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PION-NUCLEUS SCATTERING IN THE ISOBAR FORMALISM[†]

E. J. Moniz

Center for Theoretical Physics
Department of Physics and Laboratory for Nuclear Science
Massachusetts Institute of Technology
Cambridge, Massachusetts, 02139

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E. J. Moniz

Massachusetts Institute of Technology

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I. INTRODUCTION

Pion-nucleus interactions play a central role in our attempts to gain a deeper understanding of nuclear systems. In some ways, this role is fulfilled by providing another "conventional" nuclear probe. For example, the pion may be used to measure nuclear matter distributions and, because of considerable variation in the mean free path over the accessible energy range, can be used to emphasize surface or volume effects. Because it comes in three charge states, double charge exchange can be used to study double analog transitions or proton rich nuclei. Because of its quantum numbers $J^P = 0^-$, reactions such as radiative capture may be used to study isovector M1 states. However, in addition to these, rather more unique opportunities are also presented. These originate in the boson nature of the pion and in its intimate connection to the nuclear force.

A brief reminder of the role played by the pion in the dynamics of the two nucleon system is in order. Of course, the long range nuclear force has been known for a long time to be given by one pion exchange (Fig. 1a). The nonrelativistic pion-nucleon vertex is dictated by the pseudoscalar nature of the pion. This same coupling led Chew and Low to explain very simply the origin of the attractive "force" in the $J = 3/2$, $T = 3/2$ πN scattering channel which leads to the famous resonance at an invariant energy of 1232 MeV. The interaction in this channel is the crossed πN Born term in Fig. 1b and we can think of the resonance as intermediate excitation of the (unstable) Δ -isobar. This is the object predicted in the quark model as the "first excited

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Figure 1: Pions and isobars in the nuclear force, in pion scattering and absorption, and in the nuclear electromagnetic current.

state" of the nucleon, although the fundamental connection between the Chew-Low and the quark model dynamics is not understood. The Δ in turn plays a very important role in the NN force; while not understood nearly as well, a large part of the intermediate range attraction which binds nuclei appears to be due to intermediate excitation of the Δ (Fig. 1c).

The possibility of pion absorption is implicit in the πNN vertex discussed above. Energy/momentum conservation requires that absorption of a physical (timelike) pion occurs on at least two nucleons. The cross section for $\pi^+ + d \rightarrow p + p$ shows a "resonance bump" just like that seen in the πN total cross section, so that it is empirically clear that the Δ is again playing a central role in the relevant energy regime. This is indicated in Fig. 1d. Finally, the pion and isobar degrees of freedom in the nucleus are known to play an important role in determining nuclear electromagnetic current distributions. Examples are indicated in Figs. 1e and 1f and are usually called exchange current contributions. The lesson in all this is that we understand at least semi-quantitatively the important role of pions and isobars in the nuclear force, in pion reactions with one and two nucleons, and in the "nuclear structure" of the two nucleon system. In these lectures, we will be concerned with using this understanding as input for a theoretical approach to the pion-nucleus many-body problem.

The conventional theoretical framework for high energy hadron-nucleus reactions is the multiple scattering or optical potential approach. Here, one focuses upon the pion degree of freedom as it works its way through the nucleus. However, this approach certainly does not emphasize the role of isobar dynamics in the nuclear medium and, in practice, this is often "frozen" out of the problem. Yet these features may be expected to be quite important. The isobar propagation distance in free space is not small compared to other length scales of interest in the nucleus. That is, if we write the Δ -propagator crudely as

$$G_{\Delta}(E, p) = (E - E_R + i\Gamma/2 - p^2/2m)^{-1}, \quad (1.1)$$

where E_R and Γ (≈ 115 MeV) are the resonance position and width, respectively, then the coordinate space propagator is

$$G_{\Delta}(r) \sim \frac{e^{-\kappa r}}{r}, \quad \kappa^2 \equiv 2m_{\Delta}(E - E_R + i\Gamma/2). \quad (1.2)$$

Near resonance, we have

$$\text{Im} \kappa \sim \sqrt{\frac{M_{\Delta} \Gamma}{2}} \sim (.7 \text{ fm})^{-1} \quad (1.3)$$

while the usual mean free path in nuclear matter is

$$\lambda = (\rho \sigma_T)^{-1} \approx \frac{1}{2} \text{ fm}. \quad (1.4)$$

at resonance. Clearly, isobar propagation should be important for a quantitative description of pion interactions at intermediate energies. Furthermore, pion absorption (which goes through the Δ) is known to constitute a very large part of the pion reaction cross section for $\pi - {}^{12}\text{C}$ scattering at 130 MeV and an even larger part at lower energies.¹ Again, this is obviously a very important part of the dynamics and must be incorporated in the formalism.

In these lectures, the isobar-hole formalism for pion reactions will be described. This is an alternate theoretical approach in which the focus is upon the isobar degree of freedom and its dynamics in the nucleus. As will become clear, the physics is not fundamentally different from that in the multiple scattering approach. However one is led to a rather different set of approximations, and we find the isobar formalism a natural way to incorporate the physics discussed above. The concept of isobar-hole doorway states was discussed first by Kisslinger and Wang² and has formed the basis of microscopic calculations of pion scattering.^{3,4,5,6} Application of the model to $\pi - {}^{16}\text{O}$ scattering will be discussed in considerable detail; these calculations have been performed in collaboration with M. Hirata, J. H. Koch and F. Lenz.⁴

The lectures are structured as follows. We shall first discuss more precisely the introduction of the isobar as an explicit degree of freedom and the connection of such a picture to one involving only pions and nucleons. The isobar-hole formalism and its relation to the usual pion optical potential will then be described. The goal here is to evaluate the isobar-hole propagator in a finite nucleus by diagonalizing the Hamiltonian in an appropriate basis. The result is a set of isobar-hole eigenstates and eigenvalues, with strong collectivity a feature of the results. This collectivity will lead to introduction of an extended schematic model for pion scattering.⁴ This is basically a generalization of the familiar Brown-Bolsterli nuclear particle-hole schematic model for excited states and allows considerable streamlining of the calculations. The collectivity will also facilitate interpretation of the role of different aspects of the isobar dynamics. In order to see explicitly how the calculations "work", a simple spinless s-wave model will be discussed.⁷ Analytic results will be obtained in the high energy limit and some insight into the collectivity gained. Finally, we discuss application of the model to $\pi - {}^{16}\text{O}$ scattering and compare with recent high precision elastic scattering data. We thus hope to show the extent to which microscopic treatment of the many-body dynamics explains the data and the extent to which additional physical input is required. Another test of the model

is provided by various inelastic processes and we discuss briefly inclusive reactions.) Hopefully, the impression left will be one in which the successes of the model can be appreciated while not disguising the need for considerably more development in the microscopic calculations.

II. THE ISOBAR

There has been considerable confusion over the introduction of the Δ isobar as an explicit degree of freedom, so a fairly detailed discussion of this point is worthwhile. On the one hand, some argue that the Chew-Low model for the 3-3 resonance indicates the absence of an elementary Δ in the strong interaction Hamiltonian; others point to the quark model as providing a three-quark Δ on a footing completely equal to that of the nucleon. In fact, this distinction is practically irrelevant for pion-nucleus physics. The point is that we can always introduce an "elementary" isobar or quasiparticle to describe a two-body interaction.^{8,9} This is not different fundamentally from the game played with the NN force of introducing separable potentials (of various rank) to describe the interaction very well in some limited energy range, with a residual interaction made up of the difference between the true and separable interactions. With a proper choice of the separable form, the residual interaction may be treated in perturbation theory. The isobar approximation for the πN interaction is more physically motivated because of the clear resonance behavior at invariant mass 1232 MeV; the separable behavior of the transition matrix close to a resonance is inherent in the isobar model.

Let us start by introducing a Δ particle with bare mass m_Δ . This particle must couple to the open πN channel, as in Fig. 2. We now have a coupled channel situation for the πN and Δ channels, and we can express the πN transition operator with either channel projected out:

$$T_{\pi N} = V_{\pi N} + V_{\pi N} G_{\pi N}^0 T_{\pi N} \quad (2.1)$$

$$= (g_{\Delta\pi N})^\dagger G_\Delta g_{\Delta\pi N} \quad (2.2)$$

$$V_{\pi N} = (g_{\Delta\pi N})^\dagger G_\Delta^0 g_{\Delta\pi N} \quad (2.3)$$

$$G_\Delta = G_\Delta^0 + G_\Delta^0 \Sigma_\Delta G_\Delta \quad (2.4)$$

$$\Sigma_\Delta = g_{\Delta\pi N} G_{\pi N}^0 (g_{\Delta\pi N})^\dagger \quad (2.5)$$

Figure 2: The $\pi N \Delta$ vertex.

Equation 2.1 expresses $T_{\pi N}$ in the usual Lippmann-Schwinger form, but the interaction $V_{\pi N}$ is provided by intermediate coupling to the isobar (Equation 2.3). These equations are represented diagrammatically in Figure 3. The free isobar propagator has the usual form $G_\Delta(E, p) = (E - m_\Delta - p^2/2m_\Delta)^{-1}$, where m_Δ is again the bare mass; note that we have used nonrelativistic notation here for simplicity, although this will be modified shortly. An entirely equivalent expression is given in Equation 2.2, where the picture is now one of exciting the Δ through the vertex operator $g_{\Delta\pi N}$ and de-exciting it with $g_{\Delta\pi N}^\dagger$; G_Δ is the full or dressed isobar propagator obtained by including the self-energy (Equation 2.4) generated by intermediate coupling to the πN channel (Equation 2.5). The equivalence of the two expressions for $T_{\pi N}$ can be checked by direct expansion in powers of $g_{\Delta\pi N}$. The different approaches to the pion-nucleus problem, pion multiple scattering versus isobar-hole dynamics, can already be seen in Equations 2.1 and 2.2: the first approach focuses upon iteration of the pion-nucleon interaction, and the second upon evaluation of the isobar propagator. However, if the basic coupling is the same, the physics is identical.

To proceed further, we must be more quantitative in describing the coupling. One concern is a proper treatment of the spin and isospin. Recall that the interaction Hamiltonian describing the πNN vertex is written as $i(f_{\pi NN}/m_\pi)f(q^2)q \cdot \vec{\sigma} \cdot \vec{\tau}_a$ where q is the

Figure 3: πN scattering in the isobar model.

pion momentum, α is the isospin label, and $f(q^2)$ is a vertex function which cuts off the coupling for large momentum transfer. We introduce an analogous description of the $\pi N \Delta$ vertex $i(f_{\pi N \Delta}/m_\pi)h(q^2)\vec{q} \cdot \vec{S} T_\alpha$, where $\vec{S}$ and $\vec{T}$ are the spin and isospin transitions operators.¹⁰ These are defined according to

$$\langle 3/2, m_\Delta | \vec{S} | 1/2, m_N \rangle \equiv \sum_\lambda \langle 1\lambda, 1/2 m_N | 3/2 m_\Delta \rangle \hat{e}_\lambda \quad (2.6)$$

where the $\hat{e}_\lambda$ are the spherical components of the unit vectors. This definition is obviously similar to the definition of the Pauli spin matrices, except that $\vec{S}$ connects spaces of spin 1/2 and 3/2. Obviously, these operators are not self-adjoint (in matrix representation, they have dimensionality 2×4). A similar definition applies to the isospin operator. The important point is that the operator $\vec{S}_1 \vec{S}_2$ operates in the nucleon space and is nothing other than the p-wave $J = 3/2$ projection operator:

$$\vec{p} \cdot \vec{S} \vec{q} \cdot \vec{S} = \frac{2}{3} \vec{p} \cdot \vec{q} - \frac{1}{3} \vec{p} \times \vec{q} \cdot \vec{\sigma} \quad (2.7)$$

The first and second terms are the spin nonflip and spin flip terms, respectively. With the use of the vertex functions, the effective πN interaction in the center of mass frame is, for invariant energy $\sqrt{s}$,

$$V_{pq}(s) = \frac{f_{\pi N \Delta}}{m_\pi} \frac{\vec{p} \cdot \vec{S} \vec{h}(p^2) \vec{q} \cdot \vec{S} h(q^2)}{S - m_\Delta^2} \frac{f_{\pi N \Delta}}{m_\pi} \quad (2.8)$$

where $(S - m_\Delta^2)^{-1}$ is the relativistic bare propagator. With a Blankenbecler-Sugar prescription¹¹ for the relativistic generalization of the Lippmann-Schwinger equation, the transition operator is

$$T_{pq}(s) = \frac{V_{pq}(s)}{D(s)} \quad (2.9)$$

$$D(s) = G(s)^{-1}$$

$$= S - m_\Delta^2 + \frac{1}{3} \left(\frac{f_{\pi N \Delta}}{m_\pi} \right)^2 \int_0^\infty \frac{dq q^2}{(2\pi)^2} \frac{\omega_q + E_q}{\omega_q E_q} \frac{q^2 h^2(q^2)}{(\omega_q + E_q)^2 - S - i\epsilon}$$

$$\omega_q = (m_\pi^2 + q^2)^{1/2} \quad E_q = (m_N^2 + q^2)^{1/2}$$

The basic structure of the self energy is obvious, involving an integral over the $\pi N \Delta$ vertex functions and the free πN propagator. The exact form of the propagator and the additional factors involving ω_q and E_q are special to the Blankenbecler-Sugar prescription, but any choice will lead to the form

$$D(s) = S - m_\Delta^2 + \int_0^\infty dq q^2 f(q, k_s) \frac{q^2 h^2(q^2)}{q^2 - k_s^2 - i\epsilon} \quad (2.10)$$

where k_s is the center of mass on-shell momentum for energy $\sqrt{s}$ and $f(q, k_s)$ is a slowly varying function of the off-shell and on-shell momenta. It is important to note that the prescription of the specific form of the self-energy cannot be accomplished without specification of the dynamics (i.e., it is not a kinematic question). We shall return to this point below.

The next requirement is specification of the parameters in Equation (2.9). Contact with experiment comes from the fact that the transition operator, and therefore the dressed propagator, carries the phase of the πN scattering amplitude:

$$G(s) = |G(s)| \exp(i\delta_{33}(s)) \quad (2.11)$$

Below pion production threshold, $\delta_{33}(s)$ is the πN scattering phase shift in the $J = 3/2$ $T = 3/2$ channel. Using the analytic properties of $G^{-1}(s)$ in the complex s -plane, we can solve the inverse scattering problem using a modified Muskhelishvili-Omnès procedure.¹² The basic idea is that the discontinuity of $G(s)$ across the unitarity branch cut is given in terms of the vertex function $h(q^2)$. After subtracting out the asymptotic behavior of the bare propagator, one can write a dispersion relation for $\ln G^{-1}(s)$; the integrand in the dispersion integral then contains the difference

$$\ln G^{-1}(s^+) - \ln G^{-1}(s^-) = -2i \delta_{33}(s) \quad (2.11)$$

The result² is an expression for the vertex function in terms of an integral over the phase shift:

$$\left(\frac{f_{\pi N \Delta}}{m_\pi} \right)^2 h^2(k_s^2) = - \frac{24\pi\sqrt{s}}{k_s \xi(k_s)} \{s - (m_\pi + m_N)^2\} \sin \hat{\delta}_{33}(s) \quad (2.12)$$

$$\theta \exp \left\{ - \frac{\mathcal{O}}{\pi} \int_{(m_\pi + m_N)^2}^\infty ds' \frac{\hat{\delta}_{33}(s')}{s' - s} \right\}$$

where the phase $\hat{\delta}$ is determined by

$$\xi(k) f(k) = \xi(k) \frac{ne^{2i\hat{\delta}} - 1}{2ik} \equiv \frac{e^{i\hat{\delta}} \sin \hat{\delta}}{k} \quad (2.13)$$

and

$$\xi(k) \equiv \text{Im} \frac{-1}{kf(k)} \quad (2.14)$$

Note that $\xi(k)$ is one for elastic scattering, and the $\hat{\delta}$ is then the scattering phase shift. Further, the bare mass is given by¹²

$$m_\Delta^2 = S_R + \frac{1}{3} \left(\frac{f_{\pi N \Delta}}{m_\pi} \right)^2 \rho \int_0^\infty \frac{dq q^2}{(2\pi)^2} \frac{\omega_q + E_q}{\omega_q E_q} \frac{q^2 h^2(q^2) \xi(q)}{(\omega_q + E_q)^2 - S_R} \quad (2.15)$$

where $\sqrt{S_R} = 1232$ MeV (i.e., the renormalized isobar "mass"). The numerical results, calculated from the experimental phase shifts, are given in Reference 12. For our purposes, it is sufficient to note that the simple parametrization¹³ (no inelasticity)

$$\frac{f_{\pi N \Delta}}{m_\pi} h(q^2) = \frac{g}{1 + q^2/\alpha^2} \quad (2.16)$$

with $\alpha = 1.8 \text{ fm}^{-1}$, $m_\Delta = 6.83 \text{ fm}^{-1} = 1348$ MeV and $g = 25.5$ reproduces the experimental phase shifts up to 300 MeV pion kinetic energy. Note that the self-energy interactions shift the bare mass by over a hundred MeV; when inelasticity is included, the shift is even larger. As an aside, this role of the dynamical self-energy shifts raises interesting questions about the meaningfulness of static quark model calculations of hadron spectroscopy.

It is important to note that the inverse scattering procedure tells us that we can describe the scattering phase shifts exactly within the isobar model even if the fundamental interaction $V_{\pi N}$ has nothing whatsoever to do with a bare isobar. Clearly, if the πN scattering amplitude did not display resonant behavior, such a parametrization would not make much sense. We might very well expect to have a rather poor off-shell extension of the amplitude. However, given the existence of the resonance and the associated time-delay, the isobar model is simply a convenient way to describe the center of mass motion of the interacting πN system. The isobar model automatically builds in the separability of the transition operator required near any pole in the energy variable. Further, there is always the opportunity to eliminate the Δ by letting the bare mass go to infinity; i.e., in this limit, Equation (2.8) becomes

$$V_{pq}(s) = - \left[\tilde{f} \vec{p} \cdot \vec{S}^\dagger h(p^2) \right] \left[\tilde{f} \vec{q} \cdot \vec{S} h(q^2) \right] \quad (2.17)$$

$$= - \left(\frac{2}{3} \vec{p} \cdot \vec{q} - \frac{1}{3} \vec{p} \times \vec{q} \cdot \vec{\sigma} \right) \tilde{f}^2 h(p^2) h(q^2)$$

which is simply a static, rank-one, separable potential. The off-shell πN amplitude has also been described within this framework via the inverse scattering procedure.¹⁴

In closing this chapter, two comments relating to use of the isobar model in the nuclear problem are in order. The first involves use of the transition operator in reference frames other

than the center of mass frame. In non-relativistic potential scattering, there is of course no problem; one simply evaluates the transition operator using relative momenta of the πN system and the energy of relative motion (i.e., total energy minus center of mass kinetic energy). Relativistically, the situation is more complicated. This problem must be addressed since the pion is already quite relativistic at the resonance ($p/E \sim .9$), although the nucleon can be treated nonrelativistically to a good approximation. Within the isobar model, in which the pion is effectively projected out, both isobars and nucleons are essentially nonrelativistic, but the $\pi N \Delta$ vertex itself demands a relativistic treatment for consistency; this is because $m_\Delta \neq (m_\pi + m_N)$, so that Galilean invariance cannot be preserved at the vertex. However, the "frame transformation" in a relativistic Hamiltonian theory again cannot be specified independent of a dynamical theory for the many-body physics.¹⁵ The problem is not kinematic and there is no real meaning to "transforming the off-shell πN transition matrix". The origin of the problem is that, in the relativistic theory, the interaction (defined as the difference between the full and noninteracting Hamiltonian), does not commute with the generators of Lorentz boosts.¹⁵ Only within a specific theoretical framework (e.g., a Blankenbecler-Sugar treatment of the πd problem^{13,16}) is the prescription for evaluating interactions of the πN subsystem unambiguous. Such a theory prescribes the relative momenta to be used in evaluating vertex functions in various frames and the Jacobian of the transformation between πN variables and relative variables (the square root of which multiplies the transition operator in different frames).

The situation is not so bleak in practice. To lowest order in ω/m , where ω is the relativistic pion energy, the relative momentum at the πN interaction vertex both in the relativistic isobar models and in direct interaction theories¹⁷ is

$$\vec{k} = \vec{q} - \frac{\omega \vec{q}}{m + \omega} \vec{P} \quad (2.18)$$

where $\vec{q}$ is the pion momentum and $\vec{P}$ is the πN center of mass momentum. A truly relativistic treatment is impractical for anything more than the two nucleon target, so the simple prescription in Eq. 2.18 will be adopted in the work described in later chapters.

The second point involves antisymmetrization of nucleons. In the nuclear medium, the "Pauli blocking" modification of the πN transition matrix will be somewhat different in isobar models or field theories. In all models, the Δ self energy arising from the πN coupling has the form given in Equation 2.10. At this level, the only difference in Pauli effects is connected to the off-shell extension of the transition matrix; i.e., different

models will favor a somewhat different range of intermediate nucleon momenta. Such differences are not terribly great if all theories yield the same phase shifts as solutions to a Lippmann-Schwinger-like scattering equation. Field theories may have an additional effect. For example, the Chew-Low driving term has a $2\pi - N$ intermediate state, which is sensitive to Pauli effects. However, the effects incorporated in the πN self energy are far more important (the nucleons have lower intermediate momenta), so that effectively the various models are not intrinsically different.

In summarizing this chapter, we have seen how to construct a representation of the πN transition operator focusing upon the isobar degree of freedom. This is not dependent on the existence of a bare Δ . The real motivation for introducing the model is convenience. It provides a mechanism for focusing attention in the many-body problem upon various important aspects of the pion-nucleus dynamics.

III. ISOBAR-HOLE FORMALISM FOR PION SCATTERING

1. The Projection Operator Formalism

The basic idea behind the isobar-hole formalism can be understood quite compactly with projection operators. Starting with the pion plus one nucleon, let P project onto the πN space and D onto the isobar space (this is a re-writing of some results from the last chapter). In this language, the interaction term (Figure 2) is an off-diagonal channel coupling H_{PD} , and the coupled-channel Lippmann-Schwinger equations read

$$T_{PP} = H_{PD} G_D^0 T_{DP} \quad (3.1)$$

$$T_{DP} = H_{DP} + H_{DP} G_P^0 T_{PP}$$

Equations (2.2), (2.4) and (2.5) then read

$$T_{PP} = H_{PD} G_D H_{DP} \quad (3.2)$$

$$G_D^{-1} = (G_D^0)^{-1} - H_{DP} G_P^0 H_{PD}$$

Naturally, we can evaluate Equation (3.2) with any complete set of isobar wavefunctions. In the nuclear case, it will prove convenient to use eigenfunctions ψ_i in a shell-model potential:

$$\langle \vec{p}', \vec{\pi}' | T_{PP}(E) | \vec{p}, \vec{\pi} \rangle = \sum_{ij} F_i^*(\vec{p}', \vec{\pi}') G_{ij}(E) F_j(\vec{p}, \vec{\pi})$$

$$G_{ij}(E) = \langle i | G_D(E) | j \rangle = \int \frac{d\vec{p}}{(2\pi)^3} \frac{\psi_i^*(\vec{p}) \psi_j(\vec{p})}{G_D^{-1}(E, \vec{p})}$$

$$F_i(\vec{p}, \vec{\pi}) = \langle i | H_{DP} | \vec{p}, \vec{\pi} \rangle$$

$$= i \frac{\tilde{f}_{\pi N \Delta}}{m_\pi} h(\kappa^2) \psi_i^*(\vec{p}) \vec{\kappa} \cdot \vec{S} \chi_p$$

$$\vec{p} \equiv \vec{p} + \vec{\pi} \quad \vec{\kappa} \equiv \vec{\pi} - \frac{\omega}{m + \omega} \vec{p}$$

The coupling constant $\tilde{f}_{\pi N \Delta}$ contains the "frame transformation" factor described in the last chapter and χ_p is the nucleon spinor. The F_i are now vertex functions for creating the isobar in the shell model state ψ_i and G_{ij} is the isobar propagator connecting various shell model states.

Finally we turn to the nucleus. The P-space will now represent the pion plus the nucleus in the ground state, which we assume to be described by a closed shell harmonic oscillator Slater determinant. The doorway or D space includes states consisting of an isobar plus one nucleon hole. The Q-space includes all the rest, such as nuclear multihole states. In elastic scattering, we wish to calculate the projected transition matrix T_{PP} and formal solution of the Lippmann-Schwinger equation gives^{2,1,18}

$$T_{PP} = H_{PD} \{E + i\epsilon - H_{DD} - H_{DD}^\dagger - H_{DD}^\dagger\}^{-1} H_{DP}$$

$$H_{DD}^\dagger = H_{DP} \{E + i\epsilon - H_{PP}\}^{-1} H_{PD}$$

$$H_{DD}^\dagger = H_{DQ} \{E + i\epsilon - H_{QQ}\}^{-1} H_{QD}$$

This result uses the doorway state hypothesis¹⁹ $H_{PD} = 0$, meaning that the complicated Q-space is reached only via the doorway states. The term in curly brackets is the isobar-hole propagator in the nucleus. The interaction $H_{DD}^\dagger$ includes intermediate scattering through the nuclear ground state and therefore corresponds to pion propagation in the elastic channel. Iteration of T_{PP} in powers of $H_{DD}^\dagger$ generates the multiple scattering series. The term H_{DD} is just the doorway space diagonal interaction and includes effects such as isobar kinetic energy and binding. The spreading interaction $H_{DD}^\dagger$ summarizes intermediate coupling to the

Q-space. Comparing Equations (3.4) and (3.2), it is evident that the nuclear transition matrix has been expressed in terms of self-energy operators for the isobar-hole states. The evaluation will proceed by again inserting a complete set of shell model isobar states in Equation (3.4) and then diagonalizing the Δ -hole propagator.

2. Connection to Pion Multiple Scattering

In order to relate directly the isobar-hole and pion multiple scattering theories, we present a brief and rather crude review of the latter. We assume that the pion-nucleus Hamiltonian can be written as

$$H = k_\pi + V + H_N, \quad V \equiv \sum_i^A V_{\pi N}(i)$$

$$H_N = \sum_i^A k_i + \sum_{i < j}^A U_{ij}$$

where k is the kinetic energy operator and H_N is the nuclear Hamiltonian (assuming two-body forces). The pion-nucleus transition operator satisfies the Lippmann-Schwinger equation

$$T(E) = V + V G(E^+) T(E)$$

$$G(E) = (E - k - H_N)^{-1}$$

and can be solved formally to give²⁰

$$T(E) = \sum_i^A \tau_i(E) \psi_i(E)$$

$$\tau_i(E) = V_{\pi N}(i) + V_{\pi N}(i) G(E) \tau_i(E)$$

$$\psi_i(E) = 1 + G(E) \sum_{j \neq i}^A \tau_j(E) \psi_j(E)$$

Note that τ_i is not the two-body πN transition matrix, since the propagator $G(E)$ contains the full nuclear Hamiltonian. The free t-matrix is defined by

$$t_i(\epsilon) = V_{\pi N}(i) + V_{\pi N}(i) g_i(\epsilon) t_i(\epsilon)$$

$$g_i(\epsilon) = (\epsilon - k_\pi - k_i)^{-1}$$

and can be related to τ_i according to

$$\tau_i(E) = t_i(\epsilon) [1 + (g_i(\epsilon) - G(E)) t_i(\epsilon)]^{-1}$$

$$= t_i(\epsilon) - t_i(\epsilon) [g_i(\epsilon) - G(E)] t_i(\epsilon) + \dots$$

Expansion of $T(E)$ in powers of τ or of t yields the multiple scattering series. The optical potential $\mathcal{V}(E)$ is defined as that potential which describes correctly pion-nucleus scattering when used in the two-body Lippmann-Schwinger equation for the pion plus ground state nucleus:

$$T(E) \equiv \mathcal{V}(E) + \mathcal{V}(E) g_0(E) T(E) \quad (3.10)$$

$$g_0(E) \equiv \frac{P_0}{E - K_\pi - K_N}$$

where K_N is the nuclear kinetic energy operator and $P_0 \equiv |\bar{\Psi}_0\rangle\langle\bar{\Psi}_0|$ projects onto nuclear ground states. Again, we can solve formally for $\mathcal{V}$ and obtain

$$\mathcal{V}(E) = T(E) (1 + g_0(E) T(E))^{-1}$$

$$= \sum_i^A \tau_i(E) + [\sum_{i \neq j}^A \tau_i(E) G(E) \tau_j(E) - \sum_{ij}^A \tau_i(E) g_0(E) \tau_j(E)] + (\text{order } \tau^3) \quad (3.11)$$

$$= \sum_i^A \tau_i(\epsilon) + \sum_{i \neq j}^A \tau_i(\epsilon) [G(E) - g_0(E)] \tau_j(\epsilon) + \sum_i^A \tau_i(\epsilon) [G(E) - g_0(E) - g_1(\epsilon)] \tau_i(\epsilon) + (\text{order } \tau^3) \quad (3.12)$$

Of course, other expansions could be written, but the question is always one of convergence. Equation (3.11) provides a simpler expansion, but the τ -matrix involves nuclear dynamics. Equation (3.12) involves the two-body t -matrix but is more complicated in having an additional self-correlation term and in requiring choice of the parametric energy ϵ . Note that choice of this parameter requires examination of the second order optical potential and adoption of some convergence criterion. Note also that no questions of coherent approximation versus closure approximation enter in Equation (3.12) until second order.

Most multiple scattering calculations to date have been performed with the first order optical potential. With the isobar model for the elementary πN interaction, we can write^{3,1}

$$\mathcal{V}^{(1)}(\vec{p}, \vec{q}; E) = A \langle \vec{p}; 0 | T(E) | \vec{q}; 0 \rangle$$

$$= A \langle \vec{p}; 0 | g_{\Delta\pi N}^\dagger \frac{1}{D(E - H_\Delta)} g_{\Delta\pi N} | \vec{q}; 0 \rangle \quad (3.13)$$

$$H_\Delta \equiv T_\Delta + V_\Delta + H_{A-1}$$

To obtain this, we have written

$$H_N = H_{A-1} + k_1 + \sum_{j \neq 1} U_{1j}$$

$$\equiv H_{A-1} + k_1 + V_\Delta \quad (3.14)$$

where it is assumed that the i^{th} nucleon is struck by the pion. The interaction energy of the nucleon constituent of the isobar has been transformed into a binding potential V_Δ acting on the isobar. The pion and nucleon kinetic energies have been decomposed into center of mass motion, given by the isobar kinetic energy operator, and relative motion, which is implicit in the calculation of D . Clearly, the resonance denominator $D(E - H_\Delta)$ accounts for isobar propagation and binding effects; in the closure approximation, the operator H_Δ is taken as a constant (see Chapter VI). Assuming a shell model wavefunction for the target and inserting a complete set of isobar shell model states, we have (more details can be found in Reference 3):

$$\mathcal{V}^{(1)}(\vec{p}, \vec{q}; E) = \sum_{NN'} \sum_{\Delta\Delta'} F_{\Delta'N'}^*(\vec{p}) \langle \Delta'N' | D^{-1}(E - H_\Delta) | \Delta N^{-1} \rangle \otimes F_{\Delta N}(\vec{q}) \quad (3.15)$$

$$F_{\Delta N}(\vec{q}) = \langle \Delta | g_{\Delta\pi N} | \vec{q}; N \rangle$$

$$= \int \frac{d\vec{k}}{(2\pi)^3} \psi_\Delta^*(\vec{k} + \vec{q}) \frac{\tilde{F}_{\pi N \Delta}}{m_\pi} h(\kappa^2) \vec{\kappa} \cdot \vec{S} \psi_N(\vec{k}) \quad (3.16)$$

As before, $\vec{\kappa}$ is the πN relative momentum. In performing calculations, we shall linearize the resonance denominator

$$D(E - H_\Delta) \approx D(E) - \gamma(E) H_\Delta$$

$$\gamma(E) \equiv \frac{\partial}{\partial E} D(E) = 1 - \frac{\partial}{\partial E} (E_R(E) - \frac{1}{2} \Gamma(E)) \quad (3.17)$$

The last form is obtained by writing the resonance denominator explicitly in Breit-Wigner form; this is not an approximation since

the proper energy dependence is retained.

The next step is to calculate the nuclear elastic scattering amplitude using Equations (3.15) and (3.10). The formal summation obviously yields

$$T(\vec{k}', \vec{k}; E) = \sum_{NN'} \sum_{\Delta\Delta'} F_{\Delta'N'}^*(\vec{k}') \quad (3.18)$$

$$\Theta \langle \Delta'N'^{-1} | [D(E) - \gamma(E)H_{\Delta} - \mathcal{W}]^{-1} | \Delta N^{-1} \rangle F_{\Delta N}(\vec{k})$$

$$\begin{aligned} \langle \Delta'N'^{-1} | \mathcal{W} | \Delta N^{-1} \rangle &= \langle \Delta'N'^{-1} | g_{\Delta\pi N} g_o(E) g_{\Delta\pi N}^{\dagger} | \Delta N^{-1} \rangle \\ &= \int \frac{d\vec{q}}{(2\pi)^3} \frac{F_{\Delta'N'}^*(\vec{q}) F_{\Delta N}(\vec{q})}{(E - q^2/2Am_N)^2 - \omega_q^2 + i\epsilon} \end{aligned} \quad (3.19)$$

Clearly, $\mathcal{W}$ plays the role of a pion "exchange" interaction between isobar-hole states and enters as an isobar-hole self energy. Comparing Equations (3.18) and (3.4), we see that we have exactly the same form. The diagonal interaction is generated by H_{Δ} while the elastic channel interaction $H_{DD}^{\dagger}$ is given by $\mathcal{W}$. However, the first order optical potential corresponds to a complete absence of the spreading interaction $H_{DD}^{\dagger}$. This is expected since the latter arises from coupling to more complicated states. We shall return to this point later.

3. Isobar-Hole Calculations

The calculations are carried out by diagonalizing the isobar-hole Hamiltonian in the shell model basis. As is usual in similar Tamm-Dancoff calculations of nuclear excited states, the basis is reduced to subspaces of fixed angular momentum. Thereby, the pion-nucleus partial wave with orbital angular momentum L is described by coupling to Δ -hole states with the same angular momentum. Since the ^{16}O ground state has isospin zero, the relevant Δ -hole states all have $T=1$. The reduction is accomplished using standard shell model techniques; for example, the pion exchange term $\mathcal{W}$ has the Δ -hole matrix elements

$$\begin{aligned} \langle \Delta' h'^{-1}; J T | \mathcal{W} | \Delta h^{-1}; J T \rangle &= - \sum_{J' T'} \frac{1}{(2J' + 1)(2T' + 1)} \begin{Bmatrix} J_{\Delta'} & J_h & J \\ J_{\Delta} & J_h & J' \end{Bmatrix} \begin{Bmatrix} 3/2 & 1/2 & 1 \\ 3/2 & 1/2 & T' \end{Bmatrix} \\ \Theta \langle \Delta' h'; J' T' | \mathcal{W} | \Delta h'; J T \rangle & \quad (3.20) \end{aligned}$$

$$\begin{aligned} \langle \Delta' h' | \mathcal{W} | \Delta h' \rangle &= \int \frac{d\vec{k} d\vec{k}' d\vec{q}}{(2\pi)^9} h(\vec{k}')^2 h(\vec{k})^2 \left(\frac{\vec{q}}{m_{\pi}} \right)^2 \\ &\Theta \frac{[\psi_{\Delta'}^*(\vec{k}') \vec{\sigma} \cdot \vec{S}^{\dagger} \phi_h(\vec{k}' - \vec{k} + \vec{p})] [\phi_h^*(\vec{p}) \vec{\sigma} \cdot \vec{S} \psi_{\Delta}(\vec{k})]}{k_o^2 - (k - p)^2 + i\epsilon} \end{aligned} \quad (3.21)$$

where k_o is the external pion wavenumber. In Equation (3.21), the ψ and ϕ are isobar and nuclear shell model wavefunctions, respectively, and must still be coupled to J' , T' with Clebsch-Gordan coefficients. The expansion in Δ wavefunctions is, of course, truncated; typically 8 to 10 $\hbar\omega$ is sufficient for the energies of interest. Diagonalization produces a set of Δ -hole eigenstates $|D_1^L\rangle$ and complex eigenvalues ϵ_1^L for each angular momentum:

$$T_L(E) = g_{\Delta\pi N}^{\dagger} \sum_i \frac{|D_1^L\rangle \langle D_1^L|}{E - \epsilon_1^L + i\Gamma/2 - \epsilon_1^L(E)} g_{\Delta\pi N} \quad (3.22)$$

The dimensionality of the space is determined by the truncation of the isobar basis set and the $\langle D_1^L |$ are the biorthogonal wavefunctions which must be introduced in the treatment of non-Hermitian Hamiltonians. The eigenstates are represented as a sum over the Δ -hole basis states

$$|D_1^L\rangle = \sum_j C_{1j} |(\Delta h) j\rangle \quad (3.23)$$

Equation (3.22) constitutes the isobar-hole representation of the pion-nucleus transition operator.

Before applying the formalism, at least two improvements in the physical input are needed. First, Pauli blocking of the isobar decay should be incorporated. That is, the free-space width of the Δ is generated by decay into a pion and nucleon, given by the imaginary part of the Δ self-energy in Equation 2.9. In the nuclear medium, this width will be quenched since some of the "decays" will lead to already filled nucleon orbitals. Consequently, we add a term $\delta\mathcal{W}$ to the isobar-hole Hamiltonian

$$\begin{aligned} \langle \Delta' N'^{-1} | \delta\mathcal{W} | \Delta N^{-1} \rangle &= - \delta_{NN'} \sum_{occ} \int \frac{d\vec{k}}{(2\pi)^3} \\ &\Theta \frac{F_{\Delta'm}^*(\vec{k}) F_{\Delta m}(\vec{k})}{(E - \epsilon_{Nm} - k^2/2Am_N) - \omega_k^2 + i\epsilon} \end{aligned} \quad (3.24)$$

where $\epsilon_{NM} = \epsilon_M - \epsilon_N$. This term is diagonal in the hole and therefore is simply a Δ self-energy term. There is an obvious similarity to the pion exchange term $\mathcal{W}$. This term can be expected to be quite important at the lower energies since the Pauli blocking effect in pion quasifree scattering is substantial for small momentum transfers.¹

Another important ingredient is the spreading interaction H_{DD} . As remarked earlier, this is not present in any treatment based upon the first order optical potential. Nevertheless, the very large cross section observed for pion absorption in the resonance region points to the importance of this damping mechanism for the isobar-hole states. The states reached by absorption are clearly in the Q-space, involving for example a nuclear $2p - 2h$ state with no pion present. Unfortunately the various Q-space couplings are too complicated to evaluate quantitatively at this time, so we resort to a phenomenology. We are guided by the expectation that absorption will be the dominant spreading interaction (at least for energies at and below the resonance) and by the fact that this process goes through the Δ . Consequently, we introduce a phenomenological isobar self-energy given by a Δ spreading potential which is local and proportional to the nuclear ground state density:

$$\mathcal{V}_{sp}(r) = \mathcal{V}_0 \rho(r)/\rho(0) \quad (3.25)$$

The complex strength of the potential $\mathcal{V}_0$ will be taken as a free parameter fit to the forward elastic amplitude. It is important to make clear the nature of the approximation made here. First, we have assumed that the dominant Q-space effects can be incorporated as isobar self-energies, diagonal in the hole label. In principle, vertex corrections arising from Δ -h potential interactions or from multihole states should be included. We shall test this assumption to some extent later on by calculating total absorption and inclusive (π, π') cross sections from the spreading width. Second, even within this framework we have assumed a local spreading potential. Clearly, the isobar self-energy interactions should be nonlocal. Nevertheless, we do not expect to be very sensitive to this approximation since only inclusive quantities are being evaluated and the pion wavelength is still quite long at intermediate energies. The danger lies less in describing elastic scattering than in possibly extracting unphysical parameters because of a poor phenomenology. We shall test this assumption in Chapter VI when comparing the detailed calculations with data on $\pi - {}^{16}\text{O}$ scattering.

It is worth collecting the various parts of the isobar dynamics now included. The Δ -hole propagator which is to be diagonalized, Equation 3.18, now reads

$$\begin{aligned} G_{\Delta h}^{-1} &= D(E) - \mathcal{H}_{\Delta} \\ &= D(E) - [\gamma(E)H_{\Delta} + \mathcal{W} + \delta\mathcal{W} + \mathcal{V}_{sp}] \end{aligned} \quad (3.26)$$

This reflects not only pion multiple scattering, as would be included in a standard first order optical potential calculation, but also isobar propagation and binding, Pauli blocking, and coupling to more complicated states (e.g., absorption) through the phenomenological spreading potential. Additional effects such as a ρ -meson effective isobar-hole force could also be included quite easily. The eigenvalues reflect directly the importance of the various dynamical effects.

Before applying the model, we summarize the approximations which have been introduced:

- (i) shell model for the nuclear ground state wavefunction
- (ii) retention of only the resonance part of the πN interaction
- (iii) neglect of Coulomb effects
- (iv) retention of H_{Δ} to first order only
- (v) truncation of the isobar shell model space
- (vi) neglect of vertex corrections
- (vii) local density approximation for the spreading interaction.

Approximations (ii) and (iii) will not be made in Chapters VI and VII, where quantitative comparison with data will be made. Approximations (iv) and (v) are amenable to quantitative evaluation and will be quite suitable for the applications discussed. We examine in Chapter VI the effects of relaxing approximation (vii) with a reasonable extension to nonlocal form. However, approximations (i) and (vi) are far more difficult to evaluate meaningfully without a significant increase in calculational complexity. The shell model approximation is probably quite good for our calculations of elastic scattering, although some uncertainty in the interpretation of the strength of the spreading potential may arise from the neglect of short range correlations. The neglect of vertex corrections may become significant at the higher energies, where multinucleon knockout by pions is known to increase. This will be discussed in Chapters VI and VII.

We now show some results⁴ for $\pi - {}^{16}\text{O}$ at 140 MeV (close to the peak of the total cross section). The calculations discussed here do not include Coulomb or nonresonant πN interaction, so comparison with data is deferred to Chapter VI. In Table 1, we give the contribution of each 0^- eigenstate to the partial wave transition matrix (see Equation 3.22) and the associated eigenvalues. The basis was 17 dimensional with truncation at 8 $M\omega$ (the dimensionalities are considerably greater for higher L).

1	ϵ_i [MeV]		t_i	
1	-38.3	-1175.2	-.124	1.675
2	+28.7	- 143.1	.043	-1.066
3	+55.3	- 15.9	.034	-1.027
4	+41.8	+ 12.2	.025	-1.016
5	+68.3	+ 14.8	-.005	-1.008
6	+11.6	- 124.9	.004	-1.007
7	+23.3	+ 18.9	.001	1.003
8	+46.2	- 135.9	-.001	-1.003
9	+46.0	+ 111.0	-.004	1.002
10	+61.3	+ 123.2	.002	-1.001
11	+28.1	+ 18.3	.002	1.001
12	+51.1	- 136.1	.000	1.001
13	+45.6	+ 119.3	.001	1.001
14	+46.5	+ 114.6	.000	-1.001
15	+39.4	+ 110.3	-.000	1.000
16	+53.0	+ 113.3	-.000	-1.000
17	+28.3	+ 111.3	.001	-1.000

Table 1: Eigenvalues ϵ_i of the Δ -hole eigenstates and contributions to the nuclear amplitude $T = \sum_i t_i = \frac{1}{2}(1 - ne^{2i\delta})$ for the 0^- partial wave at $T_\pi = 140$ MeV. With the truncation described in the text, the 0^- Δ -hole space is 17-dimensional.

The interesting result here is the strong "collectivity" of one eigenstate⁴; i.e., one eigenstate provides by far the largest contribution to the transition matrix and has an eigenvalue significantly split off from the rest of the spectrum. Note that this splitting occurs principally in the imaginary part of the eigenvalue. Indeed, $\text{Im}(\sum_i \epsilon_i) = -193.9$ MeV, so that $\text{Im} \epsilon_1 \approx 0.9 \text{ Im}(\text{Tr} \mathcal{H}_\Delta)$. This collectivity is expected through analogy with the nuclear Brown-Bolsterli particle-hole schematic model; this will be discussed in detail in the next chapter. In any case, it is clear from Equation 3.22 that the various eigenstates contribute to the elastic transition matrix only to the extent that they have reasonable overlap with the (normalized) state

$$|\Delta_o^L\rangle \equiv \frac{g_{\Delta\pi N}|0;(\vec{k})_L\rangle}{[\langle(\vec{k})_L|g_{\Delta\pi N}^\dagger g_{\Delta\pi N}|0;(\vec{k})_L\rangle]^{1/2}} \quad (3.27)$$

This state is found to have an overlap of 0.9 with the leading eigenstate and represents the linear response of the nuclear ground state to the incoming pion. It is a coherent superposition of Δ -hole states. For example, the coefficients of the Δ -hole basis states in the doorway state $|\Delta_o\rangle$ are given in Table 2.⁴ Note that all the allowed isobar angular momentum states have appreciable contributions in the doorway, with the s-shell isobar having the largest weight. Convergence comes about in the expansion in the Δ principal quantum number. We note that the same oscillator parameter was used for the isobar and nucleon wavefunctions, and it is conceivable that convergence could be aided by using a different parameter or basis set for the isobar.

Some insight into the dynamics can be obtained by examining the decomposition of the expectation value of $\mathcal{H}_\Delta$ in the doorway state:

$$\begin{aligned} \epsilon_A &\equiv \langle\Delta_o|\mathcal{H}_\Delta|\Delta_o\rangle \\ &= \langle\Delta_o|\mathcal{W}|\Delta_o\rangle + \langle\Delta_o|\delta\mathcal{W}|\Delta_o\rangle + \gamma(E)\langle\Delta_o|\mathcal{H}_\Delta|\Delta_o\rangle \\ &\quad + \langle\Delta_o|\mathcal{V}_{sp}|\Delta_o\rangle \end{aligned} \quad (3.28)$$

In these calculations, we have used a spreading potential of central strength $\mathcal{V}_0 \approx -70i$ MeV. We show the different contributions⁴ to $\text{Im} \epsilon_A$ in Figure 4, together with the value for the leading collective 0^- eigenstate. The dominant width $\text{Im} \epsilon_\pi$ comes from the pion rescattering term $\mathcal{W}$. This is basically the origin of the collectivity: $\mathcal{H}_{pp}$ enters in Equation 3.4 both in generating the doorway state and, in second order, the elastic channel interaction $\mathcal{W} = \mathcal{H}_{DD}^\dagger$. However, the other aspects of the isobar dynamics are far from negligible. The spreading interaction width $\text{Im} \epsilon_A$ is larger than the free width, providing an important damping mechanism for the isobar-hole states. Further, the Pauli term $\text{Im} \epsilon_p$ quenches the free width substantially. These terms will be seen in Chapter VII to have important consequences for the inelastic reactions.

4. Optical Potential Games

Before developing further and applying the isobar-hole formalism, it is instructive to translate crudely the isobar-hole doorway state parameters back into an optical potential language.

Figure 4: Decomposition of the imaginary part of the doorway expectation value of the isobar-hole Hamiltonian. The eigenvalue of the leading eigenstate is denoted by $\hat{\epsilon}$, and Γ is the free-space isobar width.

i	Δ			hole			c_i
	n	l	j	n	l	j	
1	0	1	1/2	0	0	1/2	.413
2	1	1	1/2	0	0	1/2	.241
3	2	1	1/2	0	0	1/2	.082
4	3	1	1/2	0	0	1/2	.019
5	0	2	1/2	0	1	1/2	.337
6	1	2	1/2	0	1	1/2	.273
7	2	2	1/2	0	1	1/2	.114
8	3	2	1/2	0	1	1/2	.031
9	0	0	3/2	0	1	3/2	-.578
10	1	0	3/2	0	1	3/2	-.118
11	2	0	3/2	0	1	3/2	.084
12	3	0	3/2	0	1	3/2	.060
13	4	0	3/2	0	1	3/2	.019
14	0	2	3/2	0	1	3/2	.337
15	1	2	3/2	0	1	3/2	.273
16	2	2	3/2	0	1	3/2	.114
17	3	2	3/2	0	1	3/2	.031

Table 2: Decomposition of the doorway state $|\Delta_0\rangle$ into Δ -hole shell model basis states for the 0^- partial wave at 140 MeV.

This also gives a semiquantitative preview of the results to be obtained later. We anticipate the results of Chapter VI to the extent that the collectivity displayed above persists in the more refined calculations and that a good description of the data is obtained. Then, since the Pauli and spreading interactions give isobar (as opposed to isobar-hole) self-energies, and since the doorway state itself already gives a reasonable approximation to the transition matrix, we carry these self-energies over into the resonance denominator in the first order pion optical potential. More specifically, we shall be extremely crude and use the high energy approximation²¹ for the nuclear amplitude

$$\begin{aligned}
 F(q) &= \frac{ik}{2\pi} \int d^2b e^{iq \cdot \vec{b}} \Gamma(b) \\
 \Gamma(b) &= 1 - e^{i\chi(b)} \\
 \chi(b) &= \frac{-1}{2k} \int_{-\infty}^{+\infty} dz v(\vec{b} + \hat{k}z)
 \end{aligned}
 \tag{3.29}$$

$$V(r) = -4\pi\rho(r)f_{\pi N}(0)$$

where $\chi(b)$ is the semiclassical phase shift for impact parameter b and $v(r)$ is the Glauber first order optical potential. Assuming resonance dominance of the πN amplitude, we have

$$f_{\pi N}(0) = -\frac{k\sigma_T^R/8\pi}{E-E_R+i\Gamma/2} \tag{3.30}$$

where $\sigma_T^R = 140$ mb is the average of the πp and πn total cross sections at resonance. Motivated by the isobar-hole results, we modify the resonance denominator by the addition of density-dependent self-energies:

$$\begin{aligned}
 f_{\pi N}(0) &= \frac{-k\sigma_T^R/8\pi}{E-\tilde{E}_R+i\Gamma/2+\frac{i}{2}[\Gamma_{\text{Abs}}\tilde{\rho}(r)+\Gamma_{\text{Pauli}}\tilde{\rho}(r)^{1/3}]} \\
 \tilde{\rho}(r) &\equiv \rho(r)/\rho(0)
 \end{aligned}
 \tag{3.31}$$

In the resonance region, the widths are most important and we put $\tilde{E}_R = E_R$. The parameters Γ_{Abs} and Γ_{Pauli} are fixed according to the s-wave doorway state expectation values of the spreading and Pauli interactions, respectively (see Fig. 4):

$$\Gamma_{\text{Abs}}/2 \approx 70 \text{ MeV}; \quad \Gamma_{\text{Pauli}}/2 \approx 20 \text{ MeV} \tag{3.32}$$

Of course, the elastic width $\text{Im } \epsilon_\pi$ does not enter since we work here with the optical potential rather than the transition matrix (see Equations 3.15 and 3.18). We have assigned the same local density dependence to the absorption width as was used in the

spreading potential and take the Pauli width proportional to the local Fermi momentum. The results for the total elastic and reaction cross sections for $\pi - 160$ are shown in Figure 5, where it is seen that the main effect is a suppression of the elastic cross section at the lower energies. An excessively large elastic cross section is characteristic of first order optical potential calculations. Furthermore, the reaction cross section can be written as directly proportional to the imaginary part of the optical potential. We interpret $(\Gamma + \Gamma_{\text{Pauli}})$ and Γ_{Abs} as partial widths into quasifree and absorption channels, respectively, and thereby are able to decompose the reaction cross section into these two channels. This is also shown in Figure 5. The absorption cross section is seen to dominate below about 120 MeV, in good agreement with the limited data available. These results are encouraging for thinking that the model incorporates the important physics and will emerge also in the more quantitative results given below.

In calculating the absorption and inelastic (i.e., quasifree) cross section, it is interesting to observe the impact parameter dependence of the associated profile functions $\Gamma(b)$. As can be seen from Equations (3.29) and (3.31), the quasifree profile function is suppressed strongly in the nuclear interior. The reaction cross section is changed only little because the absorption profile function is large for small impact parameter. Consequently the rough picture which emerges is of comparatively strong central absorption and peripheral quasifree knockout.

One last exercise is to calculate the elastic differential cross section given by Equation 3.39. In particular, analysis of the differential cross section up to the first minimum has been proposed as a good means for extracting the matter radius of nuclei. We show in Figure 6 the first order Glauber results for two different values of the 160 harmonic oscillator parameter. As expected, the larger radius leads to a narrower forward peak in the differential cross section. On the other hand, inclusion of the Δ self-energy modifications also changes the shape of the cross section. The lesson appears to be that the "higher order effects" in the optical potential need very careful treatment before a meaningful extraction of neutron radii at the level of say 0.1 fm can be made with pions in the resonance region. This holds also for π^+/π^- comparisons on $N \neq Z$ nuclei, since the absorption term has strong isospin dependence.

Figure 5: Elastic, reaction and total cross sections for $\pi^{-16}\text{O}$ scattering, calculated in the eikonal approximation. Solid curves result from using the free πN amplitude, Equation 3.30; dot-dash curves result from including absorption and Pauli widths, Equation 3.31. Dashed curves represent decomposition of reaction cross section into total absorption and total inelastic pion scattering cross section.

Figure 6: Elastic differential cross section in the eikonal approximation. Solid and dashed curves give results for harmonic oscillator parameter $c = 1.81 \text{ fm}$ and for πN amplitude with and without absorption and Pauli widths, respectively. Dot-dashed line is first order result with a larger oscillator parameter.

IV. EXTENDED SCHEMATIC MODEL

We remind the reader of the important result obtained in the isobar-hole framework in the last chapter: the isobar-hole "spectrum" is collective in the sense that one or two eigenstates, described as coherent superpositions of Δ -hole shell model basis states, dominate the nuclear partial wave transition matrix. This specifies that the leading eigenstate is very similar to the doorway state. This state is split off from the rest of the Δ -hole spectrum in the imaginary part of the eigenvalue. These features led to reformulation of the isobar-hole formalism in Reference 4. The approach, called the extended schematic model, uses the results summarized above to achieve rapid convergence without diagonalizing in the full isobar-hole space.

1. The Brown-Bolsterli Model

It is useful to begin with a brief review of the Brown-Bolsterli degenerate schematic model²² for nuclear particle-hole states. We might imagine as an application the calculation of collective states contributing to a low energy photoreaction, as indicated in Figure 7a. Absorption of the photon leads to a doorway state described as a linear superposition of the particle-hole basis states:

$$\begin{aligned} \langle (\text{ph})i | H_Y | 0 \rangle &\equiv Y_i \\ \bar{\Psi}_D &= N \sum_i Y_i \phi_i = N \vec{Y} \cdot \vec{\phi} \end{aligned} \quad (4.1)$$

where $\phi_i = |(\text{ph})i\rangle$ represents the i^{th} p-h basis state and $\bar{\Psi}_D$ is the doorway state. We also need the nuclear eigenstates and, in the degenerate schematic model, the interaction in the p-h basis is assumed to have the form

$$V_{ij} = g D_i D_j \quad (4.2)$$

This structure emerges naturally from p-h force of the type depicted in Fig. 7b, where the dashed line represents the interaction (e.g., meson exchange). The D_i play the role of vertex functions and g incorporates coupling constants and the propagator of the (static) exchanged particle. With the further assumption of degeneracy of the unperturbed p-h states ϕ_i , the Hamiltonian is diagonalized easily. There is one collective state

$$\begin{aligned} \bar{\Psi}_0 &= N \sum_i D_i \phi_i = N \vec{D} \cdot \vec{\phi} \\ \epsilon_0 &= g \sum_i D_i^2 = \text{Trace } (V) \end{aligned} \quad (4.3)$$

Figure 7: (a) Photoexcitation of a particle-hole state. (b) Exchange diagram for the particle-hole interaction.

The remaining eigenstates are orthogonal to $\bar{\Psi}_0$ and remain degenerate ($\epsilon_i = 0, i > 0$). Clearly, if the Y_i are proportional to the D_i , the collective state $\bar{\Psi}_0$ becomes identical to the doorway state and concentrates all the photoexcitation strength. This state has been shifted "maximally" from its unperturbed position. This simple model was used to show simply the origin of giant resonance states in a shell model picture.

One might imagine that further residual p-h interactions will weaken the collectivity. The simplest example of this is obtained by adding an additional "separable" interaction

$$V_{ij} = g D_i D_j + h E_i E_j \quad (4.4)$$

Note that the spectrum in an N-dimensional basis can always be reproduced trivially by going to a rank N separable Hamiltonian, so that this procedure is very analogous to the construction of low rank separable potentials to reproduce matrix elements of more complicated forces. In the degenerate limit, this problem is also solved easily²³; the eigenstates and eigenvalues are

$$\begin{aligned}
\bar{\Psi}_\alpha &= \bar{C}^\alpha \cdot \bar{\Phi} \\
\bar{C}^\alpha &= \begin{cases} N_\alpha [g \bar{D} \bar{D} \cdot \bar{E} - (\epsilon_\alpha - g \bar{D}^2) \bar{E}], & \alpha = 1, 2 \\ N_\alpha \bar{D} \times \bar{E}, & \alpha \geq 3 \end{cases} \\
\epsilon_\alpha &= \begin{cases} \frac{1}{2} [g \bar{D}^2 + h \bar{E}^2 \pm \{(g \bar{D}^2 - h \bar{E}^2)^2 + 4gh(\bar{D} \cdot \bar{E})^2\}^{1/2}], & \alpha = 1, 2 \\ 0, & \alpha \geq 3 \end{cases}
\end{aligned} \quad (4.5)$$

The residual interaction has now fragmented the strength into two states. Still assuming that $\bar{\gamma} \propto \bar{D}$, the doorway state is no longer an eigenstate but has overlap with $\bar{\Psi}_{1,2}$; therefore, each of these will contribute to the photoreaction. Note that the new state mixed in "points" in the direction $\bar{E}$ in the particle-hole Hilbert space. This is to be expected since operation of the Hamiltonian on the doorway state and orthogonalization leads to a new state

$$\begin{aligned}
\bar{\Psi}_D' &\equiv V \bar{\Psi}_D - \bar{\Psi}_D \langle \bar{\Psi}_D, V \bar{\Psi}_D \rangle \\
&\sim \bar{E}
\end{aligned} \quad (4.6)$$

In other words, diagonalization of the Hamiltonian leads to a rotation in the $\bar{D}, \bar{E}$ plane of the Hilbert space.

How does this apply to pion scattering? The analogy between pion scattering and the Brown-Bolsterli model is clear from a comparison of Figures 7 and 8. The isobar-hole doorway state is created by absorption of an on-shell pion. The degeneracy assumption corresponds to the closure limit ($H_\Delta \rightarrow \text{constant}$) and the residual interaction corresponds to nonstatic pion "exchange" between Δ -hole states. For on-shell pion exchange (i.e., on-shell pion multiple scattering), the Δ -hole interaction has the same relation to the doorway excitation vertex as does Equation (4.2) to Equation (4.1). Note however that the on-shell condition implies that the interaction is imaginary, so that the eigenvalue splitting is in the imaginary part. In this limit, the doorway state has all the pion excitation strength in pion-nucleus scattering. The residual interaction which fragments the strength then includes the off-shell pion rescattering as well as binding effects, the Pauli blocking, and the spreading interaction. The collectivity will remain if the imaginary part of the pion exchange interaction $\mathcal{W}$ contributes the largest self-energy in the isobar-hole propagator. This is the case for the central par-

Figure 8: (a) Pion excitation of a Δ -hole state. (b) Pion "exchange" between Δ -hole states.

tial waves over the entire energy range of interest, as seen in Figure 4.

2. The Extended Schematic Model for Pion Scattering

To have an effective theory built explicitly upon the collectivity in the isobar-hole approach, one needs a systematic approach for evaluating corrections at any level of approximation. We accomplish this by introducing a change of basis in the Δ -hole space.⁴ Clearly, the doorway state plays a special role in the theory and we use this as a "starting vector". The construction of the rest of the basis is motivated by Equation 4.6 of the degenerate p-h model: the "next" contributing state is obtained by applying the Hamiltonian to the doorway state and orthogonalizing. We continue this procedure (in principle) until the entire Δ -hole space is spanned:

$$\begin{aligned}
|\Delta_\alpha^L\rangle &= N_\alpha \{ \mathcal{H}_\Delta |\Delta_{\alpha-1}^L\rangle - \sum_{\beta=0}^{\alpha-1} |\Delta_\beta^L\rangle \mathcal{H}_{\beta, \alpha-1} \}, \quad \alpha > 0 \\
\mathcal{H}_{\alpha, \beta} &\equiv \langle \Delta_\alpha^L | \mathcal{H}_\Delta | \Delta_\beta^L \rangle.
\end{aligned} \quad (4.7)$$

Note that this procedure will span only that part of the Hilbert space which has some overlap with the doorway state, but this is

anyway the only part which can contribute to the nuclear transition matrix. This construction has the tridiagonal property

$$\mathcal{H}_{\alpha\beta} = 0 \quad \text{for } |\alpha - \beta| > 1 \quad (4.8)$$

and generates a continued fraction representation of the doorway matrix element of the Δ -hole propagator:

$$\langle \Delta_0^L | G_{\Delta h} | \Delta_0 \rangle = \langle \Delta_0 | (E - E_R + i\Gamma/2 - \mathcal{H}_\Delta)^{-1} | \Delta_0 \rangle$$

$$= \frac{1}{E - E_R + \frac{1}{2}\Gamma - \mathcal{H}_{00} - \mathcal{H}_{01}^2 \frac{1}{E - E_R + \frac{1}{2}\Gamma - \mathcal{H}_{11} - \mathcal{H}_{12}^2 \frac{1}{E - E_R + \frac{1}{2}\Gamma - \mathcal{H}_{22} - \dots}}}$$

(4.9)

The procedure described by Equations 4.7 through 4.9 is familiar in many branches of physics, variously passing under the names method of minimized iterations, Lancsöz method, coupled quasi-particle method, the method of Padé approximants. It has been applied in both nuclear structure and nuclear reaction theory. Extensive shell model calculations²⁴ have used the construction of Equation 4.7, starting from an ansatz for $|\Delta_0\rangle$, to calculate low-lying collective states. The continued fraction representation Equation (4.9) is familiar from reaction theory¹⁹, reflecting sequential coupling to states of higher and higher complexity. We shall refer to the states $|\Delta_0\rangle$ as the doorway basis.

The point of introducing this basis is the expectation that the continued fraction will converge very rapidly; i.e., we may expect that construction of only the first few doorway states will be sufficient for high accuracy. This can be checked by systematically extending the basis. Further, within a doorway basis of dimensionality N , the eigenvalue problem can be seen from Equation (4.9) to consist of solving for the roots of an N th order polynomial in $(E - E_R + i\Gamma/2)$. Also, it is worth noting that the first off-diagonal matrix element corresponds directly to the fluctuation of the Hamiltonian in the doorway state

$$\mathcal{H}_{01} = [\mathcal{H}^2_{00} - \mathcal{H}_{00}^2]^{1/2} \quad (4.10)$$

Obviously this reflects the extent to which the doorway state is an eigenstate.

A test of the convergence is shown in Table 3. The exact results refer to the transition matrix obtained by diagonalizing in

	$\epsilon(J^\pi)$ [MeV]		$\tau(J^\pi)$	
	0^-	4^-	0^-	4^-
Exact	-38.4 - 1175.2	18.2 - 128.0 46.1 - 122.9	-.021 + 1.553	.108 + 1.189
Schematic N=1	-25.8 - 1185.2	20.85 - 115.90	-.022 + 1.640	.114 + 1.198
N=2	-38.2 - 1175.4	11.6 - 129.2 41.9 - 110.9	-.019 + 1.549	.108 + 1.188
N=3	-38.4 - 1175.2	17.3 - 127.1 49.3 - 127.2	-.021 + 1.553	.108 + 1.188

Table 3: Eigenvalues of the dominant eigenstates and partial wave transition matrix for $J^\pi = 0^-, 4^-$ at 140 MeV. "Exact" refers to the results of the full isobar-hole calculations; "schematic" refers to the extended schematic model calculation using $N = 1, 2$ and 3 doorway basis states. Taken from Reference 4.

the full Δ -hole shell model basis and to the eigenvalues of the leading eigenstates. For the $L^{\pi} = 4^{-}$ partial wave, two eigenstates have substantial contributions. The schematic results refer to calculations done in the doorway basis truncated at dimensionality N . Clearly, the transition matrix for both the central and the peripheral partial wave is essentially exact in the two-dimensional doorway basis. The eigenvalues converge slightly more slowly, but even here it is clear that the "eigenstates" obtained by diagonalizing the first few doorway states are very similar to the "true" eigenstates calculated in the full Δ -hole space. The extended schematic model simplifies the calculations significantly and facilitates insight into the reaction dynamics. This is the calculational approach used for obtaining the results in Chapters VI and VII.

V. A MODEL PROBLEM

We conclude this formal part of the lectures by examining a simple model problem in the isobar-hole framework. This model has been worked out in collaboration with F. Lenz and K. Yazaki¹⁷ and affords an opportunity to see explicitly the connection between multiple scattering and isobar dynamics. The model consists of the simplest, non-trivial physical situation: pion scattering from an uncorrelated nucleus ($A \gg 1$) in the fixed nucleon approximation, with zero-range s -wave πN interactions. The nuclear multiple scattering series for the elastic amplitude is then

$$F(\vec{k}', \vec{k}) = A \int d\vec{x}_1 \rho(\vec{x}_1) e^{i(\vec{k}' - \vec{k}) \cdot \vec{x}_1} f(\vec{k}', \vec{k}) \\ + A^2 \int d\vec{x}_1 d\vec{x}_2 \rho(\vec{x}_1) \rho(\vec{x}_2) \int \frac{d\vec{p}}{(2\pi)^3} e^{i(\vec{k}' - \vec{p}) \cdot \vec{x}_1} f(\vec{k}', \vec{p}) \frac{4\pi}{p^2 - k^2 - i\epsilon} \\ \otimes e^{i(\vec{p} - \vec{k}) \cdot \vec{x}_2} f(\vec{p}, \vec{k}) + \dots \quad (5.1)$$

Here, $f(\vec{p}, \vec{q})$ is the off-shell πN scattering amplitude and we assume

$$f(\vec{p}, \vec{q}; E) = - \frac{g^2/4\pi}{E - E_R + i\Gamma(E)/2} \quad (5.2)$$

With such a simple assumption, the equivalent optical potential would be local and proportional to the ground state density.

In performing isobar-hole calculations, we are effectively evaluating the Δ -hole spectrum for various values of the total angular momentum. This leads us to consider the partial wave decomposition of the nuclear amplitude:

$$F(\vec{k}', \vec{k}) = \sum_{\ell m} 4\pi Y_{\ell m}^*(\hat{k}') Y_{\ell m}(\hat{k}) F_{\ell}(k) \\ F_{\ell}(k) = \int_0^{\infty} dx_1 x_1^2 \frac{[g \phi(x_1) j_{\ell}(k x_1)]^2}{E - E_R + i\Gamma/2} \\ + \int_0^{\infty} dx_1 x_1^2 \int_0^{\infty} dx_2 x_2^2 \frac{[g \phi(x_1) j_{\ell}(k x_1)] \mathcal{W}_{\ell}(x_1, x_2) [g \phi(x_2) j_{\ell}(k x_2)]}{(E - E_R + i\Gamma/2)^2} \\ + \dots \quad (5.3)$$

$$\mathcal{W}_{\ell}(x_1, x_2) \equiv 4\pi \int \frac{d\vec{p}}{(2\pi)^3} \frac{j_{\ell}(p x_1) j_{\ell}(p x_2)}{p^2 - k^2 - i\epsilon} g^2 \phi(x_1) \phi(x_2) \\ = ik [g \phi(x_1)] h_{\ell}^{(1)}(k x_1) j_{\ell}(k x_2) [g \phi(x_2)] \quad (5.4)$$

We have defined $\Delta p(x) \equiv \phi^2(x)$, so that $\phi(x)$ plays the role of a nucleon orbital wavefunction in this model. We can now sum the entire multiple scattering series to obtain

$$F_{\ell}(k) = - \int_0^{\infty} dx_1 x_1^2 \int_0^{\infty} dx_2 x_2^2 \psi_0^{\ell}(x_1) \Delta_{\ell}(x_1, x_2) \psi_0^{\ell}(x_2) \\ \psi_0^{\ell}(x) = g \phi(x) j_{\ell}(k x) \quad (5.5)$$

$$\Delta_{\ell} = [E - E_R + i\Gamma/2 + \mathcal{W}_{\ell}]^{-1}$$

Therefore, we have an expression entirely equivalent to the multiple scattering series but with the multiple scattering incorporated into an isobar self-energy $\mathcal{W}_{\ell}$. It is clear from Equation (5.4) that $\mathcal{W}_{\ell}$ is the non-static pion exchange interaction and that the on-shell piece is

$$\text{Im } \mathcal{W}_{\ell}(x_1, x_2) = k \psi_0^{\ell}(x_1) \psi_0^{\ell}(x_2) \quad (5.6)$$

This is again the condition for the degenerate schematic model, since the problem is to evaluate the expectation value of Δ_{ℓ} in the doorway ψ_0^{ℓ} . In other words, if only on-shell rescattering is kept, we have

$$[F_{\ell}(k)]_{\text{on-shell}} = \frac{- \int_0^{\infty} dx x^2 [\psi_0^{\ell}(x)]^2}{E - E_R + i\Gamma/2 + ik \int_0^{\infty} dx x^2 [\psi_0^{\ell}(x)]^2} \quad (5.7)$$

Clearly, $\text{Im } \mathcal{W}$ generates the partial wave elastic width and the

isobar free width the inelastic width.

We now apply the extended schematic model to this problem. We drop the angular momentum superscripts and start by constructing the doorway basis. The normalized doorway state is

$$\phi_0(x) \equiv N \psi_0(x) \quad (5.8)$$

$$\int_0^\infty dx x^2 [\phi_0(x)]^2 = 1$$

The remaining states are given by

$$\phi_n = N_n [\omega_{n-1} - \sum_{i=0}^{n-1} \phi_i \omega_{i,n-1}] \quad (5.9)$$

and the partial wave amplitude is

$$F_\ell(k) = \frac{-1}{k} \frac{\text{Im } \omega_{0,\ell}}{E - E_R + i\Gamma/2 + \omega_{0,\ell} + \omega_{0,1}} \frac{1}{E - E_R + i\Gamma/2 + \omega_{0,1} + \dots} \quad (5.10)$$

These expressions are carried over directly from Chapter IV.

All the integrals can be evaluated in closed form in the high energy limit.⁷ The result for the doorway state is

$$\phi_n^{(\ell)}(x) = \left[\frac{2(2n+1)}{f(\infty)} \right]^{1/2} \phi(x) P_n(\rho) \begin{cases} k j_\ell(kx) & , n \text{ even} \\ -k n_\ell(kx) & , n \text{ odd} \end{cases}$$

$$\rho \equiv \frac{f(x)}{f(\infty)}, \quad f(x) \equiv \int_0^x dy A_\ell(y) \quad (5.11)$$

The nonvanishing expectation values of ω_ℓ (Equation 5.4) are

$$\omega_{n,n} = i \gamma \delta_{no} \quad (5.12)$$

$$\omega_{n,n+1} = \frac{\gamma}{\sqrt{4(n+1)^2 - 1}}$$

where we have defined

$$\gamma \equiv - \frac{g^2 f(\infty)}{2k(E - E_R + i\Gamma/2)} \quad (5.13)$$

Comparison with Equation (3.29) reveals that γ is just the semiclassical phase shift. Substituting the results for $\omega_{i,j}$ into Equation 5.10, one finds that the continued fraction in the latter is essentially the expansion of the exponential:

$$F_\ell(k) = \frac{1}{2k} (1 - e^{2i\gamma}) \quad (5.14)$$

This is, of course, just the high energy scattering amplitude which could have been calculated directly from the optical potential. It should be clear that the multiple scattering and isobar approaches are completely equivalent.

The convergence of the continued fraction representation of the exponential is quite rapid even for large values of the exponent. Convergence is obtained by keeping $n \approx \gamma$ terms in the fraction. For $\pi - 160$ scattering, the semiclassical phase shift $\gamma \approx 2$, so that very few doorway states are needed for convergence. Further, we can expect the number of states needed to grow roughly like $A^{1/3}$ for larger nuclei.

One last exercise is to write the pion wavefunction in the isobar representation. The pion outgoing scattering wavefunction is conveniently written in terms of the half-off-shell nuclear scattering amplitude as

$$\psi_\ell^{(+)}(k;r) = j_\ell(kr) + \frac{2}{\pi} \int_0^\infty dp p^2 \frac{j_\ell(pr)}{p^2 - k^2 - i\epsilon} F_\ell(p,k;E_k) \quad (5.15)$$

$$F_\ell(k,k;E_k) \equiv F_\ell(k) \equiv \frac{\eta_\ell \exp(2i\delta_\ell) - 1}{2i}$$

We now write the amplitude in the isobar representation

$$F_\ell(p,k;E_k) = - \sum_n \langle p|n;k \rangle \langle n;k | \frac{1}{E - E_R + i\Gamma/2 + \omega_\ell} | 0;k \rangle \quad (5.16)$$

$$\theta < 0; k|k \rangle$$

$$\langle p|n;k \rangle \equiv f_n(p)$$

$$= \int_0^\infty dx x^2 g \phi(x) j_\ell(px) \phi_n^{(\ell)}(x) \quad (5.17)$$

$$f_0(k) = \frac{g}{k} \sqrt{\frac{f(\infty)}{2}}$$

Note that the vertex function $f_n(p)$ is non-zero for all the doorway states (in general). Consequently, reactions other than elastic scattering are sensitive to off-diagonal doorway expectation values of the isobar-hole propagator. Thus, the fact that

the doorway expansion of the elastic on-shell nuclear scattering amplitude converges rapidly does not necessarily mean that other amplitudes (such as the half-off-shell amplitude) will do so. We now have for the pion wavefunction

$$\psi_{\ell}^{(+)}(kr) = j_{\ell}(kr) [1 + ik F_{\ell}(k)] \quad (5.18)$$

$$-g^2 \sqrt{\frac{f(\omega)}{2}} \sum_{n=0}^{\infty} \int_0^{\infty} dr' \phi(r') \bar{\phi}_n^{(\ell)}(r') n_{\ell}(kr_{>}) j_{\ell}(kr_{<}) \langle n | G_{\Delta n} | 0 \rangle$$

Of course, only the $n=0$ term in Equation (5.18) contributes to the wavefunction outside the nuclear density (i.e., the asymptotic wavefunction can be written in terms of the scattering phase shift). For our simple model, the wavefunction does not converge rapidly in the high energy limit.⁷ We shall return to this point in Chapter VII in calculating the absorption cross section.⁴

It is amusing to comment on the phenomenological implications of having both the on- and off-shell amplitude converge rapidly in the doorway expansion. This would imply that a rank N separable potential, with N equal to the number of doorways, would suffice for the nuclear (partial wave) optical potential. Of course, the extreme nonlocality implicit in a very low rank separable potential already leads to the conclusion that the number of states needed should grow with the nuclear dimension. For reactions with very large momentum mismatches, there will generally be strong coupling to higher doorway states. For this case, convergence may be supplied more readily by the off-diagonal matrix elements of the Δ -hole propagator rather than the overlap integral in Equation 5.18. Detailed consideration of these questions is in progress.

VI. π - ^{16}O ELASTIC SCATTERING

We now apply the isobar-hole formalism to elastic π - ^{16}O scattering over a wide energy range. All these calculations have been carried out in collaboration with M. Hirata, J. Koch and F. Lenz.⁴ Comparison will be made with recent high precision data from S.I.N.²⁵ over the energy range 79 to 343 MeV and with data from Los Alamos²⁶ at 50 MeV. In making such a comparison, it is important to include nonresonant πN interactions and the Coulomb interaction. We include these within the two-potential framework:

$$T \equiv T_{\text{background}} + T_{\text{resonant}} \quad (6.1)$$

with $T_{\text{background}}$ calculated from the background interaction and

T_{resonant} calculated with the resonant interaction and pion waves distorted by the background potential. For the nonresonant πN interaction, we take a conventional first order static optical potential.

The first step is to determine the phenomenological spreading potential strength by fitting the forward elastic amplitude (i.e., total cross sections and/or forward differential cross sections); where available data in the Coulomb/nuclear interference region have been used. The resulting spreading potential is shown as a function of energy in Figure 9. The error bars indicate very crudely the uncertainty in the total cross section data. We note that the fit is made effectively to the quantity $(\gamma V_{\Delta} + V_{\text{sp}})$, since these have the same density dependence. We have taken the depth of the binding potential as 55 MeV, and this choice obviously affects somewhat the spreading potential parameter (especially the real part). In any case, the result is that the spreading potential is roughly -50i MeV, with a slightly attractive real part, over a wide energy range centered close to the resonance energy. On the other hand, there are clearly peculiarities in the fit at the highest energy; not only is the spreading potential changing rapidly, but the imaginary part is even slightly positive. While this does not violate unitarity, we nevertheless are caused difficulty in Chapter VII because of this. This will be discussed below. From now on, all calculations will use the spreading potential given in Figure 9. It should be noted that the same spreading potential is found for π^{+} and π^{-} scattering whenever the data for both charges exists at more or less the same energy.

The central spreading potential strength $V_{\text{sp}}(0) \sim -50i$ MeV may seem at first a bit large, but in fact it is consistent with the absorption widths in pionic atoms. If we write the usual p-wave pionic atom optical potential as

$$V(r) \equiv 4\pi \vec{r} \cdot \alpha(r) \vec{r} \quad (6.2)$$

$$\alpha(r) = c_0 \rho(r) + C_0 \rho^2(r)$$

then phenomenological fits give²⁷

$$c_0 = 0.21 \text{ m}_{\pi}^{-3} \quad (6.3)$$

$$\text{Im } C_0 = (.04 \text{ to } .07) \text{ m}_{\pi}^{-6}$$

The real part of C_0 is generally prescribed to be $\text{Re } C_0 = -\text{Im } C_0$, although this is in fact quite arbitrary. In the threshold situation appropriate to pionic atoms, the imaginary part of the

Figure 9: Isobar spreading potential (in MeV) fit to forward elastic scattering amplitude. Dashed lines are intended only as a guide to the eye.

optical potential is generated only by pion absorption. For nuclear matter densities, the ratio of the second to first order terms in the phenomenological optical potential is given by

$$\frac{c_0}{c_0} \rho \approx (.10 \text{ to } .17)(1 - 1) \quad (6.4)$$

We now want to calculate this same quantity assuming that the spreading potential represents p-wave absorption mediated by the isobar (which we assume to be the dominant p-wave absorption mechanism). We write for the first order term

$$\alpha(r) = \rho(r)[c_{NR} + c_\Delta] \quad (6.5)$$

where c_Δ is the contribution to the zero-energy p-wave scattering volume coming from the " Δ -pole", while c_{NR} represents the remainder (e.g., a crossed pion graph). Various authors²⁸ give $c_\Delta \approx .55 c_0$, and we adopt this value. It should be clear that we do not pretend to supply here more than an order-of-magnitude estimate, and the uncertainty in even defining c_Δ must be weighed at the end. Given this, we now modify the resonant part in a way similar to that used in the Glauber calculation at the end Chapter III:

$$\begin{aligned} \alpha(r) &\equiv \rho(r) \left[c_{NR} + c_\Delta \frac{E - E_R}{E - E_R + \frac{1}{2} \Gamma_{Abs} \tilde{\rho}(r)} \right] \\ &= \rho(r) c_0 \left[1 - \frac{c_\Delta}{c_0} \frac{1(\Gamma_{Abs}/2) \tilde{\rho}(r)}{E - E_R + 1(\Gamma_{Abs}/2) \tilde{\rho}(r)} \right] \end{aligned} \quad (6.6)$$

Taking $\Gamma_{Abs} = 100$ MeV (recall that $E = 0$ for the pionic atom situation), we have from the second term in square brackets

$$\frac{c_0}{c_0} \rho \approx .14(1 - .3) \quad (6.7)$$

which obviously is in rather close agreement with Equation 6.4. We repeat that the exercise is intended only to demonstrate that the spreading potential strength is in the right ballpark.

1. The Resonance Region

We start our detailed comparison with the data close to the free resonance energy. Figure 10 shows the differential cross section for $\pi^+ - {}^{16}O$ scattering at 163 MeV²⁵ together with the Δ -hole theoretical prediction. Also shown is the calculation

Figure 10: Elastic differential cross section for $\pi^+ - {}^{16}\text{O}$ scattering at 163 MeV. Solid and dotted curves result from the isobar-hole and PIPIT calculations, respectively. The dashed curve is the isobar-hole result modified by inclusion of "higher order" Coulomb effects (see discussion on p. 46). Data from Reference 25.

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given by PIPIT²⁹; this is a widely available momentum-space, first order optical potential code which uses a nonlocal separable πN transition matrix extended off-shell via the inverse scattering solution.¹⁴ It includes the "frame transformation" of the elementary amplitude and some correction for Fermi averaging.³⁰ Obviously, the isobar-hole results provide a very good description of the data except in the vicinity of the second minimum. Nevertheless, this is not really very instructive, and we turn to an analysis of the partial wave amplitudes.

Figure 11 is the Argand plot for the partial wave amplitude $T_L = (\eta_L \exp(2i\delta_L) - 1)/2i$ at 163 MeV. Again, the Δ -hole and PIPIT results are given. The deep diffraction structure in the latter arises because of the small real parts of the PIPIT partial wave amplitudes. Also shown is the result of a partial wave fit to the data ($\chi^2/\text{degree of freedom} = 1.1$). An important point is the comparatively good agreement between the Δ -hole calculations and the partial wave fit in the peripheral partial waves. For these partial waves, the spreading potential and Pauli blocking terms are comparatively unimportant, but a good treatment of the isobar propagation and binding effects is very important.

Table 4 gives the decomposition of the doorway expectation value of the isobar-hole Hamiltonian

$$\langle \Delta_0 | \mathcal{H}_\Delta | \Delta_0 \rangle = \langle \mathcal{W} \rangle + \langle \delta \mathcal{W} \rangle + \gamma \langle H_\Delta \rangle + \langle V_{sp} \rangle \quad (6.8)$$

for each partial wave. The first column gives the root-mean-square radius for the isobar in the doorway state and therefore a measure of "peripherality"; recall that the root-mean-square radius of the ${}^{16}\text{O}$ ground state density is 2.7 fm. In the closure limit (which is appropriate to PIPIT), $\langle H_\Delta \rangle$ is a constant for all partial waves (approximately 55 MeV in this case). On the other hand, this quantity obviously varies considerably with L in the Δ -hole calculations. The attraction for $L = 0$ is understood by recalling that the biggest component in the 0^- doorway state (see Table 2) has the isobar in the $0S_{3/2}$ shell model state. Clearly, the differences in $\langle H_\Delta \rangle$ between various partial waves are not negligible compared to other isobar self-energies.

The central partial waves are more complicated and the Δ -hole results do not agree with the partial wave fit in detail. The dominant interaction is generated by pion exchange. The real part of $\langle \mathcal{W} \rangle$, which roughly corresponds to off-shell propagation, is substantially smaller than the imaginary part for the central partial waves, implying that the scattering amplitude is fairly insensitive to reasonable changes in the off-shell extension of the πN amplitude. For the peripheral partial waves, $\langle \mathcal{W} \rangle$ is

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Figure 11: Argand plot corresponding to Figure 10. The points are the partial wave amplitudes, with the lines serving only to guide the eye. The dashed lines correspond to a partial wave fit to the data.

L^π	$\bar{R}_\Delta$ (fm)	$\langle \mathcal{H} \rangle$ (MeV)	$\langle \delta \mathcal{H} \rangle$ (MeV)	$\langle H_\Delta \rangle$ (MeV)	$\langle \mathcal{H}_{sp} \rangle$ (MeV)
0^-	2.24	-48 - 140i	2 + 25i	- .8 + .6i	-7 - 40i
1^+	2.31	-36 - 99i	10 + 20i	8.6 + .8i	-7 - 38i
2^-	2.49	-28 - 76i	7 + 20i	10.7 + .4i	-7 - 36i
3^+	2.90	-28 - 47i	7 + 14i	23.5 + .1i	-5 - 26i
4^-	3.36	-23 - 22i	6 + 8i	37.3 + .1i	-3 - 17i
5^+	3.80	-14 - 8i	4 + 4.5i	49.3	-2 - 10i
6^-	4.19	-7 - 2i	3 + 2i	58.6	-1 - 6i
7^+	4.63	-3 - .5i	1.5 + .7i	65.1	-.5 - 3i

Table 4: Decomposition of the doorway state expectation value of the isobar-hole Hamiltonian for π^+_{160} scattering.

is no longer small, but here single scattering provides a good approximation to the amplitude so that again off-shell effects are not too important. Returning to the central partial waves, the "higher order effects" of Pauli blocking and the spreading potential modify the dynamics in an important way. The Pauli term $\langle \delta \mathcal{H} \rangle$ provides a significant quenching of the free isobar decay width ($\Gamma/2 \approx 59$ MeV). Since the latter corresponds to the inelastic width for quasi-free scattering, this effect increases the ratio of elastic/inelastic pion scattering. On the other hand, the spreading interaction provides a large damping mechanism, suppressing the ratio of elastic to inelastic scattering. The net result is a suppression of elastic scattering and considerable loss of flux into the Q-space (see Figure 5 and Chapter VII).

Another interesting quantity is the off-diagonal doorway coupling. This determines convergence of the extended schematic model. We give the Hamiltonian matrix among the first two doorway states in Table 5 for the 0^- and 4^- partial waves. Recall that $\langle \mathcal{H}_\Delta \rangle_{01}$ is the fluctuation of the Hamiltonian in the doorway state (Equation 4.10). For the central 0^- partial wave, $|\langle \mathcal{H}_\Delta \rangle_{01} / \langle \mathcal{H}_\Delta \rangle_{00}| \approx 1/3$, implying that the collective doorway state is "close" to an eigenstate. This leads to the rapid convergence of the doorway expansion. On the other hand, the fluctuation in the 4^- doorway state is not small compared to the diagonal value $\langle \mathcal{H}_\Delta \rangle_{00}$. However, $\langle \mathcal{H}_\Delta \rangle_{01}$ is small compared to

	$L^\pi = 0^-$	$L^\pi = 4^-$
$\langle \mathcal{H}_\Delta \rangle_{00}$	-53 - 154i	13.9 - 14.4i
$\langle \mathcal{H}_\Delta \rangle_{01}$	59 + 8.2i	22.1 + 10i
$\langle \mathcal{H}_\Delta \rangle_{11}$	8.2 + 3.4i	25 + 4i

Table 5: Hamiltonian matrix $\langle \tilde{\Delta}_i | \mathcal{H}_\Delta | \tilde{\Delta}_j \rangle$ in MeV among the first two doorway states for $L^\pi = 0^-, 4^-$ at 163 MeV.

the free half width and this leads again to convergence of the doorway expansion. Clearly, the nature of the convergence is quite different for the central and peripheral partial waves. In the former, the isobar-nuclear dynamics is extremely important and leads to a "true" collective state. For the peripheral partial waves the isobar-hole states simply are damped weakly except for the usual quasifree decay; i.e., impulse approximation eventually becomes adequate for large angular momenta.

Within the Δ -hole framework, it is relatively simple to examine the importance of various dynamical modifications. Because of the collectivity, evaluation of the associated self-energies in the doorway state provides an immediate gauge of the role played by each effect. For example, one can calculate the effects of "higher order" Coulomb effects on π^+/π^- differences. These can arise from mass differences for the different Δ charge states, differences in the binding energy of protons and neutrons in the 160 ground state, and Coulomb energy differences for the different Δ charge states. The latter is the most important, and the Coulomb energy for Δ^{++} at the center of the residual nucleus (i.e., 160 minus one proton) is easily calculated to be 8 MeV. Comparing this to the scale fixed by the $L = 0$ results in Table 4, it is clear that such effects will not be large for the differential cross section. This can be seen in Figure 10. On the other hand, the effects are not completely negligible for π^+/π^- differences and, in fact, make the theory/experiment comparison virtually the same for $\pi^+/\pi^- - ^{160}$ scattering at 163 MeV (with the same spreading potential). Similarly, a nuclear structure correction would be inclusion of hole widths for the s-shell. Taking a half width $\Gamma_s/2 \approx 8$ MeV, the magnitude of the effect on the differential cross section is roughly the same as that of the Coulomb effects just described, although the cross section is now

suppressed slightly at the first minimum.

To summarize this subsection, it is clear that a fairly successful description of pion scattering in the resonance region is offered by the Δ -hole calculations. There are clearly discrepancies in the central partial waves. Here, the physics is complicated, and our simple assumption that the Q-space coupling proceeds through a local density-dependent spreading potential may be inadequate.

2. Scattering at Lower Energies

In Figure 12, we show the differential cross section for $\pi^+ - ^{160}$ scattering at 79 MeV²⁵ together with Δ -hole and PIPIT results. The PIPIT results give a much too large elastic cross section (recall the similar results in Figure 5). The isobar-hole calculations, while fitting the total elastic cross section quite well because of the spreading potential, do not do terribly well in the differential cross section. The source of the difficulty is seen by examining the Argand diagram, Figure 13. Comparing the Δ -hole partial wave amplitudes with those from the partial wave fit to the data, it is clear that the central partial wave amplitudes (i.e., 0^- and 1^+) must be reduced considerably. This is not surprising. At the lower energies, the background interaction is comparatively more important and we have used a simple first order optical potential. Phenomenological optical potential fits³¹ indicate the need for a substantial, repulsive second order, s-wave optical potential. We have included such a term of central strength 10 MeV in the background potential and show the resulting partial wave amplitudes in Figure 13. This clearly goes in the right direction and in fact results in a good fit to the data except in the region of the first minimum. It is interesting that an important effect of introducing the background potential is to reduce the resonant part of the amplitude. This is because the repulsive potential "pushes out" the pion wavefunction and thus reduces the isobar excitation strength.

Further improvement is obtained by including a binding energy shift and Pauli suppression in the first order background potential.³² This is seen most clearly at 50 MeV, where the "background" interaction is very much in the foreground. In Figure 14 we show the data²⁶ and a theoretical calculation. The latter includes the same type of p^2 -background potential described above, with central strength (20 - 3i) MeV. In addition, the energy at which the first order optical potential is evaluated has been shifted by about 20 MeV and the imaginary part has been quenched (to simulate the role of the Pauli principle). The last modifications were important for obtaining the right ratio of

Figure 12: Elastic differential cross section for $\pi^+ - {}^{16}\text{O}$ scattering at 79 MeV; data from Reference 25.

Figure 13: Argand plot corresponding to Figure 12. Dot-dashed line represents the isobar-hole results modified by including an s-wave repulsive second order optical potential of central strength 10 MeV in the background interaction.

Figure 14: Elastic differential cross section for $\pi^+ - {}^{16}\text{O}$ scattering at 49.7 MeV; data from Reference 26.

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forward to back angle scattering. The ingredients necessary for reproducing the data are obviously present.

3. Scattering Above the Resonance

Figure 15 shows the differential cross section for $\pi^+ - {}^{16}\text{O}$ elastic scattering at 240 MeV.²⁵ The theoretical curves are again the isobar-hole and PIPIT results. The latter obviously appear to be better, but a rather different light is cast on the comparison by examining the Argand plot, Figure 16. Here, we show also the result of a partial wave fit ($\chi^2/\text{degree of freedom} \approx 1.1$). Notice that the peripheral partial waves ($L \geq 5$) are quite different. We stress that the isobar propagation and binding effects needed for a good calculation of the peripheral partial waves are not "speculative"; they are calculated just as reliably in the nuclear shell model framework as are the pion rescattering and Pauli corrections. Their absence in the PIPIT calculation is simply a shortcoming of static optical potential calculations. However, the absence of deep minima in the PIPIT differential cross section, which obviously improves the appearance in comparison with the data, is aided significantly by the too large real parts of the peripheral amplitudes.

The isobar-hole calculations clearly have a significant problem with the central and semi-peripheral partial waves. In particular, the relative position of the $L = 0$ and 1 amplitudes from the partial wave fit cannot be reproduced in the Δ -hole approach without some appreciable change. It is certainly true that the partial wave fit is far from unique, but all good fits which we found had some "non-smooth" L -behavior for the first few partial waves. We note that some phenomenological optical potential fits which do include binding effects^{31,33} do seem to get better descriptions of the data. However, these calculations also give total cross sections about 15% too big. If we allow the total cross section to be this big in fitting the spreading potential, the Δ -hole results also appear quite good for the angular distribution. However, such values for the total cross section are not reasonable and point to the importance of including this constraint when evaluating the success of theoretical calculations. It also points to the need for accurate total cross section measurements.

Since the πN resonance interaction clearly dominates in this energy regime, we must assume that our treatment of the spreading potential needs refinement. As stressed earlier, this can take two forms: introduction of vertex corrections and/or inclusion of nonlocality in the isobar spreading potential. Apart from

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Figure 15: Elastic differential cross section for $\pi^+ - {}^{16}\text{O}$ scattering at 240 MeV; data from Reference 25. The dashed curve is the isobar-hole result modified by inclusion of nonlocality in the spreading potential (see discussion on pg. 53).

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Figure 16: Argand plot corresponding to Figure 15.

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simple Δ -hole "potential" interactions (which is a vertex correction), microscopic calculation of such terms requires consideration of two-hole diagrams. Simple examples of isobar self-energy terms are given in Figure 17, corresponding to pion absorption and pion multiple reflection (local field effects). Some examples of vertex corrections are given in Figure 18; these are obtained by hole exchange of the diagrams indicated in Figure 17. Quantitative evaluation of these terms may be expected to be very difficult; for example, NN and Δ N short-range forces are probably very important for quantitative accuracy.³⁴ Nevertheless, we can expect this "two-hole physics" to become very important above the resonance. For example, multinucleon knockout certainly increases strongly in this energy regime.¹ This is not surprising for several reasons: with more energy available, multinucleon knockout is easier; above the resonance, loss of energy in one quasifree interaction takes the pion into the resonance region, where another scattering is highly probable, the backward peaking of the π N scattering amplitude in the resonance region makes substantial energy transfer to the nucleon more likely.

The self energy corrections are the easiest to incorporate phenomenologically. We have estimated the role of nonlocality by expanding

$$\mathcal{V}_{sp}(r) = (1 + \frac{1}{6}\ell^2 \nabla^2 + \dots) \mathcal{V}_{sp}^{(0)}(r) \quad (6.9)$$

where ℓ plays the role of a nonlocality range. This range can be estimated from the absorption diagram (Figure 17a) to be $\ell \approx 1$ fm.³⁴ The resulting modification of the differential cross section is shown in Figure 15. There is clearly substantial improvement in raising the cross section at back angles, and this appears encouraging for examining Δ -dynamics in detail. Unfortunately, the total cross section is increased by about 8% and, as mentioned above, such a modification of the angular distribution could be achieved with the original spreading potential if a larger total cross section is allowed. Therefore, the lessons from this exercise are that smooth (or long-range) modifications of the dynamics do not change substantially the angular distribution while maintaining the constraint offered by the forward amplitude but that the interpretation of the phenomenological spreading potential strength may be altered somewhat. It is conceivable that very different forms for the spreading interaction could have substantial effects, but it is very difficult to think of any physical process which might generate these. This takes us to the point of evaluating the vertex corrections, but no progress has been made here yet.

Figure 17: Simple examples of isobar self-energies: (a) absorption; (b) multiple reflection. The double-dashed lines represent particle-particle scattering.

Figure 18: Simple example of vertex corrections obtained by hole exchange in the self-energy diagrams of Figure 17.

At the highest energy studied, 343 MeV, the isobar-hole calculations yield excellent agreement with the differential cross section data.²⁵ However, the spreading potential has changed drastically (See Figure 9) and even has a slightly positive imaginary part. We repeat that this does not violate unitarity (the imaginary part is still small in magnitude compared to the free Δ half-width), but physical interpretation is difficult and the phenomenology is therefore poor.

VII. THE REACTION CROSS SECTION

We have seen that the phenomenological spreading potential plays a crucial role in the description of pion scattering throughout the resonance region. One test of the model is provided, of course, by the description provided for the elastic angular distribution. This was reproduced reasonably well at all energies except 240 MeV. A more direct test came from the comparison with pionic atom level widths. However, this test is only semi-quantitative. We provide a further test by calculating the partial cross section into the Q-space. Further, we shall assume that this cross section is predominantly that for "true" pion absorption. As discussed already, inelastic data implies that this should be a good assumption from threshold up to at least the resonance energy.

The basic idea is understood easily in the limit that the scattering amplitude is given by the leading doorway state. We then have

$$T_L(E) = \frac{-\Gamma_{\text{Elas}}/2}{E - E_R + i\Gamma/2 - \langle \tilde{\Delta}_0 | \mathcal{H}_\Delta | \tilde{\Delta}_0 \rangle} \quad (6.1)$$

$$= \frac{-\Gamma_{\text{Elas}}/2}{E - E_R + \frac{1}{2}[(\Gamma - \Gamma_{\text{Pauli}}) + \Gamma_{\text{Elas}} + \Gamma_{\text{Abs}}]}$$

where the Pauli, elastic and absorption half-widths are given by the imaginary part of the expectation value of $\delta\mathcal{W}$, $\mathcal{W}$, and $\mathcal{V}_{\text{sp}}$ respectively. The real part of the expectation value is incorporated into E_R . The partial wave contribution to the total cross section is given by

$$\sigma_{\text{Total}}^L = \frac{4\pi}{k^2} \text{Im } T_L \quad (6.2)$$

$$= \frac{4\pi}{k^2} \frac{\frac{1}{4} \Gamma_{\text{Elas}} ((\Gamma - \Gamma_{\text{Pauli}}) + \Gamma_{\text{Elas}} + \Gamma_{\text{Abs}})}{(E - E_R)^2 + \frac{1}{4} ((\Gamma - \Gamma_{\text{Pauli}}) + \Gamma_{\text{Elas}} + \Gamma_{\text{Abs}})^2}$$

The interpretation in terms of partial cross sections is now clear. The contributions proportional to Γ_{Elas} , $(\Gamma - \Gamma_{\text{Pauli}})$ and Γ_{Abs} give total elastic, total inelastic (quasifree), and total absorption cross sections, respectively. This decomposition is very similar to that performed in the simple Glauber calculation in Chapter III.

For quantitative results, the one doorway approximation is not sufficient. We evaluate the reaction cross section as the expectation value of the imaginary part of the pion optical potential between pion distorted waves:

$$\sigma_{\text{Reaction}}^L = \frac{4\pi}{k} \langle 0, \chi_k^{(-)} | \text{Im } \mathcal{V}_{\text{PP}} | 0, \chi_k^{(+)} \rangle \quad (6.3)$$

$$= \frac{4\pi}{k} \langle 0, (\vec{k})_L | (1 + T_{\text{PP}}^\dagger G_{\text{PP}}^\dagger) \text{Im } \mathcal{V}_{\text{PP}} (1 + G_{\text{PP}} T_{\text{PP}}) | 0, (\vec{k})_L \rangle$$

The elastic pion transition matrix operator has the isobar-hole representation

$$T_{\text{PP}} = g_{\Delta\pi N}^\dagger [D(E) - \mathcal{H}_\Delta]^{-1} g_{\Delta\pi N} \quad (6.4)$$

Finally, we relate the isobar spreading potential to the term in the optical potential generated by pion absorption:

$$\text{Im } \mathcal{V}_{\text{sp}} \equiv g_{\Delta\pi N} G_{\text{PP}}^\dagger \text{Im } \mathcal{V}_{\text{PP}}^{\text{Abs}} G_{\text{PP}} g_{\Delta\pi N}^\dagger \quad (6.5)$$

Combining Equations (6.3) through (6.5), we have⁴

$$\sigma_{\text{Abs}}^L = \frac{4\pi}{k} \sum_j \alpha_j^* \langle \tilde{\Delta}_j | \text{Im } \mathcal{V}_{\text{sp}} | \tilde{\Delta}_j \rangle \alpha_j \quad (6.6)$$

$$\alpha_j = \frac{1}{\langle \tilde{\Delta}_j | [D(E) - \mathcal{H}_\Delta]^{-1} | \tilde{\Delta}_0 \rangle} \quad (6.7)$$

$$= \frac{\langle \tilde{\Delta}_j | [D(E) - \mathcal{H}_\Delta]^{-1} | \tilde{\Delta}_0 \rangle}{\langle 0, (\vec{k})_L | g_{\Delta\pi N}^\dagger g_{\Delta\pi N} | 0, (\vec{k})_L \rangle}$$

Note that the vertex function α_j is just the off-diagonal expectation value of the isobar-hole propagator in the doorway basis. This is the same quantity which entered in the pion wavefunction or half-off-shell elastic transition matrix (Equations 5.16 and 5.18). The vertex functions α_j provide a rapid convergence for

the calculation of the absorption cross section; for example, $|\alpha_1/\alpha_0| \approx 10^{-2}$ for both central and peripheral partial waves. Consequently the absorption cross section, which involves the half-off-shell transition matrix, converges almost as rapidly as the on-shell elastic scattering amplitude.

The decomposition of the reaction cross section into absorption and inelastic scattering cross sections is shown in Figure 19. Note that the inelastic (quasifree) cross section peaks at about 200 MeV, reflecting the binding energy shift discussed previously. The absorption cross section peaks at about 100 MeV and falls slowly with increasing energy. Unfortunately, there is essentially no data for the total absorption cross section on ^{16}O . However, an estimate based upon results available with other targets indicates that the cross section is about 200 mb between 80 and 220 MeV. Our calculation is more or less consistent with this.

At the highest energies, there are problems. Since the imaginary part of the spreading potential is positive at 343 MeV, the interpretation of the absorption cross section given above is no longer possible. In fact, even at 240 MeV, the calculated absorption cross section is likely too small. We have discussed already the problems at high energy in discussing the elastic cross section. Presumably, all of these problems could be related to a breakdown of the assumption that Q-space coupling can be summarized via an isobar self-energy. This is difficult to evaluate quantitatively and would require consideration of dynamics such as those indicated in Figure 18. Unfortunately, the needed reaction data which might resolve the problem are not available. For example interpretation of the spreading potential above the resonance would be aided by experimental resolution of the question as to whether absorption occurs after quasifree nucleon knockout.

VIII. CONCLUDING REMARKS

We conclude by summarizing the main features of the isobar-hole formalism and of the comparison with data. It is important to stress again that the descriptions of pion-nucleus reactions in the multiple scattering/optical potential language and in the isobar-hole language contain, in principle, the same physics. The fundamental advantage of the isobar-hole formalism is that the nuclear degrees of freedom are not "frozen out" as they are in most multiple scattering calculations. Thus, isobar propagation and binding effects and Pauli-blocking effects are calculated within the same framework and as reliably as the pion multiple scattering itself. In the optical potential language, this corresponds to evaluation, within the framework of the

Figure 19: Decomposition of the reaction cross section into absorption and inelastic scattering cross sections.

nuclear shell model, of the πN transition operator in the medium (i.e., $\tau(E)$ in Equation 3.7). The higher order effects are handled phenomenologically, as they are in optical potential calculations. We believe that the most important "spreading interaction", at least for energies at and below the 3-3 resonance, is pion absorption. The isobar dominance of this process leads to a "natural" phenomenology for incorporating these effects as an isobar self-energy. This is reinforced by the good description of pion elastic scattering and by the evaluation of the absorption cross section and pionic atom level widths in terms of the phenomenological spreading interaction. We note that this phenomenology does not correspond to adding a simple local p^2 term in the optical potential. Above the resonance, there are unresolved problems involving the elastic angular distribution at 240 MeV and the abrupt change in the spreading potential strength at 340 MeV.

With regard to the calculational technology, we find the similarity to the many body techniques used in nuclear structure calculations and to the concepts employed in low energy nuclear reaction theory very satisfying. In particular, the extended schematic model or doorway expansion makes this similarity quite clear and somewhat simplifies the calculations. This doorway approach is based explicitly on the "collectivity" expected by analogy with the Brown-Bolsterli model. The dominance of the elastic transition matrix by one or two Δ -hole doorway states also allows clear insight into the role of the various aspects of pion-nuclear dynamics in determining the scattering. Further, to the extent that calculations of various "off-shell" reactions converge similarly in the doorway expansion, a simple unification of these reactions is afforded in terms of only a few parameters (doorway expectation values and coupling strengths). The absorption cross section offered an example of this rapid convergence.

Returning to the dynamics, the isobar propagation and binding effects were crucial for calculation of the peripheral partial wave elastic scattering amplitudes. The results seem to be in good agreement with the data. The Pauli blocking effects were important for suppressing the inelastic or quasifree pion scattering cross section. This increases the ratio of elastic to reaction cross sections. However, this is more than compensated by the spreading interaction which summarizes loss of flux into the Q-space. This finally allowed for a good description both of the elastic to reaction cross section ratio and of the inelastic to absorption cross section ratio. At the lower energies, our original treatment of the nonresonant interactions through a static first order optical potential was not adequate, but addition of a repulsive s-wave term and inclusion

of a binding energy shift and Pauli quenching led to a very good description of the data. All of these features are known to be important from optical potential fits to the data. Above the resonance, the situation was not so good. Our simple treatment of the Q-space coupling needs refinement, and this presumably requires some serious consideration of various "two-hole diagrams". However, these are very difficult to evaluate quantitatively, since this would require consideration of the strong short-distance interactions among nucleons and isobars. There is no reason to believe that a simple treatment based upon static NN correlation functions is sufficient. Considerable theoretical effort is needed here, whether in the isobar-hole or multiple scattering approach.

Finally, we note that much of the "testing" of the model came from a consideration of inelastic reactions. For example, the binding and Pauli blocking effects are reflected directly in the (π, π') inclusive spectrum; the spreading potential strength is reflected in the total absorption cross section (although this connection is far from model-independent). The most urgent theoretical need for further development of the microscopic description of pion-nucleus interactions is more reaction data. The nuclear response function for pions is still very poorly known. Systematic studies of the pion quasielastic spectrum throughout the resonance region are essential; this is the channel which leads to attenuation of the elastic channel wavefunction in a first order optical potential. Good measurements of the total absorption cross section are needed, as well as more exclusive reactions such as $\pi(\text{in flight}) \rightarrow NN$. The role of multinucleon knockout is crucial for interpretation of the second order optical potential. This was relevant directly to our discussion of the spreading interaction above the resonance. The goal is to understand the reaction mechanism by which the pion transfers energy and momentum to the nucleus, particularly in the absorption process. The role of photoinduced reactions as a complementary probe of the isobar-nuclear dynamics should not be overlooked. Hopefully, the rather meager amount of such reaction data will be expanded greatly in the next few years and will provide guidance for the theoretical developments. An understanding of the behavior of pions and Δ 's in the nucleus will both allow detailed study of nuclear bound states with pions and provide some measure of unification between descriptions of the strong force and of nuclear structure. This is a fundamental challenge of intermediate energy physics.

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