) = a_0(\sin \theta \hat{r} + \cos \theta \hat{\theta}) + \sum_{s=1}^{\infty} \frac{a_s}{I_1'(k_s b)} \left\{ \left[I_1(k_s r) \sin \theta \hat{r} + \frac{I_1(k_s r)}{k_s r} \cos \theta \hat{\theta} \right] \cos k_s z - I_1(k_s r) \sin \theta \sin k_s z \hat{z} \right\}. \quad (3.4)$$

In this expression I_1 is the first order modified Bessel function of the first kind and the prime denotes differentiation with respect to the argument. The quantities k_s are given by $k_s = 2\pi s/L$ with s an integer. The coefficients a_s are to be determined.

From the continuity of B_{1r} at $r = b$ we have

$$a_0 + \sum_{s=1}^{\infty} a_s \cos k_s z = \begin{cases} A, & |z| < w/2 \\ 0, & |z| > w/2 \end{cases}. \quad (3.5)$$

We integrate this equation over z from $-L/2$ to $L/2$ and obtain

$$a_0 L = Aw. \quad (3.6)$$

Multiplying Eq. (3.5) by $\cos k_q z$, with q any integer, and again integrating over z from $-L/2$ to $L/2$ we obtain

$$a_s = \frac{2Aw}{L} \frac{\sin(k_s w/2)}{(k_s w/2)}. \quad (3.7)$$

We now turn to the second condition that must be satisfied at $r = b$, $|z| < w/2$, namely

$$B_{1\theta}(II) - B_{1\theta}(I) = 4\pi K_1/c. \quad (3.8)$$

We insert Eqs. (3.6) and (3.7) into Eq. (3.4) and use the resulting expression along with Eq. (3.3) to obtain :

$$\frac{Aw}{L} \left\{ 1 + 2 \sum_{s=1}^{\infty} \frac{\sin(k_s w/2)}{(k_s w/2)} \frac{I_1(k_s b) \cos k_s z}{k_s b I_1'(k_s b)} \right\} = -A - \frac{4\pi K_1}{c \cos \theta} \quad (3.9)$$

For $|z| > w/2$ there is no condition on the left hand side of Eq. (3.9).

If the left hand side were indeed independent of z for $|z| < w/2$, we would have an exact solution. But of course it is not, and there is the additional problem of the continuity of B_z . These two problems arise because we have not included the complete field B_1 for $r > b$.

We evaluate the constant A by taking the average of the series in Eq. (3.9) over z . Integrating Eq. (3.9) from $z = -w/2$ to $z = w/2$ we obtain

$$A \left\{ 1 + \frac{w}{L} \left[1 + 2 \sum_{s=1}^{\infty} \frac{\sin^2(k_s w/2)}{(k_s w/2)^2} \frac{I_1(k_s b)}{k_s b I_1'(k_s b)} \right] \right\} = -\frac{4\pi K_1}{c \cos \theta} \quad (3.10)$$

The series in Eq. (3.10) is easily evaluated numerically. For $w = 3$ cm, $b = 7.5$ cm and $L = 30$ cm, we get

$$A = -2.94\pi K_1 / c \cos \theta \quad (3.11)$$

The only part of B_1 that contributes to Δp_x is the z independent term in Eq. (3.4). We define

$$I_2 = \int_{-L/2}^{L/2} B_{1y} dz \quad (3.12)$$

Employing Eqs. (3.4), (3.6), (3.11) and (3.2) we obtain

$$I_2 = -2.94\lambda \epsilon B w/b^2 \quad (3.13)$$

IV. Change in Transverse Momentum

A particle in the displaced beam traveling from $z = -L/2$ to $z = L/2$ experiences a change Δp_x in its transverse momentum, with Δp_x given by

$$\Delta p_x = \frac{e}{v} \int_{-L/2}^{L/2} (E_x - \beta B_y + E_{x1} - \beta B_{y1}) dz. \quad (4.1)$$

From Eqs. (1.6) (2.7) and (3.12) we have

$$\Delta p_x = \frac{2e \lambda \xi L}{v \beta^2 \gamma^2} + \frac{e}{v} (I_1 - \beta I_2). \quad (4.2)$$

Inserting the values of I_1 and I_2 from Eqs. (2.10) and (3.13) respectively, we have the final result,

$$\Delta p_x = \frac{e \lambda \xi}{v \beta^2} \left[\frac{2L}{\gamma^2} - 1.06 + 2.94 \beta^2 w \right]. \quad (4.3)$$

Recall that the constant 1.06 has dimensions of cm. This term represents the decrease in the electric image force because of the gap. The third term, $\sim 3\beta^2 w$, represents the decrease in the magnetic image force. In reference 1 the first term, $2L/\gamma^2$ was neglected entirely, and Δp_x written in terms of an "effective" gap width W so that (Eq. 2 of Ref. 1 in present notation)

$$\Delta p_x = 2e \lambda \xi W / (c\beta)^2.$$

Comparing with Eq. (4.3) we see that, at least for the particular geometry treated here,

$$W \approx (3w/2) - (1/28^2)$$

Although the solution for $\underline{E}_1$ found in Section II is exact, the solution for $\underline{B}_1$ found in Section III is not. The neglected portion of $\underline{B}_1$ could possibly lead to a term in Eq. (4.3) comparable to the term from the correction to the electric force, but this is still an order of magnitude less than the term from the correction to the magnetic force.

For similar geometries, the correction to the electric force may be calculated and (with some careful measurements) the correction to the magnetic force may be determined experimentally.

Figure 1
Geometry treated in this work.

Figure 2

Plot of the potential function $U(r, z)$
that satisfies Equation (2.3).