STEADY STATE DISTRIBUTION FOR UNBUNCHED BEAMS
COLLIDING IN A DRIFT SPACE[†]

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Summary

A self-consistent time-independent solution of the system of coupled equations for the distribution of two colliding unbunched beams of opposite charges is found. The solution for each beam contains an arbitrary constant which characterizes the lateral size of the beam. On the other hand, the angular divergences are uniquely determined by the line charge density of the opposite beam in case the beams have cylindrical symmetry. The possible implications of this solution for a linac collider are discussed. Further work on the stability of this solution is needed.

Formulation of the Problem

In the proposed linac collider,¹ two very intense beams collide at the point of interaction. The electromagnetic fields of the bunches are strong enough to cause the particle distributions, and hence the fields themselves, to change significantly during one collision. This change in the fields and distributions should be taken into account in a self-consistent manner in order to properly evaluate the luminosity since the luminosity strongly depends on the change of particle distributions during the collision.² An attempt is made in this work to study this problem. One aim is to look for such a distribution for which its

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spreading due to the angular divergence in the beam is exactly balanced by the focusing produced by the beam-beam force, and thus reaching an equilibrium state.

Consider two monochromatic unbunched beams with opposite charges colliding head-on. Let the negatively charged beam move in the z direction and the positively charged beam in the $-z$ direction. We assume that the beams are relativistic ($\beta \approx c$) and the transverse components of particle velocities are small ($\dot{x}, \dot{y} \ll c$).

We describe the particle distributions of the $\pm$ beams in the transverse phase space $(x, \dot{x}, y, \dot{y})$ by two distribution functions $f^\pm(x, \dot{x}, y, \dot{y}, z, t)$ which satisfy the following Vlasov equations

$$\frac{\partial f^\pm}{\partial t} + \dot{z} \frac{\partial f^\pm}{\partial z} + \dot{y} \frac{\partial f^\pm}{\partial y} + \frac{c^2}{E} F_y^\pm \frac{\partial f^\pm}{\partial \dot{y}} + \dot{x} \frac{\partial f^\pm}{\partial x} + \frac{c^2}{E} F_x^\pm \frac{\partial f^\pm}{\partial \dot{x}} = 0, \quad (1)$$

where E is the particle energy, $F_{x,y}^\pm$ are the x - and y - components of the force acting on a particle from the opposite beam. For each of the functions we use the normalization

$$\int f^\pm(x, \dot{x}, y, \dot{y}) dx d\dot{x} dy d\dot{y} = 1. \quad (2)$$

Rigorously speaking, the forces should be calculated from retarded potentials. However, with the assumption $\dot{x}, \dot{y} \ll c$, the field propagation is much faster than the rate of change in particle distribution. Hence it is a good approximation to find the forces from static potentials:

$$F_{x,y}^\pm = -2e \frac{\partial V^\mp}{\partial (x,y)}. \quad \text{DISCLAIMER} \quad (3)$$

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The factor of 2 on the right hand side is a result of adding the electric and the magnetic forces. The potentials $V_{\pm}(x,y)$ satisfy the Laplace equations

$$\frac{\partial^2 V_{\pm}}{\partial x^2} + \frac{\partial^2 V_{\pm}}{\partial y^2} = 4\pi\lambda_{\pm}e \int f^{\pm}(x,\tilde{x},y,\tilde{y})d\tilde{x}d\tilde{y}, \quad (4)$$

where $\lambda_{\pm}$ are the line density of the particle numbers in the $\pm$ beams. We will choose the additive arbitrary constant in $V_{\pm}$ in such a way that $V_{\pm}(0,0) = 0$.

For the motion of a particle in the potential $V(x,y)$, the energy E is not conserved ($E+V$ is conserved), but in the relativistic limit we can ignore V as compared with E and get $E = \text{constant}$.

Equations (1) through (4) constitute a system of coupled non-linear equations that has to be solved.

Steady State Solution For Round Beams

For unbunched beams, the steady state solution for $f^{\pm}$ must not depend on z or t . The system of equations for the steady state solution becomes

$$\tilde{y} \frac{\partial f^{\pm}}{\partial \tilde{y}} - b_{\pm}^2 \frac{\partial \phi^{\mp}}{\partial \tilde{y}} \frac{\partial f^{\pm}}{\partial \tilde{y}} + \tilde{x} \frac{\partial f^{\pm}}{\partial \tilde{x}} - b_{\pm}^2 \frac{\partial \phi^{\mp}}{\partial \tilde{x}} \frac{\partial f^{\pm}}{\partial \tilde{x}} = 0, \quad (5)$$

and

$$\frac{\partial^2 \phi^{\pm}}{\partial \tilde{x}^2} + \frac{\partial^2 \phi^{\pm}}{\partial \tilde{y}^2} = \int f^{\pm}(x,\tilde{x},y,\tilde{y})d\tilde{x}d\tilde{y}, \quad (6)$$

where we have introduced two strength parameters

$$b_{\pm}^2 = 8\pi c^2 e^2 \lambda_{\pm} / E \quad (7)$$

and the normalized potentials

$$\phi^{\pm} = V_{\pm} / 4\pi e \lambda_{\pm} \quad (8)$$

In addition to Eqs. (5) and (6), we also have the normalization condition (2) to satisfy.

It is easy to verify by direct substitution that any function of the effective Hamiltonian

$$H_{\pm}^{\pm} = \frac{\dot{x}^2 + \dot{y}^2}{2b_{\pm}^2} + \phi_{\pm}^{\pm} \quad (9)$$

satisfies Eq. (5). In particular we will choose

$$f_{\pm}^{\pm} = P_{\pm} \frac{K_{\pm}}{2\pi b_{\pm}^2} \exp(-K_{\pm} H_{\pm}^{\pm}) \quad (10)$$

where $P_{\pm}$ and $K_{\pm}$ are some constants.

Substituting Eqs. (9-10) into Eq. (6) gives

$$\frac{\partial^2 \phi_{\pm}^{\pm}}{\partial x^2} + \frac{\partial^2 \phi_{\pm}^{\pm}}{\partial y^2} = P_{\pm} e^{-K_{\pm} \phi_{\pm}^{\pm}} \quad (11)$$

It turns out that there exists a family of cylindrically symmetric solutions to this system of nonlinear equations (11). Let us denote

$$P_{-}K_{+} = C_1, \quad (12)$$

$$P_{+}K_{-} = C_2, \quad (13)$$

$$\xi = \exp(-K_{+}\phi^{-}), \quad (14)$$

$$\eta = \exp(-K_{-}\phi^{+}), \quad (15)$$

then Eqs. (11) are equivalent to

$$\xi''\xi - (\xi')^2 + \frac{\xi\xi'}{\xi} + C_1\eta\xi^2 = 0, \quad \xi(0) = 1, \quad \xi'(0) = 0, \quad (16)$$

$$\eta''\eta - (\eta')^2 + \frac{\eta\eta'}{\eta} + C_2\xi\eta^2 = 0, \quad \eta(0) = 1, \quad \eta'(0) = 0. \quad (17)$$

where $r = \sqrt{x^2 + y^2}$ and a prime means taking the derivative with respect to r . From (2) we have

$$k_{\eta} \equiv K_+/2\pi = C_1 \int_0^{\infty} \eta \, r \, dr , \quad (18)$$

$$k_{\xi} \equiv K_-/2\pi = C_2 \int_0^{\infty} \xi \, r \, dr . \quad (19)$$

Distribution functions (10) can be expressed in terms of ξ and η .

$$f^{\pm} = \frac{1}{2\pi} R_{\pm}(r) Q_{\pm}(\dot{r}) , \quad (20)$$

$$R_+ = \frac{C_2}{k_{\xi}} \xi(r, C_1, C_2) , \quad (21)$$

$$R_- = \frac{C_1}{k_{\eta}} \eta(r, C_1, C_2) , \quad (22)$$

$$Q_+ = \frac{k_{\eta}}{2} \exp(-\pi k_{\eta} \dot{r}^2/b_-^2) , \quad (23)$$

$$Q_- = \frac{k_{\xi}}{2} \exp(-\pi k_{\xi} \dot{r}^2/b_+^2) , \quad (24)$$

where $\dot{r} = \sqrt{\dot{x}^2 + \dot{y}^2}$. The constants C_1 and C_2 have an arbitrary factor connected with the choice of units for the distance r . If we choose it in such a way that

$$C_1 + C_2 = 2 \quad (25)$$

then the functions (20) represent a one-parameter family of solutions to

equations (5,11). Figs. 1-4 illustrate the radial parts $R_{\pm}$ of the distribution functions $f^{\pm}$ for different values of parameter C_1 . Corresponding radial components of the forces $F^{\pm}$ (in units $4e^2c^2\lambda_{\pm}/E$) are also shown.

From the asymptotic behavior of functions ξ and η

$$\xi \sim r^{-k_{\eta}} \quad (26)$$

$$\eta \sim r^{-k_{\xi}} \quad (27)$$

one finds that the normalized solutions of equations (16, 17) exist only if

$$k_{\eta} > 2 \quad (28)$$

and

$$k_{\xi} > 2 \quad (29)$$

Table 1 gives parameters k_{ξ} and k_{η} for different values of parameter C_1 ($C_2 = 2 - C_1$).

One particular member of this family when $C_1 = C_2 = 8\pi P$ (P is introduced to bring back the unit here) can be shown to be the following elementary function (in this case, $k_{\xi} = k_{\eta} = 4$):

$$R_{\pm} = 2\pi P / (1 + \pi P r^2)^2 \quad (30)$$

or

$$\phi^{\pm} = \frac{1}{2} \ln(1 + \pi P (x^2 + y^2)) \quad , \quad (31)$$

where we have demanded $\phi^{\pm}(0,0)=0$. The two beams have identical distribution and produce identical normalized potential. That is, in case the

two beams are identical and behave exactly with a mirror symmetry, the steady state distribution for the two beams is

$$f_{\pm}^{\pm}(x, \dot{x}, y, \dot{y}) = \frac{4 P \exp(-4\pi(\dot{x}^2 + \dot{y}^2)/b_{\pm}^2)}{b_{\pm}^2 [1 + \pi P (x^2 + y^2)]^2} \quad (32)$$

in which P is arbitrary, corresponding to an arbitrary scale in the lateral beam size, and $b_{\pm}$ are given by Eq. (7). The solution (30) is essentially the same as one found by W. H. Bennett⁶ for the case of fast electron stream passage through a dense cloud of positive ions.

Two more parameters of interest are the small-amplitude oscillation frequencies of a single particle in the potential of the opposite beam. They are given by $\omega_{+}^2 = 4 c^2 e^2 \lambda_{-} C_1 / E k_{\eta}$ and $\omega_{-}^2 = 4 c^2 e^2 \lambda_{+} C_2 / E k_{\xi}$. We give in Table 1 the values of C_1/k_{η} and C_2/k_{ξ} which are useful for calculating $\omega_{\pm}$. In the case when $C_1 = C_2, \lambda_{-} = \lambda_{+} = \lambda$, $\omega_{+}^2 = \omega_{-}^2 = 8\pi c^2 e^2 \lambda P/E$. This frequency is directly related to the plasma frequency defined in Ref. 3.

Steady State Solution for Flat Beams

Another case of interest which can be solved is the collision of two flat beams. If the lateral dimension in one direction, let it be the vertical one, is much less than the beam size in the other (horizontal) direction, then both the distribution functions and the potentials should not depend on x . We get from Eqs. (5,6):

$$\dot{y} \frac{\partial f^{\pm}}{\partial y} - b_{\mp}^2 \frac{\partial \phi^{\mp}}{\partial y} \frac{\partial f^{\pm}}{\partial \dot{y}} = 0 \quad (5')$$

and

$$\frac{d^2 \phi^\pm}{dy^2} = \int f^\pm(y, \dot{y}) d\dot{y} \quad (6')$$

The strength parameters now are

$$b_{\pm f}^2 = 8\pi c^2 e^2 \sigma_\pm / E \quad (7')$$

where $\sigma_\pm$ are the surface densities of the particle numbers of the beams.

Again it is easy to see that functions

$$f^\pm = P_\pm \frac{\sqrt{k_\pm}}{\sqrt{2\pi} b_{\pm f}^2} \exp(-k_\pm H^\pm) \quad (10')$$

with

$$H^\pm = \frac{\dot{y}^2}{2b_{\pm f}^2} + \phi^\pm \quad (9')$$

satisfy Eq. (5'). Analogous to (11) we now have

$$\phi^{\pm'} = P_\pm e^{-k_\pm \phi^\pm}, \quad (11')$$

where a prime means taking the derivative with respect to y . These equations also have a one-parameter family of solutions. Using relations (12-15) we get

$$\xi'' \xi - (\xi')^2 + C_1 \eta \xi^2 = 0, \quad \xi(0) = 1, \quad \xi'(0) = 0, \quad (16')$$

$$\eta'' \eta - (\eta')^2 + C_2 \xi \eta^2 = 0, \quad \eta(0) = 1, \quad \eta'(0) = 0 \quad (17')$$

with the normalization conditions

$$k_\xi \equiv k_-/2 = C_2 \int_0^\infty \xi \, dy, \quad (18')$$

$$k_\eta \equiv k_+/2 = C_1 \int_0^\infty \eta \, dy. \quad (19')$$

The structure of equations (16') and (17') together with the initial conditions suggest that functions ξ and η are symmetric in y . From now on we consider only $y \geq 0$.

Distribution functions (10') can be expressed again in terms of ξ and η

$$f^\pm = \frac{1}{2} R_\pm(y) Q_\pm(\dot{y}) \quad (20')$$

$$R_+ = \frac{C_2}{k_\xi} \xi(y, C_1, C_2) \quad (21')$$

$$R_- = \frac{C_1}{k_\eta} \eta(y, C_1, C_2) \quad (22')$$

$$Q_+ = \frac{\sqrt{k_\eta}}{\sqrt{\pi} b_{-f}} \exp(-k_\eta \dot{y}^2 / b_{-f}^2) \quad (23')$$

$$Q_- = \frac{\sqrt{k_\xi}}{\sqrt{\pi} b_{+f}} \exp(-k_\xi \dot{y}^2 / b_{+f}^2) \quad (24')$$

Choosing unit for the distance y in such a way that $C_1 + C_2 = 2$ we get for the flat beam a one-parameter family of solutions (20').

Figs. 5-9 give the vertical parts $R_{\pm}$ of steady state distribution functions $f_{\pm}^+$ for different values of parameter C_1 . Corresponding vertical components of the forces $F_{\pm}^+$ (in unit $4\pi e^2 c^2 \lambda_{\pm}^2 / E$) are also shown.

Asymptotic behavior of the function ξ and η

$$\xi \sim e^{-k_{\eta} y}, \quad (26')$$

$$\eta \sim e^{-k_{\xi} y} \quad (27')$$

limit possible values of k_{ξ} and k_{η} in the following way

$$k_{\eta} > 0, \quad (28')$$

$$k_{\xi} > 0. \quad (29')$$

Table 2 gives parameters k_{ξ} , k_{η} , C_1 / k_{η} and C_2 / k_{ξ} for different values of parameter C_1 . In the symmetric case $C_1 = C_2 = 8P^2$. (In this case $k_{\xi} = k_{\eta} = 8P$) the solutions are the following elementary functions

$$f_{\pm}^{\pm} = \frac{4P \exp(4Py)}{(1 + \exp(4Py))^2} \frac{\sqrt{8P}}{\sqrt{\pi} b_{\pm} f} \exp(-8Py^2 / b_{\pm}^2 f^2). \quad (32')$$

Comparing (32') with the cylindrically symmetric case, Eq. (32), one notices that the angular divergence is no longer uniquely determined by the surface charge density in the opposite beam; it contains a free parameter P which is related to the steady state beam size.

Discussion

In the following we assume that the two beams are cylindrically symmetric and identical and behave with a mirror symmetry. Under these assumptions, the steady state distribution of the two beams is given by Eq. (32).

The steady state solution (32) implies that the two beams have the same spacial distribution. The FWHM in the spacial distribution is $2/\sqrt{nP}$. Since P is arbitrary in the solution, the beam-beam force does not uniquely determine the actual beam size. The functional form of the space distribution, however, does indicate a large tail population. (This is indicated by the fact that the rms values $\langle x^2 \rangle = \langle y^2 \rangle = \infty$.) This feature might be interesting from the point of view of beam-beam blow-up in tails observed in storage rings.⁴ Furthermore, the fact that there exists a family of equilibrium distributions for which the two beams have different lateral sizes is consistent with the "flip-flop" phenomenon observed in SPEAR.⁵ Further studies are needed to apply these results to storage rings.

The angular distribution, on the other hand, contains no arbitrary constants. The steady state requires an angular divergence with rms value

$$\sigma_{x'} = \sigma_{y'} = \sqrt{e^2 \lambda_{\perp} / E} \equiv \sigma \quad (33)$$

This angular divergence is the value for which the corresponding spreading of the beam is exactly balanced by the pinching effect resulted from the attractive beam-beam force.

In the case of the linac collider, if the two colliding beams have an initial angular divergence much less than σ , one may benefit from the pinch effect since the effective luminosity increases substantially from the reduced beam size.² It is therefore probably desirable to prepare the beam in such a way that the angular divergence of the beams is not larger than σ before the beams are brought into collision. For the linac collider,¹ the angular divergence at the interaction point $\sqrt{\epsilon_y / \beta_y}$ is 0.2 mrad where ϵ_y is the emittance of the beams and β_y is the β -function at

the interaction point (assuming $\pi r_y = 3 \times 10^{-5} \text{ m}$ and $\beta_y = 1 \text{ cm}$), while Eq. (33) gives $\sigma = 0.7 \text{ mrad}$ (assuming 5×10^{10} particles/bunch, bunch length 3 mm and energy 50 GeV). It seems therefore that there will be a serious pinch effect under these conditions. (*)

We have not studied the dynamic stability of the solution; in case there is a deviation in the initial condition from the steady state solution, we have not determined the subsequent behavior of this deviation at later times. Such an analysis is needed in order to study the stability of the steady state solutions found.

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(*) If the number of particles is less than the design value, or if we decide to have a smaller β_y (2mm for example), the pinch effect will become much weaker.
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TABLE 1. Round Beam

C_1	C_2	k_{ξ}	k_{η}	C_1/k_{η}	C_2/k_{ξ}
1.0	1.0	4.0	4.0	.25	.25
1.05	.95	3.19	5.37	.20	.29
1.10	.90	2.63	8.37	.13	.34
1.15	.85	2.21	20.8	.06	.38
1.20	.80	no solution			

TABLE 2. Flat Beam

C_1	C_2	k_{ξ}	k_{η}	C_1/k_{η}	C_2/k_{ξ}
1.0	1.0	2.83	2.83	.35	.35
1.2	0.8	1.98	4.04	.30	.40
1.4	0.6	1.34	5.99	.23	.45
1.6	0.4	0.82	9.81	.16	.49
1.8	0.2	0.38	21.1	.09	.53

$C1=0.1000+0I$ $C2=0.1000+0I$ $KSI=0.4000+0I$ $ETA=0.4000+0I$
 1 - 80 3768A1

Fig. 1

$C1=0.105D+01$ $C2=0.950D+00$ $KSI=0.319D+01$ $ETA=0.537D+01$
 1-80 3765A2

Fig. 2

$C1=0.110D+01$ $C2=0.900D+00$ $KSI=0.263D+01$ $ETA=0.837D+01$
 1-80 3768A3

Fig. 3

$C1=0.115D+01$ $C2=0.850D+00$ $KSI=0.221D+01$ $ETA=0.208D+02$
 1-80 3768A*

Fig. 4

$C1=0.100D+01$ $C2=0.100D+01$ $KSI=0.283D+01$ $ETA=0.283D+01$
 1-80 3768A-

Fig 5

$CI=0.120D+01$ $C2=0.800D+00$ $KSI=0.198D+01$ $ETA=0.404D+01$
 1 - 40 3758AA

Fig. 6

$C1=0.140D+01$ $C2=0.600D+00$ $KSI=0.134D+01$ $ETA=0.599D+01$
 - 80 3768A7

Fig. 7

$C1=0.160D+01$ $C2=0.400D+00$ $KSI=0.815D+00$ $ETA=0.981D+01$
 1-80 2768A8

Fig. 8

$C1=0.180D+01$ $C2=0.200D+00$ $KSI=0.378D+00$ $ETA=0.211D+02$
 1 - 80 3768A9

Fig. 9