WAKEFIELDS IN A PILLBOX CAVITY*

Bruno Zotter

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Stanford Linear Accelerator Center
Stanford University, Stanford, California 95305

I. Introduction

The wakefields due to a bunch in a pillbox cavity can be found by mode-expansion of the fields given by the wave-equation.¹ Usually one considers first the wakefield due to an infinitely short pulse and then integrates over the current distribution. Since the series for the delta function pulse converge rather poorly, it is preferable to use the wakefields due to a step-function current,² which can also be integrated to give the result for an arbitrary current distribution.

For the most important case of fields with azimuthal symmetry, the results are expressed as double series over the radial mode number n ($1, 2, \dots, \infty$), and the longitudinal mode number p ($0, 1, 2, \dots, \infty$). For a beam traveling with light-velocity, these series have been summed analytically.² However, the results are restricted to observation times before reflections from the cylindrical side-wall reach the axis, i.e., they are actually valid only for a parallel-plate geometry³ (cavity with side-wall removed to infinity). In order to find the wakefields in cylindrical cavities of finite radius we have to re-examine the infinite series. Although no closed-form expression could be found for the side-wall reflections, they can be determined from a function defined by a single infinite sum with only one variable.

Extension to modes with azimuthal variation is straightforward. However, extension to beam velocities smaller than light-velocity is

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more complicated and again would require the use of functions defined by infinite series.

The influence of finite holes in the end-walls has not been investigated in this report, since no analytic solutions are known for this case. However, the character of the solutions for the closed cavity should help in understanding the numerical results obtained for this case.

II. Wakefields in a Pillbox Cavity^{*}

We study fields in a circular cylindrical cavity of length L and radius R with perfectly conducting walls. First we investigate azimuthal symmetric fields (mode-number $m=0$). The wakefield is defined as the voltage gain (integral over the longitudinal electric field) of a test particle on the axis of the cavity, which is a distance z_0 behind an infinitely short pulse of unit charge. For a beam with velocity $v = \beta c$ ($\gamma = (1-\beta^2)^{-1/2}$) it becomes

$$V'(z_0) = -\frac{2\beta^2}{\pi\epsilon_0 L} \sum_{n=1}^{\infty} \frac{\epsilon_p j_n^2}{\left[j_n^2 + \left(\frac{\pi p}{\gamma\epsilon}\right)^2\right]^2 J_1^2(j_n)} \begin{cases} f_1(z_0) & (0 < z_0 \leq L) \\ f_2(z_0) & (z_0 \geq L) \end{cases} \quad (2.1)$$

^{*} The formulae in this section were derived by P. Morton and K. Bane.

where

$$f_1(z_0) = \left[2 \cos kz_0/\beta - (-)^p \cos k(z_0 + L)\beta - \cos \pi p z_0/L \right] - \frac{\pi p}{2\beta^2 \gamma^2} \left[1 + \left(\frac{\pi p}{\gamma \epsilon j_n} \right)^2 \right] \left(1 - \frac{z_0}{L} \right) \sin \pi p z_c/L \quad (2.2)$$

$$f_2(z_0) = 2 \cos kz_0/\beta \left[1 - (-)^p \cos kL/\beta \right]$$

with $\epsilon = L/R$ (2.3)

and the propagation constant ($k = \omega/c$)

$$k_{(np)} = \sqrt{j_n^2 + \left(\frac{\pi p}{\epsilon} \right)^2} / R \quad (2.4)$$

For beams with light velocity ($\beta=1, \gamma \rightarrow \infty$) the expressions simplify considerably and can be written

$$v'(z_0) = - \frac{2}{\pi \epsilon_0 L} \sum_{p=1}^{\infty} \frac{\epsilon_p}{j_n^2 j_1^2(j_n)} \left\{ \begin{array}{l} 2 \cos kz_0 - (-)^p \cos k(z_0 + L) - \cos \pi p z_0/L \\ 2 \cos kz_0 \left[1 - (-)^p \cos kL \right] \end{array} \right\} \quad (2.5)$$

Since the sum over p converges only marginally, it is preferable to express the wakefield due to a step-function pulse obtained by integrating over z_0 . For beams with light-velocity we then obtain

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$$V(z_0) = -\frac{2}{\pi \epsilon_0 c} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{\epsilon_p}{j_{n1}^2 J_n^2(j_{n1})} \left\{ \begin{aligned} & \frac{2 \operatorname{sinc} z_0}{kL} - (-)^p \frac{\operatorname{sinc}(z_0+L) - \operatorname{sinc} L}{kL} - \frac{\sin \pi p z_0 / L}{\pi p} \\ & \frac{2 \operatorname{sinc} z_0}{kL} - (-)^p \frac{\operatorname{sinc}(z_0+L)}{kL} - (-)^p \frac{\operatorname{sinc}(z_0-L)}{kL} \end{aligned} \right\}$$

for $\begin{cases} 0 < z_0 \leq L \\ L \leq z_0 \end{cases}$ (2.6)

We find that the two expressions for $z_0 \leq L$ and $z_0 \geq L$ are very similar, except that the "driven term" $\sin \pi p z_0 / L$ is replaced by a "reflected term" $\sin k(z_0 - L)$. The expression for $z_0 \geq L$ agrees with Eq. (6) of Ref. 2 (except for a missing factor g/c in that expression).

The extension of these expressions to modes with azimuthal variation (mode number $m \neq 0$; $m = 1$; dipole mode, $m = 2$; quadrupole mode, etc.) is quite straightforward. We need only replace the Bessel function of order zero $J_0(x)$ and its zeros j_{0n} with Bessel functions of order m ($J_m(x)$ and their zeros j_{mn}). Since the longitudinal wakefield vanishes on the axis of the cavity for $m \neq 0$ (as $J_m(0)$ for $m \neq 0$), we have to evaluate the azimuthal varying fields for a displaced beam. There may be a transverse (deflecting) field on the axis, but this is not part of the present investigation.

III. Summation of the Infinite Series

The two expressions given in Eq. 7 of Ref. 2 to sum the mode-expansions for the fields when $v = c$ can be written in dimensionless form. We put $z_0/R = x$, $g/R = \alpha$, $v_{on} = j_n$, $\alpha_p = 2 \epsilon_p$ and define

$$\omega_{np} = \sqrt{j_n^2 + \left(\frac{\pi p}{\alpha}\right)^2} \quad (3.1)$$

to get

$$\sum_{p=0}^{\infty} \sum_{n=1}^{\infty} \frac{\epsilon_p}{j_n^2 j_1^2(j_n)} \frac{\sin \omega_{np} x}{\omega_{np}} = -\frac{\alpha}{4} \begin{cases} \ln x & (0 < x < 2\alpha < 1) \\ \ln x + \ln(x^2 - 4\alpha^2) & (2\alpha < x < 4\alpha < 1) \end{cases} \quad (3.2)$$

$$\sum_{p=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-)^p \epsilon_p}{j_n^2 j_1^2(j_n)} \frac{\sin \omega_{np} x}{\omega_{np}} = -\frac{\alpha}{4} \begin{cases} 0 & (0 < x < \alpha < 1) \\ \ln(x^2 - \alpha^2) & (\alpha < x < 3\alpha < 1) \\ \ln(x^2 - \alpha^2) + \ln(x^2 - 9\alpha^2) & (3\alpha < x < 5\alpha < 1) \end{cases} \quad (3.3)$$

The limits of validity are those quoted in the reference. The formulae are given without derivation, and we therefore attempt to derive them in order to check their limits of validity.

We start from the Fourier-Bessel expansion of $\ln x$ (see Appendix A) in the interval $0 < x < 1$

$$\sum_{n=1}^{\infty} \frac{J_0(j_n x)}{j_n J_1(j_n)} = -\frac{1}{2} \ln x \quad (3.4)$$

Computer evaluation of this series shows that it is actually valid for $0 < x < 2$.

Since the expansion of any function into a Fourier-Bessel series - (of a given order) is unique,⁴ Eqs. (3.2) to (3.4) show that there must exist relations of the form

$$\sum_{p=0}^{\infty} \epsilon_p \frac{\sin w_{np} x}{w_{np}} = \frac{\alpha}{2} \begin{cases} J_0(j_n x) & (0 < x < 2\alpha) \\ J_0(j_n x) + 2 J_0(j_n \sqrt{x^2 - \alpha^2}) & (2\alpha < x < 4\alpha) \end{cases}$$

(3.5)

$$\sum_{p=0}^{\infty} \epsilon_p (-)^p \frac{\sin w_{np} x}{w_{np}} = \alpha \begin{cases} 0 & (0 < x < \alpha) \\ J_0(j_n \sqrt{x^2 - \alpha^2}) & (\alpha < x < 3\alpha) \\ J_0(j_n \sqrt{x^2 - \alpha^2}) + J_0(j_n \sqrt{x^2 - 9\alpha^2}) & (3\alpha < x < 5\alpha) \end{cases}$$

(3.6)

Since the summation index p is hidden in the square root of w_{np} (see Eq. (3.1)) these sums are not simple Fourier series. However, it is possible to derive such sums as coefficients of a Fourier series as shown in Appendix B.

$$\sum_{p=0}^{\infty} \epsilon_p \frac{\sin y \sqrt{\beta^2 + \pi^2 p^2}}{\sqrt{\beta^2 + \pi^2 p^2}} \cos \pi p z = \frac{1}{2} \begin{cases} J_0 \left(\beta \sqrt{y^2 - z^2} \right) & (0 \leq z < y < 1) \\ 0 & (0 < y < z \leq 1) \end{cases} \quad (3.7)$$

where β is unlimited. Putting $y = x/a$ and $\beta = j_n \alpha$ we obtain the expressions of Eqs. (3.5/3.6) for the lowest range of validity;

$$z = 0: \sum_{p=0}^{\infty} \epsilon_p \frac{\sin \omega_{np} x}{\omega_{np}} = \frac{\alpha}{2} J_0(j_n x) \quad 0 < x < a \quad (3.8)$$

$$z = 1 \quad \sum_{p=0}^{\infty} \epsilon_p (-)^p \frac{\sin \omega_{np} x}{\omega_{np}} = 0 \quad 0 < x < a \quad (3.9)$$

In order to extend the range of validity to larger values of x , we can substitute the integral Eq. (B4) directly into the infinite sum. Taking out terms which do not depend on the summation index yields

$$\sum_{p=0}^{\infty} \epsilon_p \frac{\sin \omega_{np} x}{\omega_{np}} = \int_0^x dt J_0 \left(j_n \sqrt{x^2 - t^2} \right) \sum_{p=0}^{\infty} \epsilon_p \cos \frac{\pi p t}{\alpha} \quad (3.10)$$

Now the sum over cosines is well known, and corresponds to a series of delta functions whenever t/α is a multiple of two, i.e.

$$\sum_{p=0}^{\infty} \epsilon_p \cos \frac{\pi p t}{\alpha} = \sum_{k=-\infty}^{\infty} \delta \left(\frac{t}{\alpha} - 2k \right) = \alpha \sum_k \delta(t - 2k\alpha) \quad (3.11)$$

We also may replace the integral from 0 to x by half the integral from $-x$ to x , since the integrand is even. Then

$$\sum_{p=0}^{\infty} \epsilon_p \frac{\sin \omega_{np} x}{\omega_{np}} = \frac{\alpha}{z} \int_{-x}^{+x} J_0 \left(j_n \sqrt{x^2 - t^2} \right) \sum_k \delta(t - 2k\alpha) dt \quad (3.12)$$

It now depends only on x how many delta functions contribute to the integral. If $x < 2\alpha$ we find again Eq. (3.8) (with extended range!), for $2\alpha < x < 4\alpha$ we have to add $\alpha J_0 \left(j_n \sqrt{x^2 - 4\alpha^2} \right)$ etc. In general, we obtain

$$\sum_{p=0}^{\infty} \epsilon_p \frac{\sin \omega_{np} x}{\omega_{np}} = \alpha \sum_{k=0}^{\left[\frac{x}{2\alpha} \right]^*} \epsilon_k J_0 \left(j_n \sqrt{x^2 - (2k\alpha)^2} \right) \quad (3.13)$$

and similarly for the alternating sum

$$\sum_{p=0}^{\infty} \epsilon_p (-)^p \frac{\sin \omega_{np} x}{\omega_{np}} = \alpha \sum_{k=1}^{\left[\frac{x+\alpha}{2\alpha} \right]} J_0 \left(j_n \sqrt{x^2 - (2k-1)^2 \alpha^2} \right) \quad (3.14)$$

where the series on the RHS terminate when k is larger than the integer part of respectively $\frac{x}{2\alpha}$ and $\frac{x+\alpha}{2\alpha}$. These functions are shown in Figs. 1 and 2.

* Square brackets indicate the integer part of the value enclosed.

Together with Eq. (3.4), these expressions yield the summation formulae Eqs. (3.2) and (3.3) with the only restriction $0 < x < 2$. The various terms on the RHS express the reflections of the fields at the two side-walls, but the condition $x < 2$ restricts the instant of observation to times before any reflections from the cylindrical wall have arrived at the axis. The expressions are valid at all times only for a cavity with the cylindrical wall removed to infinity, i.e., for two parallel plates. This case has also been solved directly in the time domain,³ and the equivalence of the two solutions has been established.

IV. Reflections from the Cylindrical Wall

In order to extend the validity of the solutions to all times for a cavity with a finite radius, we need the sum of the series given in Eq. (3.4) for arbitrary values of x . Unfortunately, we did not find an analytic expression, but this function of a single variable is easily evaluated by computer.

The function is shown in Fig. 3, and we see that it changes its character at every even integer value of x . We are thus led to define

$$\sum_{n=1}^{\infty} \frac{J_0(j_n x)}{j_n^2 J_1^2(j_n)} = f_0(x) \quad (4.1)$$

$$\text{where} \quad f_0(x) \approx \frac{1}{2} \ln x \quad \text{for } 0 < x < 2 \quad (4.2)$$

while it is only defined by the sum for $x \geq 2$.

Using this function, we are now in a position to obtain the general result for the two double series,

$$\sum_{p=0}^{\infty} \sum_{n=1}^{\infty} \frac{\epsilon_p}{j_n^2 j_1^2(j_n)} \frac{\sin \omega_{np} x}{\omega_{np}} = \alpha \sum_{k=0}^{\left[\frac{x}{2\alpha}\right]} \epsilon_k f_0 \left(\sqrt{x^2 - (2k\alpha)^2} \right) \quad (4.3)$$

and

$$\sum_{p=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-)^p \epsilon_p}{j_n^2 j_1^2(j_n)} \frac{\sin \omega_{np} x}{\omega_{np}} = \alpha \sum_{k=1}^{\left[\frac{x+\alpha}{2\alpha}\right]} f_0 \left(\sqrt{x^2 - (2k-1)^2 \alpha^2} \right) \quad (4.4)$$

The fields can thus be obtained from a single finite sum, which usually consists only of a few terms if the instant of observation is not too long after the passage of the pulse.

V. Wakefields for $v = c$

We now have all the expressions required to calculate wakefields for beams traveling with light-velocity. For $z_0 < L$ we use Eq. (2.8a), which we rewrite with the normalized variables

$$\xi = \frac{z_0}{R}, \quad \alpha = \frac{L}{R}, \quad \omega_{(np)} = k_{(np)} R \quad (5.1)$$

to get for $0 < \xi < 1$

$$v_a(\xi) = \frac{2}{\pi \epsilon_0 c} \sum_{n=1}^{\infty} \frac{1}{j_n^2 J_n^2(j_n)} \sum_{p=0}^{\infty} (-)^p \frac{\sin \omega(\xi + \alpha)}{\omega^p} - 2 \frac{\sin \omega \xi}{\omega \alpha} \\ + (-)^p \frac{\sin \omega \alpha}{\omega \alpha} - \frac{\sin \pi p \xi}{\pi p} \quad (5.2)$$

We discuss the last two terms first. Equation (3.14) has a discontinuity at $x = \alpha$, and the sum over 0 for the last-but-one term converges to

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2} [f(\alpha - \epsilon) + f(\alpha + \epsilon)] = \frac{1}{2} \left[0 + J_0 \left(j_n \sqrt{\alpha^2 - \alpha^2} \right) \right] = \frac{1}{2} \quad (5.3)$$

The Fourier-series for the last term is well known

$$\sum_{p=0}^{\infty} \epsilon_p \frac{\sin \pi p \xi}{\pi p} = \frac{1}{2} \quad (0 < \xi < 2) \quad (5.4)$$

Hence the last two terms cancel each other, and we are left with the first two terms which yield with Eqs. (4.3) and (4.4)

$$v_a(\xi) = \frac{2}{\pi \epsilon_0 c} \left[f_0 \left(\sqrt{\xi(\xi + 2\alpha)} \right) - f_0(\xi) \right] \quad (0 < \xi < \alpha) \quad (5.5)$$

For $\alpha < \frac{2}{\sqrt{3}} (R > \frac{\sqrt{3}}{2} L)$ the argument of $f_0(x)$ remains below two. Reverting to the original variables we get

$$v_a(z_0) = -\frac{1}{2\pi \epsilon_0 c} \ln \left(1 + 2 \frac{L}{z_0} \right) \quad (0 < z_0 < L) \quad (5.6)$$

in agreement with Ref. (2) (which was actually not derived for this region of $z_0 < L$!)

For $z_0 > L$ ($\xi > \alpha$) we rewrite Eq. (2.8b) with normalized variables

$$v_b(\xi) = \frac{2}{\pi \epsilon_0 c} \sum_{n=1}^{\infty} \frac{1}{j_n^2 J_1^2(j_n)} \sum_{p=0}^{\infty} \epsilon_p \left[(-)^p \frac{\sin \omega(\xi+\alpha)}{\omega} - 2 \frac{\sin \omega \xi}{\omega} + (-)^p \frac{\sin \omega(\xi-\alpha)}{\omega} \right] \quad (5.7)$$

Performing the summation over p with Eqs. (3.13) and (3.14) we obtain

$$v_b(\xi) = \frac{2}{\pi \epsilon_0 c} \sum_{n=1}^{\infty} \frac{1}{j_n^2 J_1^2(j_n)} \left[\sum_{k=1}^{\left[\frac{\xi}{2\alpha} + 1 \right]} J_0(j_n a_{1k}) - 2 \sum_{k=0}^{\left[\frac{\xi}{2\alpha} \right]} \epsilon_k J_0(j_n a_{2k}) + \sum_{k=1}^{\left[\frac{\xi}{2\alpha} \right]} J_0(j_n a_{3k}) \right] \quad (5.8)$$

Where

$$\begin{aligned} a_{1k}^2 &= (\xi + \alpha)^2 - (2k - 1)^2 \alpha^2 \\ a_{2k}^2 &= \xi^2 - (2k)^2 \alpha^2 \\ a_{3k}^2 &= (\xi - \alpha)^2 - (2k - 1)^2 \alpha^2 \end{aligned} \quad (5.9)$$

For $\xi < 2\alpha$ ($z_0 < 2L$) only the first two sums contribute a single term each. If furthermore $\sqrt{\xi(\xi + 2\alpha)} < 2$, corresponding to $z_0 < \sqrt{L^2 + 2R^2} - L$, both arguments a_1 and a_2 are less than 2 and we find with Eq. (4.2)

$$v_b(\xi) = \frac{2}{\pi \epsilon_0 c} \left[-\frac{1}{2} \ln \sqrt{\xi (\xi + 2\alpha)} + \frac{1}{2} \ln \xi \right] \quad (5.10)$$

Reverting to the original variables, we thus find again

$$v_b(z_0) = -\frac{1}{2\pi \epsilon_0 c} \ln \left(1 + \frac{2l}{z_0} \right) \quad (5.11)$$

Valid for $L < z_0 < \min(2L, \sqrt{L^2 + 2P^2} - L)$. This is the same expression as Eq. (5.6) and in full agreement with Ref. 2 where R was assumed to be larger than 2L.

For any value $z_0 > L$ ($\xi > \alpha$) we obtain with Eqs. (4.3) and (4.4)

$$v_b(\xi) = \frac{2}{\pi \epsilon_0 c} \left[\sum_{k=1}^{\left[\frac{\xi}{2\alpha} + 1\right]} f_0(a_{1k}) - 2 \sum_{k=0}^{\left[\xi/2\right]} \epsilon_k f_0(a_{2k}) + \sum_{k=1}^{\left[\xi/2\right]} f_0(a_{3k}) \right] \quad (5.12)$$

where the a_{1k} are given by Eq. (5.9).

Sometimes it is more convenient to rewrite the expression for the wakefield in terms of

$$\zeta = z_0/L (= \xi/\alpha) \quad (5.13)$$

We can combine Eqs. (5.6) and (5.12) to write

$$\begin{aligned}
\frac{\pi \epsilon_0 c}{2} V(\zeta) &= f_0 \left(\alpha \sqrt{\zeta(\zeta + 2)} \right) - f_0(\alpha \zeta) \\
&+ \sum_{k=1}^{\lfloor \zeta/2 \rfloor} \left[f_0 \left(\alpha \sqrt{(\zeta + 1)^2 - (2k + 1)^2} \right) - 2f_0 \left(\alpha \sqrt{\zeta^2 - (2k)^2} \right) \right. \\
&\left. + f_0 \left(\alpha \sqrt{(\zeta - 1)^2 - (2k - 1)^2} \right) \right] \quad (5.14)
\end{aligned}$$

These finite series are readily evaluated by computer, and the wake field is shown for three different values of α in Figs. 4 to 7. The program "WAKEFUN" can be used to evaluate $V(\zeta)$ for any value of α , and could be easily modified to calculate modes with $m \neq 0$.

For $\alpha \rightarrow 0$ ($R \rightarrow \infty$), we can replace all $f_0(x)$ in Eq. (5.14) by $-1/2 \ln x$. Combining the various logarithmic terms we find

$$V(\zeta) = \frac{1}{2\pi\epsilon_0 c} \ln \left(1 + \frac{2}{\zeta + 2 \lfloor \zeta/2 \rfloor} \right) \quad (5.15)$$

valid for all $\zeta > 0$, which is the general solution for the parallel plate resonator.

VI. Higher Azimuthal Modes and $v \neq c$

Only the aximuthally symmetric mode ($m = 0$) has a longitudinal electric field on the axis of the cavity. Higher-order azimuthal modes thus can only be excited if the beam is displaced from the axis (or is oscillating transversely).

For fields in a pillbox cavity, we essentially need only change the order of the Bessel function $J_0(x)$ to $J_m(x)$ to generalize the fields to

an arbitrary mode-number m ($m=1$ dipole, $m=2$ quadrupole mode, etc.) However, if we evaluate the fields for a displaced beam, we have an additional factor $J_m\left(j_{mn} \frac{d}{R}\right)$ in the expansion. The sums over p can be performed as for $m=0$, and we can use Eqs. (3.13) and (3.14) but now $\omega_{np}^2 = j_{mn}^2 + \left(\frac{\pi p}{\alpha}\right)^2$. For the sum over n we thus have to evaluate series of the form

$$\sum_{n=1}^{\infty} \frac{J_0(j_{mn}x) J_m(j_{mn}y)}{j_{mn}^2 J_{m-1}^2(j_{mn})} \quad (6.1)$$

For small values of $y = d/R$, we can approximate $J_m(j_{mn}y)$ by $j_{mn}^m y^m / 2^m m!$ and use Eqs. (4.3) and (4.4) if we replace $f_0(x)$ by $J_m(j_{mn}y)$

$$f_m(x) = \sum_{n=1}^{\infty} \frac{J_0(j_{mn}x)}{j_{mn}^{2-m} J_{m-1}^2(j_{mn})} \quad (6.2)$$

The function $f_1(x)$ has been evaluated by computer and is shown in Fig. 2.

For the case $v \neq c$, the main difficulty comes from the term $\left[j_n^2 + (\pi p/\epsilon\gamma)^2\right]^2$ in the denominator of Eq. (2.3). In order to perform the summation over p , we would now need an analytic summation of the two series

$$g_{\pm}'(a,b,x) = \sum_{p=0}^{\infty} (\pm)^p \epsilon_p \frac{\cos x \sqrt{\pi^2 p^2 + b^2}}{(\pi^2 p^2 + a^2)^2} \quad (6.3)$$

In the limited time available we have not found a closed-form expression for this series. However, the series converges quite rapidly and can easily be evaluated by computer. Using this function, we can perform the summation over n in Eq. (2.3) except for the last term for $z_0 < L$, which turns out to be a Fourier series which can be summed analytically.⁶ However, the series over n converge very slowly, and it is again better to integrate Eq. (2.3) over z_0 to obtain the wakefield of a step function. Then we need the two functions

$$g_{\pm}(a,b,x) = \sum_{p=0}^{\infty} \epsilon_p \frac{(\pm)^p}{(\pi^2 p^2 + a^2)^2} \frac{\sin x \sqrt{\pi^2 p^2 + b^2}}{(\pi^2 p^2 + a^2)^2} \quad (6.4)$$

Because of the two parameters a and b , the computational effort is much larger, and has not been carried out as yet.

VII. Conclusions

The wakefields of a step-function current in a pillbox cavity have been calculated for $v = c$. The expressions are valid for cavities of arbitrary length and radius, thereby removing the limitations of Ref. 2, which essentially gave only the results for a parallel-plate resonator. The wake field has been derived explicitly for azimuthally symmetric modes, but the generalization to azimuthally varying modes has been indicated. Inspection of Figs. 5 to 8 shows that the wakefields have many discontinuities, corresponding to the effects of reflected waves from the cylindrical wall and side-walls. For realistic cavities with

side-holes, the reflections from the cylindrical wall should not be changed considerably, while the side-wall reflections will be delayed and reduced in strength as they come only from annular regions excluding the axis. However, the character of the wake fields should essentially remain unchanged.

For beam velocities smaller than c , there should again be little change from the results shown if $\gamma \gg 1$. No explicit expressions have been obtained for small γ , but the time-consuming evaluation of the double series can again be reduced by defining auxiliary functions with a reduced number of parameters.

Note added after proof: The derivation of the summation formulae in Eqs. (3.2) and (3.3) has been published in Report 73-A1 of the University of Limoges, France, by M. Chatard-Moulin, B. Jecko, and A. Papiernik (1973).

Appendix A: Fourier-Bessel Expansion of $\ln x$

The Fourier-Bessel expansion of a properly behaved function defined for $0 < x < 1$ can be obtained from the series

$$f(x) = \sum_{n=1}^{\infty} a_n J_0(j_n x) \quad (A.1)$$

where the j_n 's are the n^{th} zeros of $J_0(x)$. Since the functions $J_0(j_n x)$ form a complete, orthogonal set in the interval $(0,1)$ with weight-function x , we can calculate the expansion coefficients a_n uniquely.

The "norm" is given by

$$\int_0^1 J_0^2(j_n t) dt = \frac{1}{2} J_1^2(j_n) \quad (A.2)$$

and we thus obtain

$$a_n = \frac{2}{J_1^2(j_n)} \int_0^1 f(t) J_0(j_n t) t dt \quad (A.3)$$

fortunately, the integral of $\ln t J_0(\lambda t) t$ can be found in tables⁵

$$\int_0^1 \ln t J_0(\lambda t) t dt = \frac{1}{\lambda^2} [J_0(\lambda) - 1] \quad (A.4)$$

with $\lambda = j_n$, the $J_0(\lambda)$ term on the RHS vanishes and we obtain

$$a_n = - \frac{2}{j_n^2 J_1^2(j_n)} \quad (A.5)$$

The series for $\ln x$ thus becomes

$$\sum_{n=1}^{\infty} \frac{J_0(j_n x)}{j_n^2 J_1^2(j_n)} = -\frac{1}{2} \ln x \quad (\text{A.6})$$

The derivation is valid for $0 < x \leq 1$, but computer evaluation of the sum shows that it is actually good for $0 < x < 2$. For $x = 0$, the series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{j_n^2 J_1^2(j_n)} = -\frac{1}{2} \ln(0) \rightarrow \infty \quad (\text{A.7})$$

An alternate way of obtaining these series is to expand

$$f(x) = \begin{cases} 1 & (0 < x < y) \\ 0 & (y < x < 1) \end{cases} \quad (\text{A.8})$$

With the integral

$$\int_0^1 f(t) J_0(j_n t) t dt = \int_0^y J_0(j_n t) t dt = \frac{y}{j_n} J_1(j_n y) \quad (\text{A.9})$$

we obtain the series

$$\sum_{n=1}^{\infty} \frac{J_1(j_n y)}{j_n^2 J_1^2(j_n)} J_0(j_n x) = \begin{cases} \frac{1}{2y} & (0 < x < y) \\ 0 & (y < x < 1) \end{cases} \quad (\text{A.10})$$

Taking $x = 0$ and integrating over y yields the same result as above.

Appendix B: Fourier Series with Square Roots as Arguments

In general, an expansion of a function $f(x)$ into a series of functions is only possible if these form a complete set. In order to calculate the expansion coefficients, it is also important that the functions are orthogonal over the interval of interest. The circular functions of argument $\sqrt{n^2 + a^2}$ unfortunately do not meet these criteria, and we cannot find an expansion in these functions directly.

It is possible, however, to find a regular Fourier series in which the expansion coefficients contain the desired circular functions with square roots as arguments.

We start by expanding the function

$$f(x) = \begin{cases} J_0 \left(\beta \sqrt{y^2 - x^2} \right) & (0 < x < y) \\ 0 & (y < x < 1) \end{cases} \quad (B.1)$$

into a cosine series

$$f(x) = \sum_{n=0}^{\infty} \epsilon_n a_n \cos \pi n x \quad (B.2)$$

The coefficients are then given by the integral

$$a_n = 2 \int_0^y J_0 \left(\beta \sqrt{y^2 - x^2} \right) \cos \pi n x \, dx \quad (B.3)$$

This integral can be found in tables.⁶

$$\int_0^y J_0 \left(\beta \sqrt{y^2 - x^2} \right) \cos \alpha x \, dx = \frac{\sin y \sqrt{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2}} \quad (B.4)$$

with $\alpha = \pi n$ we obtain the coefficients a_n and hence the series

$$\sum_{n=0}^{\infty} \epsilon_n \frac{\sin y \sqrt{\pi^2 n^2 + \beta^2}}{\sqrt{\pi^2 n^2 + \beta^2}} \cos \pi n x = \begin{cases} \frac{1}{2} J_0 \left(\beta \sqrt{y^2 - x^2} \right) & (0 \leq x < y < 1) \\ 0 & (0 < y < x \leq 1) \end{cases}$$

used in the text. This series has been summed by computer in order to verify the expansion and to find the extension for $x, y > 1$ discussed in the text.

The same series can be obtained as a Fourier-Bessel series using Sonine's second integral,⁴ with half-integer values for the Bessel functions.

References

1. P. Morton, K. Neil, LRL Report 18103 (1968) p. 365.
2. Papiernik et al., Proc. 9th Inter. Conf. Part. Accel., 1974, 375.
3. A. Chao, P. Morton, SPEAR Report 105 (1975).
4. G. Watson, Theory of Bessel Functions, Cambridge, 1962, Section 18.6.
5. A. Wheelon, Table of Summable Series, Holden-Day, 1969.
6. B. Zotter, CERN-ISR/78-12 (unpublished).
7. I. Gradshteyn, I. Ryzhik, Tables of Integrals, Eq. 6.677.6.

Fig. 1

Fig. 2

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Fig. 3

Fig. 4

Fig. 5

Fig. 6

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Fig. 7

Fig. 7