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Radial Diffusion of Resonant Particles in Tandem Mirrors*

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ABSTRACT

The radial diffusion of resonant particles in the banana regime and the plateau regime is derived in detail from kinetic theory. The plateau diffusion, calculated in the τ -approximation, confirms the result given by Ryutov-Stupakov without derivation. The banana diffusion also agrees with the R-S result provided the latter is corrected for some dependence of the collision frequency on ε and ϕ .

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I. Introduction and Summary

Owing to the lack of axial symmetry associated with the quadrupole fields in a tandem mirror system, ions can diffuse radially with an enhanced rate.¹⁻³ Depending on whether the longitudinal bounce frequency ω_{bounce} is greater or smaller than the azimuthal drift frequency ω_{drift} , this diffusion can be classified as the "neoclassical" diffusion and the "resonant" diffusion. In the case of $\omega_{\text{bounce}} \gg \omega_{\text{drift}}$, the radial displacements of the ions Δr incurred in the short transition regions at either end of the tandem system tend to cancel each other because of the relative orientations of the fields in the two regions (omnigeneity), rendering the neoclassical diffusion less important. On the other hand, when $\omega_{\text{drift}} > \omega_{\text{bounce}}$ the radial displacements at both ends can reinforce each other when the change of the azimuthal drift angle $\Delta\psi$ during the ion motion from one end to the other is just right, ie. $\Delta\psi(r, \mu, \epsilon) = k\pi/2$, $k = \text{odd integers}$. Those particles which satisfy the above condition almost exactly will accrue the radial excursions continuously to a large maximum value $x_{\text{res}} \sim \sqrt{r \cdot R}$, where R is the radius of the plasma, provided the radial drift motion is not interrupted by collisions. Consequently, when the collision frequency ν_i is sufficiently small, ie. $\nu_i t_{\parallel} \ll (\Delta r/R)^{3/2}$, the radial diffusion is dominated by such resonant particles with large step sizes, leading to a rate proportional to ν_i the banana diffusion. The "mildly resonant" particles which lie a little farther away from the resonance curve can also have the radial displacement built up to an appreciable value $x_{M,R} \sim \Delta r(\epsilon/\Delta\epsilon) \sim \Delta r(\nu_i t_{\parallel})^{1/3}$ where $\Delta\epsilon$ is the deviation from the resonance energy ϵ_k . When collisions are sufficiently frequent to force the resonant particles into a local Maxwellian distribution but not frequent enough to do so for the mildly resonant particles because the latter occupy

a larger volume in phase space, the diffusion will be dominated in the regime $(\Delta r/R)^{3/2} \ll v_i t_{\parallel} \ll 1$ by the mildly resonant particles - the plateau diffusion. This regime is characterized by a diffusion rate independent of v_i , and, according to present thinking,⁴ is most relevant to the tandem mirror system.

The kinetic-theoretic derivation of the radial diffusion can be carried out for the different regimes separately. In the resonant plateau regime, we can pattern after Galeev and Sagdeev⁵ in a first calculation by neglecting the $\dot{r} \frac{\partial \delta f}{\partial r}$ term in comparison with the $\dot{\psi} \frac{\partial \delta f}{\partial \psi}$ term and using the τ -approximation to the collision integral in the drift-kinetic equation. The detailed expression for the diffusion coefficient contains the qualitative features expected on physical ground, $D_{\text{plat}} \sim (\Delta r)^2 / t_{\parallel}$, and confirms that given without derivation in Ref. 3. The calculation for the banana diffusion is more complicated mathematically. The τ -approximation can no longer be used. Instead, recognizing the rapid variation of the resonant particle distribution across the narrow strip of resonance in phase space, we can approximate the collision integral by just keeping the highest order derivatives in the direction perpendicular to the resonance curve. A change of variables with one of them arranged to vary in just that direction greatly simplifies the calculation. Furthermore, we shall not neglect the $\dot{r} \frac{\partial \delta f}{\partial r}$ term in the kinetic equation since the other term, $\dot{\psi} \frac{\partial \delta f}{\partial \psi}$, proportion to $\dot{\psi}$, can vanish for particles at the turning points of their banana orbits. The left hand side of the kinetic equation for $\delta f \equiv f - f_M$, now containing derivatives of δf with respect to both r and ψ , can yet be simplified and reduced to one derivative (with respect to ψ) term by changing the variable r to $J(r, \psi)$, a constant of the unperturbed orbital motion. However, such a change also splits the single collision term (the

right hand side) into many terms with different derivatives. Fortunately, in the limit of rare collisions, a perturbation expansion in terms of v_i can be used. The lowest order component $g^{(0)}$ of δf , which is odd with respect to the sign of ψ , can be solved for with the help of the boundary conditions that it vanishes at the turning point where $\dot{\psi} = 0$, remains finite as $J \rightarrow \infty$ and is continuous across the separatrix. The averaged radial flux which can be shown to depend only on $h^{(1)}$, the lowest order component of δf that is even with respect to the sign of $\dot{\psi}$ and odd with respect to $\psi \rightarrow -\psi$, can be immediately calculated by using the known relation coupling $h^{(1)}$ to $g^{(0)}$. The resulting expression for the diffusion coefficient contains the expected qualitative features:⁶

$$D_{\text{ban}} \sim \frac{|\Delta r|^{\frac{1}{2}} \left| \frac{\partial \Delta \psi}{\partial \epsilon} \right|}{\left| \frac{\partial \Delta \psi}{\partial r} \right|^{3/2}} \epsilon_k v_i$$

Although the above expression appears quite different from that given in Ref. 3, the actual difference lies in just a modification of the dependence of v_i on the resonant particle energy ϵ_k and the electric potential ϕ . In the high collision frequency regime, $v_i t_{\parallel} \gg 1$, the particles have to random-walk from one transition region to the other, leading to a diffusion coefficient $D_{\text{coll.}} \sim (\Delta r)^2 / \tau_{\text{ran.walk}} \sim (\Delta r)^2 / t_{\parallel}^2 v_i$.

Section II describes the drift orbits. Section III is devoted to the kinetic-theoretic derivation of the plateau diffusion, Sec. IV to that of the banana diffusion. Some comments and remarks are made in the final section.

II. The Drift Orbits

In the long axially symmetric central section of the tandem mirror system (see Fig. 1) the particle will drift, on account of the radial electric field and the radial gradient of the magnetic field amplitude, around the mirror axis, resulting in a change $\Delta\psi(r, \epsilon, \mu)$ in the azimuthal angle as it goes from one end to the other. Owing to the axial symmetry of this central section no radial drift occurs. Upon entering and re-emerging from the short transition region at either end, the particle will incur a radial displacement Δr while the simultaneous azimuthal displacement is negligible compared to $\Delta\psi$. As shown in Refs. 3 and 4, Δr generally depends on r, ψ, μ, ϵ and, in the lowest order expansion in our asymmetry parameter R/L_{tr} , can be written as

$$\Delta r \approx a_1(r, \mu, \epsilon) \cos 2\psi \quad (1)$$

where

$$a_1 \sim t_{tr} v_r \sim \frac{L_{tr}}{v_{||}} \frac{B_{||}}{B^2} \frac{\frac{B_{\psi}}{L_{tr}}}{\Omega} \frac{v_{\perp}^2}{\Omega} \\ \sim \frac{L_{tr}}{v_{||}} B_{||} \frac{\left(B_{||} \frac{R}{L_{tr}} \right) \frac{1}{L_{tr}} v_{\perp}^2}{B^2 \Omega} \sim \rho_i \frac{R}{L_{tr}} \quad (2)$$

Note that $\Delta r \neq 0$ can be entirely ascribed to the asymmetric quadrupole field in the transition region.

In one axial round trip, the changes in r and ψ are given by

$$r_{n+1} - r_n = a_1(r_n) \cos 2\left(\psi_n + \frac{1}{2} \Delta\psi(r_n)\right) - a_1 \cos 2\left(\psi_n + \frac{3}{2} \Delta\psi\right) + \mathcal{O}(a_1^2) \quad (3)$$

$$\psi_{n+1} - \psi_n = 2\Delta\psi(r_n). \quad (4)$$

where the opposite signs in the two terms on the right hand side of Eq. (3) reflect the relative orientations of the quadruple fields in the two transition regions. When $\Delta\psi$ is small, the radial displacements nearly cancel. On the other hand, when the resonance condition

$$\Delta\psi(r, \epsilon, \mu) = k\pi/2, \quad k = 1, 2, 5, \dots \quad (5)$$

or

$$\epsilon = \epsilon_k(r, \mu) \quad (6)$$

is satisfied, they reinforce each other. When μ is given, a particle in the neighborhood of r and ϵ_k will have

$$\Delta\psi(r+x, \epsilon_k + \Delta\epsilon, \mu) = k\frac{\pi}{2} + \psi'_\epsilon \Delta\epsilon + \psi'_r x + \dots \quad (7)$$

where

$$\psi'_\epsilon \equiv \left[\frac{\partial \Delta\psi(r, \epsilon, \mu)}{\partial \epsilon} \right]_{\epsilon=\epsilon_k}, \quad \psi'_r \equiv \left[\frac{\partial \Delta\psi(r, \epsilon, \mu)}{\partial r} \right]_{\epsilon=\epsilon_k}$$

It then follows that

$$\psi_{n+2} - \psi_n = 4\psi'_\epsilon \Delta\epsilon + 4\psi'_r x + 2k\pi, \quad (8)$$

$$x_{n+2} - x_n = 4(-1)^{\frac{k+1}{2}} a_1 \sin 2\psi, \quad (9)$$

leading to the velocities

$$\dot{r} = \frac{\psi'_r x + \psi'_\epsilon \Delta\epsilon}{t_{||}}, \quad (10)$$

$$\dot{x} = \frac{(-1)^{\frac{k+1}{2}}}{t_{||}} a_1 \sin 2\psi, \quad (11)$$

where $t_{\parallel}$ is one axial transit time. The trajectory is represented by the differential equation

$$\frac{d\psi}{dx} = \frac{\psi_r' x + \psi_e' \Delta \epsilon}{(-1)^{\frac{k+1}{2}} a_1 \sin 2\psi}, \quad (12)$$

or its solution

$$J \equiv (x + x_0 \xi)^2 + y(\psi) = \text{const.} \quad (13)$$

where

$$\left. \begin{aligned} x_0 &\equiv \epsilon_k \psi_e' / \psi_r' \\ \xi &\equiv \frac{\epsilon - \epsilon_k(r, \mu)}{\epsilon_k} \\ y(\psi) &\equiv (-1)^{\frac{k-1}{2}} \frac{2a_1(r)}{\psi_r'} \sin^2 \psi \end{aligned} \right\} \quad (14)$$

As usual, the orbits described by Eq. (13) can be divided into the banana orbits and the transiting orbits for $J < J_0$ and $J > J_0$, respectively, where J_0 is given by Eq. (16) and $y(\psi)$ assumed positive.

We now ask: For a particle in the neighborhood of r having energy $\epsilon_k(r, \mu)$ and magnetic moment μ , what is the range $\Delta x \equiv |x|_{\max} - |x|_{\min}$ of its radial displacements? It is easy to see from Eq. (13) that

$$\begin{aligned} x_{\max}^2 &= J \\ x_{\min}^2 &= \begin{cases} J - J_0 & \text{for } J > J_0 \\ 0 & \text{for } J < J_0 \end{cases}, \end{aligned} \quad (15)$$

where

$$J_0 \equiv 2 \left| \frac{a_1}{\psi_r'} \right| = x_{\text{res}}^2 \quad (16)$$

It follows that Δr takes on its maximum value $\Delta x_{\max} = x_{\text{res}}$ when $J = J_0$. Roughly speaking we shall refer to those particles having energy $\epsilon_k(r, \mu)$ and in the x_{res} -neighborhood of r as the "resonant" particles. We may also characterize this resonance region in terms of an energy range at given r, μ , i.e. $\Delta\epsilon < \epsilon_{\text{res}} \equiv \psi_r' x_{\text{res}} / \psi_\epsilon'$, as depicted in Fig. 2. A particle in the neighborhood of r but with an energy ϵ such that $\epsilon > \Delta\epsilon > \Delta\epsilon_{\text{res}}$ will be called a "mildly resonant" (MR) particle. Its orbit can be obtained from Eqs. (10)-(13) by neglecting $\psi_r' x$ compared to $\psi_\epsilon' \Delta\epsilon$, resulting in

$$\dot{\psi} = \psi_\epsilon' \Delta\epsilon / t_{\parallel}$$

and

$$x = \frac{(-1)^{\frac{k-1}{2}} a_1 \cos 2\psi}{2\psi_\epsilon' \Delta\epsilon} + \text{const.} \quad (17)$$

The corresponding radial range $\Delta x_{\text{MR}} = |a_1 / 2\psi_\epsilon' \Delta\epsilon|$ is smaller than that for the resonant particles,⁷ x_{res} , implying a smaller step size for diffusion. However, the energy range for the MR-particles is much larger. The nonresonant (NR) particles with $\Delta\epsilon \sim \epsilon$ have $\Delta x_{\text{NR}} \sim a_1$, which is smaller than Δx_{MR} .

III Kinetic Theory-Plateau Diffusion

We first consider the case in which collisions are sufficiently frequent that the "resonant" particles are forced into a local Maxwellian distribution within the narrow region in phase space depicted in Fig. 2 rather than follow their unperturbed orbits to any significant degree of completion. On the other hand we assume that collisions are negligible for the MR- and the NR-particles. Thus we require $v_{res}^{eff} \tau_{res} \gg 1$ but $v_i t_{\parallel} \ll 1$ or $(a_1/R)^{3/2} \ll v_i t_{\parallel} \ll 1$, where $v_{res}^{eff} \sim v_i (\epsilon/\Delta\epsilon_{res})^2$ is the effective collision frequency and $\tau_{res} \sim x_{res} (a_1/t_{\parallel})^{-1}$ the drift period for the resonant particles.

With the resonance orbits sporting the largest step size demolished, the diffusion in the regime is expected to be dominated by the MR- particles.

The drift-kinetic equation ⁸ is

$$\dot{r} \frac{\partial f}{\partial r} + \dot{\psi} \frac{\partial f}{\partial \psi} = \left(\frac{\partial f}{\partial t} \right)_{coll}. \quad (18)$$

We write the distribution function in the form

$$f(r, \psi, \mu, \epsilon) = f_M(r, \epsilon) + \delta f(r, \psi, \mu, \epsilon) \quad (19)$$

where

$$f_M = n(r) \left[\frac{m_i}{2\pi T_i(r)} \right]^{3/2} \exp \left[- \frac{\epsilon - e\phi_s(r)}{T_i(r)} \right]$$

is the local Maxwellian distribution including, in particular, the thermalized resonant particles and δf describes essentially the MR-particles.

Using Eqs. (17) and (11) for MR-particles in the plateau regimes we can estimate the ratio $\dot{r}/r\dot{\psi}$ and show that $(a_1/R)^{1/2} > \dot{r}/r\dot{\psi} > a_1/R$, where we have also used $[v_i (\epsilon/\Delta\epsilon)^2] \cdot [\Delta x_{MR} (a_1/t_{\parallel})^{-1}] \sim 1$ or $\Delta\epsilon/\epsilon \approx (v_i t_{\parallel})^{1/3}$ for the group of MR-particles contributing dominantly to the diffusion. Substitution of Eqs. (17), (11) into the linearized Eq. (18) and the neglect of $\dot{r} \frac{\partial f}{\partial r}$ compared to $r\dot{\psi} \frac{\partial \delta f}{\partial \psi}$ lead to the simplified equation

$$\frac{1}{t_{\parallel}} \psi_{\epsilon}^* (\epsilon - \epsilon_k) \frac{\partial \delta f}{\partial \psi} + \frac{a_1}{t_{\parallel}} (-1)^{\frac{k+1}{2}} \sin 2\psi \frac{\partial f_M}{\partial r} = -\nu \delta f \quad (20)$$

where the collision integral has been written in the τ -approximation.⁵

The reason why we can use such a crude approximation is that δf for the MR-particles is expected to vary less sharply over a larger volume of phase space (than the resonant particles, say), rendering the exact form of the collision term less crucial.⁹ Equation (20) can be solved immediately to yield

$$\delta f = \frac{(-1)^{\frac{k+1}{2}} a_1}{2} \left\{ \frac{e^{2i\psi}}{2\psi_{\epsilon}^* (\epsilon - \epsilon_k) - i\nu t_{\parallel}} + \frac{e^{-2i\psi}}{2\psi_{\epsilon}^* (\epsilon - \epsilon_k) + i\nu t_{\parallel}} \right\} \frac{\partial f_M}{\partial r} \quad (21)$$

The radial particle flux averaged over ψ can be written as

$$\begin{aligned} \langle q_r \rangle &= \int \langle \dot{r} \, d n \rangle \\ &= \sum_{k=1,3,5,\dots} \int d\mu \, d\epsilon \, \frac{2\pi H}{L m_i^2} (-1)^{\frac{k+1}{2}} a_1 \langle f \sin 2\psi \rangle, \end{aligned} \quad (22)$$

where the different resonances are assumed non-overlapping and contribute independently to the flux. Since $\langle f_M \sin 2\psi \rangle = 0$, the integrand in Eq. (22) is proportional to $\langle \delta f \sin 2\psi \rangle$, given by

$$\langle \delta f \sin 2\psi \rangle = -\frac{1}{4i} (-1)^{\frac{k+1}{2}} a_1 \frac{\partial f_M}{\partial r} \left[\frac{1}{2\psi_{\epsilon}^* (\epsilon - \epsilon_k) - i\nu t_{\parallel}} - \text{c.c.} \right].$$

We may now substitute the above into Eq. (22) and take the limit

$\nu t_{\parallel} \rightarrow 0+$ to obtain

$$\langle q_r \rangle = - \frac{\pi^2 H}{2m_i^2 L} \sum_{k=1,3,5,\dots} \int d\mu \left[\frac{a_1^2(r, \epsilon, \mu)}{|\psi_{\epsilon}|} \frac{\partial f_M}{\partial r} \right]_{\epsilon = \epsilon_k(r, \mu)} \quad (23)$$

Note that the limit of vanishing collision frequency can only be taken at this stage after making all the provisions which ensured that Eq. (20) and its solution δf are for the MR-particles and not for the resonant particles. Otherwise, the 0^+ -collision frequency would certainly lead to diffusion dominated by the resonant particles with large step sizes in their rarely perturbed orbits rather than by the MR-particles in theirs. The fact that $\langle q_r \rangle$ is independent of the collision frequency comes about,⁹ not because collisions play no role, but because the width $\Delta\epsilon$ of the distribution scales as $\nu^{1/3}$ while its magnitude scales as $\nu^{-1/3}$, as in the case of Landau damping.¹⁰

Qualitatively the diffusion coefficient is governed by those MR-particles which have a drift times $\tau_{MR} \approx \Delta x_{MR} (a_1/t_{\parallel})^{-1}$ comparable to the inverse effective collision frequency $(\nu_{MR}^{eff})^{-1} \approx (\nu_i \frac{\epsilon^2}{\Delta\epsilon^2})^{-1}$

Thus,

$$D_{plat} \sim (\Delta x_{MR})^2 (\Delta\epsilon/\epsilon) \nu_{MR}^{eff} \sim a_1^2/t_{\parallel},$$

which is consistent with Eq. (23). Our result also confirms the one given without derivation in Ref. 3.

IV Kinetic Theory - Banana Diffusion

In the regime $v_{\text{res}}^{\text{eff}} \tau_{\text{res}} \ll 1$ or $(a_1/R)^{3/2} \gg v_{i\parallel}$ the resonant particles only suffer rare collisions and are expected to dominate the diffusion because of their large step sizes in the unperturbed orbits. As discussed in Sec. II, the width of the resonant particle distribution is $\Delta \epsilon_{\text{res}} = \frac{\psi_r x_{\text{res}}}{\psi_e}$ which is extremely small. Therefore, the deviation δf from the local Maxwellian distribution will vary sharply across this narrow-resonance strip. This fact may be exploited to approximate the collision integral. At a given point r , the resonance strip runs along the curve $\epsilon = \epsilon_k(r, \mu)$ in the (μ, ϵ) -plane. If we perform a change of variables from (μ, ϵ) to $(\mu, \xi = \frac{\epsilon - \epsilon_k(r, \mu)}{\epsilon_k})$ the resonance region will be a straight strip along the $\xi = 0$ axis, essentially independent of μ in the (μ, ξ) plane. In the collision integral involving derivatives with respect to the velocity components we may first transform them into derivatives with respect to μ and ξ and then use the fact that $\frac{\partial}{\partial \mu} \ll \frac{\partial}{\partial \xi}$ to eliminate all but the second derivative terms with respect to ξ .

The linearized drift-kinetic equation appropriate for the resonant particles takes the form

$$\frac{\omega}{r} \left[\frac{\partial J}{\partial \psi} \frac{\partial f_M}{\partial r} + \frac{\partial J}{\partial \psi} \frac{\partial \delta f}{\partial x} - \frac{\partial J}{\partial x} \frac{\partial \delta f}{\partial \psi} \right] = v_i(r, \mu) \frac{\partial^2 \delta f(x, \psi, \xi)}{\partial \xi^2} \quad (24)$$

where the collision term on the right hand side is derived in the Appendix. In Eq. (24) the velocities $\vec{r}$ and $\vec{\psi}$ given by Eqs. (10) and (11) are now expressed in terms of the invariant $J(x, \psi, \xi)$ of Eq. (13). The variable μ actually plays the role of a parameter and will often not be written out explicitly for convenience. Also, r refers to a fixed spatial point while x is the variable; $\omega \equiv -r\psi_r/2t_{\parallel}$.

We construct from δf two functions in terms of the variables (u, ψ, ξ) where $u = x + x_0 \xi \propto \dot{\psi}$,

$$\begin{aligned} g(u, \psi, \xi) &= \frac{1}{2} [\delta f(u, \psi, \xi) - \delta f(-u, \psi, \xi)], \\ h(u, \psi, \xi) &= \frac{1}{2} [\delta f(u, \psi, \xi) + \delta f(-u, \psi, \xi)], \end{aligned} \quad (25)$$

which have definite symmetry with respect to the change of sign of $\dot{\psi}$. At the turning point of a banana orbit where $u = \dot{\psi} = 0$ or $\psi = \psi_{\text{turn}}$ with $J - y(\psi_{\text{turn}}) = 0$, we see that

$$g(u = 0, \psi, \xi) = 0. \quad (26)$$

The left hand side of Eq. (24) contains derivative terms with respect to both x and ψ . We may reduce it to one derivative term by making a further change¹¹ of variables from (u, ψ, ξ) to $(J = u^2 + y(\psi), \psi, \xi)$, exploiting the property of J as an invariant of the unperturbed motion. Equation (24) then gives rise to two coupled equations¹² for g and h :

$$\begin{aligned} -\frac{\omega}{r} \frac{\partial J}{\partial x} \frac{\partial g(J, \psi, \xi)}{\partial \psi} &= -\frac{\omega}{r} \frac{\partial J}{\partial \psi} \frac{\partial f_M}{\partial r} + \\ &+ \nu_i \left\{ \left[x_0^2 \left(2u \frac{\partial}{\partial J} \right)^2 + \frac{\partial^2}{\partial \xi^2} \right] h + 4x_0 \frac{\partial}{\partial \xi} \left[u \frac{\partial}{\partial J} g \right] \right\}, \end{aligned} \quad (27)$$

$$-\frac{\omega}{r} \frac{\partial J}{\partial x} \frac{\partial h(J, \psi, \xi)}{\partial \psi} = \nu_i \left\{ \left[x_0^2 \left(2u \frac{\partial}{\partial J} \right)^2 + \frac{\partial^2}{\partial \xi^2} \right] g + 4x_0 \frac{\partial}{\partial \xi} \left[u \frac{\partial}{\partial J} h \right] \right\}, \quad (28)$$

where $u = \sigma \sqrt{J - y(\psi)}$,
 $\sigma = \text{sign}(x + x_0 \xi)$.

Although the left hand sides of Eqs. (27) and (28) are simplified by the transformation the collision terms on the right hand sides have proliferated. These equations still involve three variables ψ, J, ξ in an essential way.¹³ The real simplification achieved is in shifting the complication from the left hand side to the terms multiplied by the small parameter ν_i .

on the right hand side. We solve for g and h in the form of perturbation expansions:

$$g = g_0 + v^2 g_2 + \dots, \quad h = h_1 + \dots \quad (29)$$

In the v^0 order, the equation

$$\frac{\partial g_0(J, \psi, \xi)}{\partial \psi} = - \frac{\sigma \frac{\partial y}{\partial \psi} \frac{\partial f_M}{\partial r}}{2 \sqrt{J - y}}$$

can be immediately integrated to yield

$$g_0 = - \sigma \sqrt{J - y(\psi)} \frac{\partial f_M}{\partial r} + C(J, \xi) \quad (30)$$

where $C(J, \xi)$ is to be determined from boundary conditions. This function $C(J, \xi)$ takes on different forms for banana (trapped) particles and transiting particles. The separatrix is determined by

$$J = J_c \equiv \max y(\psi) = \begin{cases} 2 \left| \frac{a_1}{\psi_r} \right|, & \text{when } (-1)^{\frac{k-1}{2}} a_1 / \psi_r > 0 \\ 0, & \text{when } (-1)^{\frac{k-1}{2}} a_1 / \psi_r < 0 \end{cases} \quad (31)$$

For the banana particles (indexed by b) or when $J < J_c$, the boundary condition (26) at the turning point $\psi = \psi_{\text{turn}}$ applies, leading to $C^b(J, \xi) = 0$ or

$$g_0^{(b)} = - \sigma \sqrt{J - y(\psi)} \frac{\partial f_M}{\partial r} \quad (32)$$

For the transiting particles (indexed by t) or when $J > J_c$ we have to resort to Eq. (28) for the determination of $C^{(t)}(J, \xi)$. Substitution of Eq. (30) into Eq. (28) yields, to order v^1 ,

$$\frac{\partial h_1^{(t)}(J, \psi, \xi)}{\partial \psi} = \frac{-r v_i \sigma}{2\omega} \frac{1}{\sqrt{J - y}} \left[4x_0^2 \left(u \frac{\partial}{\partial J} \right)^2 + \frac{\partial^2}{\partial \xi^2} \right] g_0^{(t)}(J, \psi, \xi). \quad (33)$$

Upon integrating over ψ from $-\pi/2$ to $\pi/2$ the periodicity of the transiting particles leads to, as long as v_i does not vanish identically,

$$4x_0^2 \frac{\partial}{\partial J} \left[\int_{-\pi/2}^{\pi/2} d\psi \sqrt{J - y(\psi)} \frac{\partial c(t)}{\partial J} \right] + \frac{\partial^2 c(t)}{\partial \xi^2} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{\sqrt{J - y(\psi)}} = 0, \quad (34)$$

which can be solved by separation of variables. However, the joining¹⁴ of $g_0^{(b)}$ to $g_0^{(t)}$ at $J = J_c$ requires that $c^{(t)}$ be independent of ξ since $g_0^{(b)}$ is independent of ξ . Thus we obtain

$$c^{(t)}(J) = K \int_{J_c}^J \frac{dJ}{E(J)}, \quad (35)$$

where

$$E(J) \equiv \int_{-\pi/2}^{\pi/2} d\psi \sqrt{J - y(\psi)}.$$

To determine the constant K in Eq. (35) we require that $g_0^{(t)}(J, \psi, \xi)$ be finite as $J \rightarrow \infty$, yielding $K = \sigma \frac{\pi}{2} \frac{\partial f_M}{\partial r}$. Hence,¹⁵

$$g_0^{(t)} = -\sigma \left[\sqrt{J - y(\psi)} - \frac{\pi}{2} \int_{J_c}^J \frac{dJ}{E(J)} \right] \frac{\partial f_M}{\partial r}. \quad (36)$$

In the limit $J \rightarrow \infty$,

$$\lim_{J \rightarrow \infty} g_0^{(t)}(J, \psi) = \sigma \frac{\pi}{2} \frac{\partial f_M}{\partial r} \left[\int_{J_c}^{\infty} dJ \left(\frac{1}{E(J)} - \frac{1}{\pi J^{1/2}} \right) - \frac{2}{\pi} J_c^{1/2} \right], \quad (37)$$

which is finite and independent of ψ, ξ .

We now turn to the calculation of the radial flux. Equation (22) gives

$$\langle q_r \rangle = -\frac{\pi H}{m_i L} \sum_{k=1,3,5,\dots} \left\langle \int \psi_r \frac{\partial J}{\partial \psi} \delta f d\mu d\varepsilon \right\rangle. \quad (38)$$

With $\delta f = g + h$ according to Eq. (25) and the fact that g_0 is even while $\frac{\partial J}{\partial \psi}$ is odd in ψ the ψ -averaged flux can be written as

$$\langle q_r \rangle = - \frac{\pi H}{m_i L} \sum_k \frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi \int d\mu \int d\xi \epsilon_k \psi_r \frac{\partial y}{\partial \psi} h(J, \psi, \xi) \quad (39)$$

where we have used the variable ξ rather than ϵ in the integration. We shall argue that $h(J, \psi, \xi)$ depends on J, ψ but only on the sign of ξ .

Firstly Eq. (28) shows that, to the v^1 order, $\frac{\partial h(J, \psi, \xi)}{\partial \psi}$ does not depend on ξ except for its sign since g_0 has been shown to be independent of ξ .

Secondly the fact that $h(u, \psi, \xi)$ is even in u according to Eq. (25) leads to the boundary condition at the turning point that $\lim_{u \rightarrow 0} \frac{\partial h}{\partial u}(u, \psi, \xi) = 0$ or, using J instead of u , that

$$\lim_{\psi \rightarrow \psi_{\text{turn}}} \sqrt{J - Y(\psi)} \frac{\partial h}{\partial J}(J, \psi, \xi) = 0.$$

Thus both the differential equation and the boundary condition for h are independent of ξ if h is self-consistently assumed to be so.

To find the flux at the spatial point r we set $x = 0$. Define

$$J_2 \equiv x_0^2 \xi^2 + y(\psi) = J(x = 0, \psi, \xi).$$

$$h_0(J_2, \psi) \equiv [h(J, \psi, \xi)]_{x=0}, \quad g_0(J_2, \psi) \equiv [g_0(J, \psi, \xi)]_{x=0}.$$

It then follows from Eq. (28) that, to order v^1 ,

$$\frac{\partial h_0(J_2, \psi)}{\partial \psi} = - \frac{2\pi r x_0^2 v_1}{\omega} \frac{\partial}{\partial J_2} \left[\sqrt{J_2 - y(\psi)} \frac{\partial g_0(J_2, \psi)}{\partial J_2} \right]. \quad (40)$$

Using ψ_2 to replace $x_0 \xi = \sigma \sqrt{J_2 - y}$ as integration variable, so that

$$\int_{-\infty}^{\infty} d\xi \longrightarrow - \int_{\infty}^{y(\psi)} \frac{dJ_2}{2|x_0| \sqrt{J_2 - y(\psi)}} + \int_{y(\psi)}^{\infty} \frac{dJ_2}{2|x_0| \sqrt{J_2 - y(\psi)}}.$$

we obtain

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi \int_{-\infty}^{\infty} d\xi \frac{\partial y}{\partial \psi} h(J_2, \psi) &= \frac{1}{2\pi |x_0|} \int_{-\pi}^{\pi} d\psi \left[\int_{-\infty}^{y(\psi)} dJ_2 \frac{\partial \sqrt{J_2 - y(\psi)}}{\partial \psi} h_-(J_2, \psi) \right. \\ &\quad \left. - \int_{y(\psi)}^{\infty} dJ_2 \frac{\partial \sqrt{J_2 - y(\psi)}}{\partial \psi} h_+(J_2, \psi) \right] \end{aligned} \quad (41)$$

Integrating by part and noting that the surface terms such as

$$\int_{-\pi}^{\pi} d\psi \int_{y(\psi)}^{\infty} dJ_2 \frac{\partial}{\partial \psi} (\sqrt{J_2 - y(\psi)} h_+) = \int_{-\pi}^{\pi} d\psi \frac{\partial}{\partial \psi} \int_{y(\psi)}^{\infty} dJ_2 \sqrt{J_2 - y(\psi)} h_+ = 0$$

where the first equality holds because $\left[h \sqrt{J_2 - y} \right]_{J_2=y} = 0$ and the second holds because of periodicity, we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi \int_{-\infty}^{\infty} d\xi \frac{\partial y}{\partial \psi} h(J_2, \psi) = \sum_{\sigma=\pm} \frac{1}{2\pi |x_0|} \int_{-\pi}^{\pi} d\psi \int_{y(\psi)}^{\infty} dJ_2 \sqrt{J_2 - y} \frac{\partial h_{\sigma}(J_2, \psi)}{\partial \psi} \quad (42)$$

Upon substituting Eq. (40) into Eq. (42) and carrying out the subsequent integral over J_2 Eq. (42) becomes

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi \int_{-\infty}^{\infty} d\xi \frac{\partial y}{\partial \psi} h(J_2, \psi) &= \sum_{\sigma} \frac{2\sigma r |x_0| v_i}{\omega} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi \int_{y(\psi)}^{\infty} dJ_2 \sqrt{J_2 - y} \frac{\partial}{\partial J_2} \left[\sqrt{J_2 - y} \frac{\partial g_0}{\partial J_2} \right] \\ &= \sum_{\sigma} \frac{\sigma r |x_0| v_i}{\omega} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi g_0^{(t)}(J_2 \rightarrow \infty) \\ &= \sum_{\sigma} \frac{\sigma r |x_0| v_i}{\omega} g_0^{(t)}(J_2 \rightarrow \infty) \end{aligned} \quad (43')$$

where we have used the facts that $g_0^{(b)}(J_2 = y) = 0$ and $g_0^{(t)}(J_2 \rightarrow \infty)$ is independent of ψ according to Eq. (37). After some algebra and the use of Eq. (32), $g_0^{(t)}(J_2 \rightarrow \infty)$ of Eq. (37) can be simplified as

$$\lim_{J_2 \rightarrow \infty} g_0^{(t)} = \sigma \frac{\pi}{2} \gamma \left| \frac{2a_1}{\psi_r} \right|^{1/2} \frac{\partial f_M}{\partial r} \quad (44)$$

where¹⁶

$$\begin{aligned} \gamma &= \int_1^\infty du \left[\frac{1}{E_-(u)} - \frac{1}{\pi u^{1/2}} \right] - \frac{2}{\pi} \\ &= \int_0^\infty du \left[\frac{1}{E_+(u)} - \frac{1}{\pi u^{1/2}} \right], \end{aligned} \quad (45)$$

and

$$E_\pm(u) \equiv \int_{-\pi/2}^{\pi/2} d\psi \sqrt{u \pm \sin^2 \psi}.$$

Substitution of Eqs. (43), (44), (14) and $\omega = -r\psi_r/2t_{||}$ into Eq. (39) yields the averaged radial particle flux at distance r from the mirror axis:

$$\langle q_r \rangle = \frac{2\sqrt{2}\pi^2\gamma H}{m_i^2 L} \sum_{k=1,3,5,\dots} \int dt_{||} \epsilon_k^2(r, \mu) \nu_i(r, \mu) \frac{|a_1|^{1/2} |\psi_r| \frac{\partial f_M}{\partial r}}{|\psi_r|^{3/2}} \quad (46)$$

Qualitatively the form of the diffusion coefficient can be arrived at by a random-walk type argument⁴:

$$D_{res.ban} \sim x_{res}^2 \frac{\Delta \epsilon_{res}}{\epsilon_{res}} \nu_{res}^{eff} \sim |a_1|^{1/2} R^{3/2} \nu_i$$

which is seen to be consistent with Eq. (46). If we interpret $\nu_i(r, \mu)$ as the 90°-collision frequency, the form of Eq. (46) regarding the dependence on $\nu_i, a_1, \psi_r, \psi_r$ also agrees with the scaling argument of R. H. Cohen⁶ and differs in appearance from the result given in Ref. 3. However, when the

detailed expression for $v_i(r, \mu)$ of Eq. (A7) is substituted into Eq. (46) the result can be converted after some algebra into a rather different form

$$\langle q_r \rangle \sim \sum_k \int d\mu t_{\parallel} \tilde{v}_i(r, \mu) \frac{|a_1|^{1/2} |\psi_r|^{1/2}}{|\psi_\varepsilon|} \frac{\partial f_M}{\partial r}. \quad (47)$$

This form agrees with that in Ref. 3. In fact, our converted $\tilde{v}(r, \mu)$ in Eq. (47) would be the same as $v_i^{R-S}(r, \mu)$ of Ryutov-Stupakov³ if the latter is corrected for some of its dependence on ε_k and ϕ . The lesson learned from the comparison of Eqs. (46) and (47) is that the form can vary from one formulation to another and, for quantitative purpose, the so-called collision frequency must be first calculated analytically.

For $\Delta\psi > 1$ and in the regime of high collision frequency, $v_i t_{\parallel} \gg 1$, hardly any of the resonant particles and the MR-particles can complete their orbits. The diffusion is then dominated by the nonresonant ($\Delta\varepsilon \sim \varepsilon$) particles which constitute the majority. Each of these will have to random-walk from one transition region to the other before it receives another kick of order a_1 . Since these kicks are phased randomly the diffusion coefficient is given by

$$D_{\text{coll.}} \sim \frac{a_1^2}{\tau_{\text{ran.walk}}} \sim \frac{a_1^2}{t_{\parallel}^2 v_i} \quad (48)$$

where $\tau_{\text{ran.walk}} \sim t_{\parallel}^2 v_i$ is the time taken by a particle to random-walk from one end to the other.

V Remarks and Comments

The radial diffusion in tandem mirrors is similar to that in toroidal systems in many physical and mathematical aspects. The role of the smallness parameter $\delta = \frac{\text{poloidal radius}}{\text{toroidal radius}}$ for the toroidal system is played by $\delta = a_1/R$ in the present case. The mathematical procedure of expanding into low-order terms of δ is carried out in both cases. As a first calculation, the major approximations made here are patterned after the pioneering toroidal work of Galeev and Sagdeev.⁵ For a more refined calculation, which may improve the numerical result somewhat, especially for the plateau regime, the variational method of Rutherford,¹⁷ Rosenbluth and Hazeltine¹⁸ and others¹⁹ should be followed.

Finally we make a remark on the effect of the asymmetry in the central solenoidal section of the tandem system. The asymmetry arises from the nonaxial-symmetrical distribution of the particle density as well as of the resultant magnetic field as a result of particle diffusion between the transition region and the central region. This effect is also of the first order in the asymmetry parameter and is indeed of importance in the neo-classical diffusion.² Although it might be argued intuitively that this effect should be less important for resonant diffusion, a more careful analysis may be worth doing.

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Fig. 1 The tandem mirror system - the central section and the transition regions.

Fig. 2 The resonance region for a given μ .

APPENDIX

Derivation of the Collision Term in Eq. (24)

We start from the Landau form of collision integral

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} = -\frac{\partial}{\partial v_{\alpha}} J_{\alpha}(\vec{v}) \quad (A1)$$

where

$$J_{\alpha}(\vec{v}) = A \int d\vec{v}' \left(\frac{\delta_{\alpha\beta}}{u} - \frac{u_{\alpha} u_{\beta}}{u^3} \right) \left[\frac{\partial f(\vec{v})}{\partial v_{\beta}} f(\vec{v}') - f(\vec{v}') \frac{\partial f(\vec{v}')}{\partial v_{\beta}} \right],$$

$$\vec{u} = \vec{v} - \vec{v}',$$

$$A = -\frac{n_0 \pi e^4}{m_i^2} \log \left[\frac{T_i^3}{4\pi n_0 e^6} \right].$$

Approximating⁵ the background particles $f(\vec{v}')$ with which the $\vec{v}$ -particle being considered is colliding by $f_M(\vec{v}')$ we then replace $f(\vec{v})$ by $\delta f(\vec{v})$ to obtain²⁰

$$\begin{aligned} J_{\alpha}(\vec{v}) &= -A \int d\vec{v}' \left(\frac{\delta_{\alpha\beta}}{u} - \frac{u_{\alpha} u_{\beta}}{u^3} \right) \left[\frac{\delta f(\vec{v}) v_{\beta} f_M(\vec{v}')}{T_i} + \frac{f_M(\vec{v}')}{m_i} \frac{\partial \delta f(\vec{v})}{\partial v_{\beta}} \right] \\ &= -A I_{\alpha\beta}(\vec{v}) \left[v_{\beta} \frac{\delta f(\vec{v})}{T_i} + \frac{1}{m_i} \frac{\partial \delta f(\vec{v})}{\partial v_{\beta}} \right] \end{aligned} \quad (A2)$$

where

$$\begin{aligned} I_{\alpha\beta}(\vec{v}) &\equiv \int d\vec{v}' \frac{u^2 \delta_{\alpha\beta} - u_{\alpha} u_{\beta}}{u^3} f_M(\vec{v}') \\ &= \frac{\partial^2}{\partial v_{\alpha} \partial v_{\beta}} \int d\vec{v}' u f_M(\vec{v}') \end{aligned} \quad (A3)$$

Using the explicit form of the local Maxwellian distribution function

$$f_M(\vec{v}') = G e^{-g v'^2}$$

and after some algebra we can reduce $I_{\alpha\beta}(\vec{v})$ to

$$I_{\alpha\beta}(\vec{v}) = \frac{2\pi G}{g} \left\{ \left(\frac{\delta_{\alpha\beta}}{v} - \frac{v_{\alpha} v_{\beta}}{v^3} \right) \left[-\frac{1}{v^2} \int_0^v dy y^2 e^{-gy^2} + \int_0^v dy e^{-gy^2} \right] \right. \\ \left. + \frac{v_{\alpha} v_{\beta}}{v^3} \frac{2}{v^2} \int_0^v dy y^2 e^{-gy^2} \right\} \quad (A4)$$

Substituting Eqs. (A4), (A2) into (A1) and after some simplification we find⁵

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll.}} = \frac{2\pi\lambda e^4}{m_i^2} \frac{\partial}{\partial v_{\alpha}} \left\{ \left[\left(\eta(x) + \eta^-(x) - \frac{\eta(x)}{2x} \right) \left(\frac{\delta_{\alpha\beta}}{v} - \frac{v_{\alpha} v_{\beta}}{v^3} \right) + \frac{v_{\alpha} v_{\beta}}{v^3} \frac{\eta(x)}{x} \right] \cdot \right. \\ \left. \cdot \left(\frac{\partial \delta f(\vec{v})}{\partial v_{\beta}} + \frac{2v_{\beta}}{v_T^2} \delta f(\vec{v}) \right) \right\} \quad (A5)$$

where

$$x \equiv \left(\frac{v}{v_T} \right)^2 = \frac{\epsilon - e\phi}{T_i}, \quad \eta(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dt t^{1/2} e^{-t}, \\ \eta^-(x) = \frac{dn}{dx} = \frac{2}{\sqrt{\pi}} x^{1/2} e^{-x}, \quad \frac{m_i v_T^2}{2} \equiv T_i, \\ 2\lambda = n_0 \log \left[\frac{T_i^3}{4\pi m_0 e^6} \right].$$

We shall now change variables from (v_x, v_y, v_z) to (ξ, μ, θ) where

$$v_x = \sqrt{2\mu H/m_i} \cos \theta, \quad v_y = \sqrt{2\mu H/m_i} \sin \theta, \quad \mu H = \frac{1}{2} m_i v_{\perp}^2,$$

$$v_z = \sqrt{(2/m_i) (\epsilon - e\phi - \mu H)}, \quad \xi = (\epsilon - e_k(r, \mu)) / \epsilon_k$$

and keep only the $\frac{\partial^2 \delta f}{\partial \xi^2}$ terms in (A5). Thus,

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} \approx \frac{2\pi\lambda e^4}{m_i^2} \left\{ \left(\eta + \eta' - \frac{\eta}{2x} \right) \frac{1}{v^3} \left(v^2 \delta_{\alpha\beta} - v_\alpha v_\beta \right) + \frac{1}{v^3} v_\alpha v_\beta \frac{\eta}{x} \right\} \frac{\partial^2 \delta f}{\partial v_\alpha \partial v_\beta}$$

$$\approx \frac{2\pi\lambda e^4}{v^3 \epsilon_k^2} \frac{\partial^2 \delta f}{\partial \xi^2} \left\{ \left(\eta + \eta' - \frac{\eta}{2x} \right) (1-a)^2 v_1^2 v_z^2 + \frac{\eta}{x} (a v_1^2 + v_z^2)^2 \right\} \quad (\text{A6})$$

where $a \equiv 1 - \frac{1}{H} \frac{\partial \epsilon_k}{\partial \mu}(r, \mu)$. We may also rewrite Eq. (A6) in the form of Eq. (24),

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} = v_i(r, \mu) \frac{\partial^2 \delta f}{\partial \xi^2},$$

where

$$v_i(r, \mu) = 4\sqrt{\frac{2\pi}{m_i}} \frac{\lambda e^4 (\epsilon_k - e\phi)^{3/2} \mu^2 H^2}{\epsilon_k^2 T_i^3} \cdot \left[\left(\frac{\epsilon_k - e\phi - \mu H}{\mu H} \right) (1-a)^2 G\left(\frac{\epsilon_k - e\phi}{T_i}\right) + \left(a + \frac{\epsilon_k - e\phi - \mu H}{\mu H} \right)^2 F\left(\frac{\epsilon_k - e\phi}{T_i}\right) \right],$$

$$G(x) = \frac{1}{x^3} \left[\left(1 - \frac{1}{2x} \right) \int_0^x e^{-t} t^{1/2} dt + e^{-x} x^{1/2} \right],$$

$$F(x) = \frac{1}{x^4} \int_0^x e^{-t} t^{1/2} dt. \quad (\text{A7})$$

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$$\frac{\partial h(\psi, J_2)}{\partial \psi} = \left[\frac{\partial h(J, \psi, \xi)}{\partial \psi} + \frac{\partial h(J, \psi, \xi)}{\partial \xi} \frac{\partial \xi(J_2, \psi)}{\partial \psi} \right]_{x=0} \neq \left[\frac{\partial h(J, \psi, \xi)}{\partial \psi} \right]_{x=0}$$

$$\text{or } J=J_2, x_0 \xi = \sigma \sqrt{J_2 - y}$$

Apparently the $\left(\frac{\partial h}{\partial \xi} \frac{\partial \xi}{\partial \psi} \right)_{x=0}$ term seems to be missing from their Eq. (62).

13. Again we note that the corresponding equations in Ref. 2, their Eqs. (62) and (63), involve only two variables, $J_2 \equiv x_0^2 \xi^2 + y(\psi)$ and ψ . This reduction of three variables x, ψ, ξ to two variables J_2, ψ was apparently arrived at by setting $x=0$ in their Eq. (61) which involves $\frac{\partial h}{\partial x}$ and $\frac{\partial g}{\partial x}$.

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