Time Series Analysis by the Maximum Entropy Method

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TIME SERIES ANALYSIS BY THE MAXIMUM ENTROPY METHOD

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ABSTRACT


The principal subject of this report is the use of the Maximum Entropy method for spectral analysis of time series. The classical Fourier method is also discussed, mainly as a standard for comparison with the Maximum Entropy method. Examples are given which clearly demonstrate the superiority of the latter method over the former when the time series is short. The report also includes a chapter outlining the theory of the method, a discussion of the effects of noise in the data, a chapter on significance tests, a discussion of the problem of choosing the prediction filter length, and, most importantly, a description of a package of FORTRAN subroutines for making the various calculations. Cross-referenced program listings are given in the appendices. The report also includes a chapter demonstrating the use of the programs by means of an example. Real time series like the lynx data and sunspot numbers are also analyzed.
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INTRODUCTION

In the past, the spectral analysis of time series has for the most part utilized two classical approaches - the power spectrum method of R. B. Blackman and J. W. Tukey (1959) and the periodogram. The Blackman and Tukey approach involves the estimation of the autocorrelation function. A shortcoming of the estimator used is its assumption of a zero extension of the given data. On the other hand, the periodogram method assumes a periodic extension of the data, the period being equal to the length of the time series. John P. Burg (1968) has presented a new technique of time series analysis which seeks to minimize assumptions about the unavailable data. The method is called maximum entropy spectral analysis (which will be referred to as MESA throughout this report). MESA emerged in an effort to achieve better resolution from short time series.

A great amount of research on MESA has been done by T. J. Ulrych. His recent publications (1972b, 1973, 1974) indicate the wide extent to which MESA can be utilized. Ulrych's contention is that the classical Fourier transform methods are inferior to MESA because of their inability to analyze low frequency variations.

MESA and Fourier analysis are compared in this report. Generally speaking, MESA is superior for short time series, but it has one major disadvantage shared also by older methods. The problem involves the determination of the length of the vector of filter coefficients used for estimating the autocorrelation function. In an attempt to solve this problem, T. J. Ulrych (1975) suggested the use of the Akaike final
prediction error (FPE), (Akaike 1969), as applied to autoregressive decomposition. A brief critical discussion of this idea is also presented here in Chapter IV.

As much as possible, the mathematical presentation of time series analysis is simplified in this report. The derivation of the maximum entropy spectrum, however, employs theorems in complex analysis. Although the mathematics becomes rigorous in some parts, it is hoped that the many examples presented will impart a fair understanding of the methods discussed to less mathematically inclined readers.

Computer programs (written in FORTRAN) which calculate both the Fourier sample spectrum and maximum entropy spectrum plus some other useful information are described and explained in Chapter V. An example is presented in Chapter VI to illustrate the use of these programs. Program listings are given in the appendices.
CHAPTER I

THE DISCRETE FOURIER TRANSFORM METHOD

Consider the functions $y_1(t)$, $y_2(t)$, $y_3(t)$ in Figures 1.1-1.3, respectively.

Fig. 1.1 The function $y_1(t)$: amplitude = 2, period = 6 years.

Fig. 1.2. The function $y_2(t)$: amplitude = 4, period = 4 years.
Fig. 1.3. The function $y_3(t)$: amplitude = 3, period = 12 years.

The sum of $y_1(t)$, $y_2(t)$, $y_3(t)$ produces a new function $y(t)$ (Fig. 1.4).

Fig. 1.4. The new function $y(t) = y_1(t) + y_2(t) + y_3(t)$, sampled at intervals $\Delta t = 1$ year.
Since \( y(t) \) is the sum of three functions whose periods are known, the analysis of its periodic components is clear. In practice, one is usually faced with the inverse problem, i.e., the analysis of a time series like \( y(t) \) in order to determine what periodic components are present. This situation leads to the utilization of the Fourier transform method.

Although many natural processes are truly harmonic, in that they can be expressed exactly as a sum of sinusoidal terms, it is more often the case that a natural process will also contain aperiodic components. Such processes can still be analyzed by harmonic methods. In fact, any aperiodic function of finite length can be approximated to any desired degree of accuracy by a sum of sinusoids, provided one includes enough sinusoidal terms in the approximating function. In most real problems one is limited to a finite, discrete time series obtained by sampling the process at say \( n \) equally spaced points in time. If the process is denoted by \( x(t) \), and if the sampling is arranged so that the sampling interval is one unit of time, i.e., \( \Delta t = 1 \), with the first sample point being taken at \( t = 0 \), then the time series can be written \( x(0), x(1), x(2), \ldots, x(n-1) \), or as is done more commonly, \( x_0, x_1, x_2, \ldots, x_{n-1} \). It can be shown that the terms of this series can be represented exactly by a finite Fourier expansion of the form

\[
(1.1) \quad x_j = a_0 + \sum_{0<k<n/2} (a_k \cos \frac{2\pi jk}{n} + b_k \sin \frac{2\pi jk}{n}) + (-1)^j a_{n/2},
\]

\( j = 0, 1, 2, \ldots, n-1 \)
where the $a_{n/2}$ term is included only if $n$ is even. The Fourier coefficients are given by

\begin{align}
(1.2) \quad a_0 &= \frac{1}{n} \sum_{j=0}^{n-1} x_j , \\
(1.3) \quad a_k &= \frac{2}{n} \sum_{j=0}^{n-1} x_j \cos \frac{2\pi jk}{n} , \quad 0 < k < \frac{n}{2} , \\
(1.4) \quad b_k &= \frac{2}{n} \sum_{j=0}^{n-1} x_j \sin \frac{2\pi jk}{n} , \\
(1.5) \quad a_{n/2} &= \frac{1}{n} \sum_{j=0}^{n/2-1} (-1)^j x_j , \quad \text{if } n \text{ is even}
\end{align}

Equation (1.1) expresses the time series $x_j$ in terms of the component frequencies $f_k = k/n$ and can also be written in terms of the equivalent periods $T_k = 1/f_k = n/k$ as

\begin{align}
(1.6) \quad x_j = a_0 + \sum_{0 \leq k \leq n/2} (a_k \cos \frac{2\pi j}{T_k} + b_k \sin \frac{2\pi j}{T_k}) + (-1)^j a_{n/2}
\end{align}

The Fourier spectrum of the time series is the discrete function represented by the set of points $(f_k, F_k)$, $k=0,1,...,[n/2]$, where $[n/2]$ is taken to be $n/2$ if $n$ is even and $(n-1)/2$ if $n$ is odd, and $F_k$ is the Fourier amplitude defined by

\begin{align}
F_k = (a_k^2 + b_k^2)^{1/2}
\end{align}
For each of the frequencies $f_k$ one can also define an associated phase by

\[ \theta_k = \tan^{-1} \left( \frac{a_k}{b_k} \right). \]

The expansion (1.6) can be written in terms of the Fourier amplitudes and phases as

\[ x_j = a_0 + \sum_{0 < k < n/2} F_k \sin \left( \frac{2\pi j}{T_k} + \theta_k \right) + (-1)^j a_{n/2}. \]

Again, this expansion is exact for all the sample points $x_0, x_1, \ldots, x_{n-1}$, and if $n$ is large enough one might reasonably hope that Equation (1.9) would give a good approximation to the underlying process $x(t)$ at other values of $t$ also. That is, for any value of $t$ in the interval $0 < t < (n-1)$, or even for $t$ outside that interval, one might hope that

\[ x(t) = a_0 + \sum_{0 < k < n/2} F_k \sin \left( \frac{2\pi t}{T_k} + \theta_k \right) \]

without too much error.

The function $(f_k, F_k)$ is the discrete Fourier spectrum of the data set $x_0, x_1, \ldots, x_{n-1}$. We note that continuous functions $x(t)$ have analogous continuous Fourier spectra, but we do not treat such cases here since we are concerned only with finite, discrete time series. For the continuous case the reader is referred to Jenkins and Watts (1968). In computing a discrete Fourier spectrum one hopes that
the result approximates the continuous Fourier spectrum of the underly-
ing process $x(t)$. The approximation is not limited to the discrete
frequencies $f_k = k/n$ but can be calculated for intermediate values of
$f$ by the formula

\begin{equation}
F(f) = \sqrt{a(f)^2 + b(f)^2},
\end{equation}

where

\begin{equation}
a(f) = \frac{2}{n} \sum_{j=0}^{n-1} x_j \cos 2\pi j f,
\end{equation}

\begin{equation}
b(f) = \frac{2}{n} \sum_{j=0}^{n-1} x_j \sin 2\pi j f.
\end{equation}

We note in passing that some writers make a distinction between the con-
tinuous function $F(f)$ and the discrete function $(f_k, F_k)$, calling the
latter a "line spectrum" and the former a "sample spectrum". Whatever
it is called, the continuous function $F(f)$ hopefully approximates the
Fourier spectrum of $x(t)$. The approximation fails to exactly represent
the desired spectrum because the time series $x_0, x_1, \ldots, x_{n-1}$
cannot exactly represent the process $x(t)$. The error arises for two
reasons:

(1) The time series is restricted to a finite interval of time, so
we do not know what the process $x(t)$ is like outside that interval, and

(2) The time series represents the process only at a finite number
of points in the interval, adjacent points being separated by the sample
spacing $\Delta t$. 
The latter restriction limits the highest frequency that is resolvable - one cannot expect to isolate components with frequencies whose corresponding periods are smaller than the sampling interval. In fact it is necessary to have at least two samples per cycle of the highest frequency (shortest period) that can be resolved. The frequency

\[ f_N = \frac{1}{2 \Delta t} \]

is called the Nyquist frequency and is the highest frequency that can be resolved with a sample interval of \( \Delta t \). The situation is actually worse than it appears on the surface because, if \( x(t) \) does have important harmonic components at frequencies greater than the Nyquist frequency, then these components will distort the Fourier spectrum values at lower frequencies. This is because the approximation cannot distinguish between the frequency \( f \) and the frequency \( (1/\Delta t - f) \). Thus, if \( \Delta t = 1 \), then the Nyquist frequency is 0.5 and if \( x(t) \) has a harmonic component with frequency \( f = 0.75 \), the estimate of the transform will be distorted at \( f = 0.25 \). This phenomenon is called aliasing and must be kept in mind in designing a sampling procedure. One should always try to choose \( \Delta t \) small enough so that \( f_N \) exceeds the frequencies of all the important harmonic components in the process.

The first restriction, i.e., the restriction to a finite time interval, produces distortions in the calculated spectrum at the low frequencies (long periods). Obviously one cannot expect to resolve frequencies corresponding to periods much longer than the total time interval represented by the sample. Most users of the Fourier method do not trust the estimate at frequencies lower than that corresponding to
periods of at most one half the total time interval, and very conserva-
tive users demand that the total interval exceed a given period by a
factor of 4 or more before they will trust the Fourier estimate at the
corresponding frequency. The Fourier technique can detect the presence
of long period components (components with periods comparable to the
total length of record), but it cannot accurately pinpoint the frequency
nor can it distinguish between two or more long period components at
frequencies that are relatively close together.

It is convenient to analyze the effect of a truncated record in
terms of a function called a data window. Consider a process \( x(t) \)
defined on the time interval \((-\infty, +\infty)\).

\[ x(t) = \begin{cases} \sin(\pi t) & |t| \leq \frac{T}{2} \\ 0 & |t| > \frac{T}{2} \end{cases} \]

\[ \Delta t = 1 \]

Fig. 1.5. The observed function \( x_0(t) \), \( \Delta t = 1 \).
In general, one can only observe within a finite interval of length $T$, say $[-T/2, T/2]$ as in Figure 1.5. The available data are then confined to the interval $[-T/2, T/2]$. The rectangular data window for the interval $[-T/2, T/2]$ is defined by

$$w(t) = \begin{cases} 1, & \text{if } -T/2 \leq t \leq T/2, \\ 0, & \text{otherwise}. \end{cases}$$

The observed function corresponding to the available data is represented by

$$x_0(t) = x(t) w(t).$$

We temporarily ignore the effect of the discrete sampling and concentrate on the effect of applying the rectangular data window, i.e., the effect of truncating the record. Subjecting the observed function $x_0(t)$ to Fourier analysis will result in a transform $\mathcal{F}_{x_0}$ of $x_0$ which is really a smoothed version of the Fourier transform $\mathcal{F}_x$ of the original signal $x(t)$. That is, if we let

$$\mathcal{F}_{x_0} = \text{Fourier transform of } x_0,$$
$$\mathcal{F}_x = \text{Fourier transform of } x, \text{ and}$$
$$\mathcal{F}_w = \text{Fourier transform of } w,$$

then it can be shown that

$$\mathcal{F}_{x_0}(f) = \int_{-\infty}^{\infty} \mathcal{F}_x(f') \mathcal{F}_w(f - f') \, df'.$$
The Fourier transform of the truncated, observed time series is said to be the convolution of the desired Fourier transform with the Fourier transform of the data window. This latter function is called the spectral window and can be shown to be

\[(1.13) \quad \mathcal{F}_w(f) = \frac{\sin 2\pi ft}{\pi ft} \quad .\]

Applying the data window (1.15), which is sometimes called the boxcar window, necessarily distorts the Fourier transform. It can be shown that \(\mathcal{F}_x(f)\) is necessarily smoother than \(\mathcal{F}_X(f)\). A common practice, therefore, is to change the shape of the data window in order to reduce the distortion in the transform. This is usually done by rounding off the corners of the boxcar window, thus damping the contribution of the data at the two ends of the record. The idea is to design a window which decreases the smoothing and hence increases the resolution of the transform. This procedure is not without its own shortcomings. Spectral windows which give good resolution of the spectral peaks are often unstable to changes in the input data. On the other hand, those spectral windows that are stable with regard to changes in the data usually give poor resolution.

A more complete discussion of data windows and their corresponding spectral windows is given by Box and Jenkins (1970). We close our discussion of the subject by noting that using any data window involves some assumption about the data outside the available record (e.g., the boxcar window assumes a periodic extension of the data with period equal to the length of the record). Furthermore, using any window, other than
the boxcar window, distorts the data that are available. As we shall see in the next chapter, the Burg maximum entropy method is motivated by an attempt to overcome these two difficulties.

As an illustration of some of the ideas incorporated in this chapter, we present a very simple example. The number of points sampled is \( n = 42 \). The function sampled was the sum of a straight line trend and a sinusoid. The linear trend was included in the example because real time series data frequently contain trend components.

The data points in Figure 1.6 are generated from the equation

\[
x_i = \int_{i-1}^{i} h(t) dt, \quad i = 1, 2, \ldots, 42,
\]

where

\[
h(t) = 7 + t + 5 \cos \frac{2\pi t}{6}
\]

The linear trend, \( x_T(t) = 7 + t \), was first removed by performing a straight line, least squares fit to the data and subtracting out the result. The function subjected to Fourier analysis was then \( x(t) - x_T(t) \). It is necessary to remove the trend line because Fourier analysis would interpret it as a component with infinite period. The spectrum of the detrended function is shown in Figure 1.7.

The highest peak in Figure 1.7 occurs at \( f = 0.1675 \) with a corresponding period of \( P = 5.97 \), which is a good estimate of \( P = 6 \), the true period of the function. The other smaller peaks are called sidelobes and do not represent significant spectral components. They are one of the results of truncating the record. The width of the main peak also depends on the record length and gives an idea of the uncertainty in the estimate of the peak frequency.
Fig. 1.6. Data $x(t)$ are histogram yearly integrals of $h(t) = 5 \cos \frac{2\pi t}{A} + 7 + t$.

Fig. 1.7. Fourier Spectrum of detrended $x(t)$.
A. A Brief Explanation of the Entropy Function.

In English, when one expresses a simple, grammatically correct, logical statement, his choice of words is governed by probability laws. For example, if a statement starts with "The mountain", one cannot just append any English phrase to have the statement make sense. "The mountain sees" or "The mountain my" do not make sense. However, "The mountain is big" is a perfectly logical statement. Thus, a person's choice of words is strongly dependent upon the logical structure of the language, and hence there is a probability that certain words are more likely than other words to follow a phrase like "The mountain". The freedom of choice in selecting a message is what is scientifically termed as information or entropy.

The word "information" must not be confused with its ordinary meaning. In the physical sciences, the entropy or information associated with a situation is a measure of the degree of randomness in the situation. Entropy is expressed in terms of probabilities - for example, the probability of getting to a certain stage in the process of forming a message and when in that stage the probability that certain words will be chosen next (Shannon and Weaver 1949).

How then can one measure the entropy or information of a process? In particular, the processes treated here are Markoff processes; that is, stochastic processes in which the probabilities depend on previous
events. Shannon and Weaver (1949) have shown that such a measure for information can best be estimated using logarithms of the probabilities.

Suppose \( p_1, p_2, \ldots, p_n \) are the known probabilities of occurrence of a set of discrete events. Then

\[
I = I(p_1, p_2, \ldots, p_n) = -K \sum_{i=1}^{n} p_i \log_a p_i,
\]

where \( K \) is a positive constant, and \( a \) is the chosen base for the logarithm, is called the entropy of the set of probabilities \( p_1, p_2, \ldots, p_n \). Similarly, the entropy of a continuous distribution with density function \( p(x) \) is defined by

\[
I = I(p(x)) = - \int_{-\infty}^{\infty} p(x) \log p(x) \, dx.
\]

Assume for example that there are two possibilities with probabilities \( p \) and \( q = 1-p \). Then the entropy is

\[
I = I(p) = -p \log_e p - (1-p) \log_e (1-p),
\]

where for convenience the natural logarithm is used, and the constant \( K \) is taken to be 1. A plot of \( I \) in Equation (2.3) appears in Figure 2.1.

\( I \) is maximized when \( q = 1-p \), that is \( q=p=\frac{1}{2} \). In a physical situation this is the case of greatest uncertainty. In tossing a coin, for
example, out of 1000 tosses, it is likely that heads will appear about 500 times and tails about 500 times, but the uncertainty for any given toss is maximized. For a biased coin the uncertainty for any given toss is less because if one simply predicted the most likely result on each toss, then one would be right more than 500 times out of 1000. The more uncertain the event is, the greater the freedom of choice in predicting the outcome. The function $I$ in Equation (2.3) is 0 when $p=0$, $q=1$ or $q=0$, $p=1$. That is $I$ assumes a minimum of 0 when the event is certain to happen. In this case, one has little freedom of choice in making a prediction. In all other cases, $I$ assumes a positive value.
The logarithm is used in the definition of entropy since it is mathematically more convenient to handle. With a logarithmic definition, information is additive, as would be expected. Assume for example that there are two independent events with probabilities $p_1$ and $p_2$. Then the probability that both events happen is $p_1 p_2$. The measured information or entropy of the two events is

$$I = -K \log_a (p_1 p_2) = I_1 + I_2,$$

where

$$I_1 = -K \log p_1, \quad I_2 = -K \log p_2.$$

The most common logarithmic bases employed in the definition of entropy are 2, 10 and e. The natural logarithm is the most convenient for use in mathematical analysis and is the one we adopted here.

From the preceding discussions, we note that $I$ is indeed a reasonable measure of entropy. To maximize $I$ is to maximize the uncertainty in the event. The constraints involved depend on the probabilities $p_1, p_2, \ldots, p_n$ with $0 \leq p_i \leq 1$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$.

In time series analysis, the relevant probabilities are not known. Instead, one is faced with the analysis of the power spectrum. In this case, the entropy is proportional to

$$I = \int_{-\infty}^{\infty} \ln P(f) \, df,$$
where \( P(f) \) is the power spectrum (see Burg 1975, Shannon and Weaver 1949).

In looking at the power spectrum of a time series, one asks how much of the original information has been lost or gained. The degree of uncertainty is measured in the entropy function \( I \) of Equation (2.6).

R. Maximum Entropy Spectral Analysis (MESA)

The basic assumption in MESA is that the stationary time series under analysis is the most random or least predictable. The classical approach involves the estimation of the power spectrum from known values of the autocorrelation function \( \rho_k \), \( |k| \leq M \), where \( M \) is less than or equal to the number of observations in the given time series. The standard assumption is that \( \rho_k = 0 \) for \( |k| > M \), but usually a weighting function is also introduced and multiplied by \( \rho_k \) and the result is Fourier transformed. Given a set of autocorrelation function values \( \rho_k \) with the imposed condition that the power spectrum be non-negative definite, there usually exist infinitely many power spectra which will be consistent with the given data. Maximum entropy spectral analysis is based on the idea of choosing the spectrum which corresponds to the most random time series whose autocorrelation function agrees with given values.

C. Statement of the Problem

Find a real positive function \( P(f) \) which maximizes the entropy function

\[
I = \int_{-f_M}^{f_M} \ln P(f) \, df
\]
under the constraints

(2.8) \[ c_k = \int_{-f_M}^{f_M} P(f) \exp(i2\pi fk\Delta t) df \]

where \(-M \leq k \leq M\), \(\Delta t\) is the sampling interval, and \(\rho_k\) the autocorrelation with time lag \(k\Delta t\), \(i = \sqrt{-1}\) and \(f_M = \frac{1}{2\pi\Delta t}\) is the Nyquist frequency.

D. Solution to the Problem

Wiener has shown that the power spectrum \(P(f)\) is the discrete Fourier cosine transform of the autocorrelation \(\rho_k\). So \(P(f)\) has a Fourier series representation

(2.9) \[ P(f) = \frac{1}{2f_M} \sum_{k=-\infty}^{+\infty} \rho_k e^{-i2\pi fk\Delta t} \]

Substituting (2.9) into (2.7), we get

(2.10) \[ I = \int_{-f_M}^{f_M} \ln(\frac{1}{2f_M} \sum_{k=-\infty}^{+\infty} \rho_k e^{-i2\pi fk\Delta t}) df \]

Since the \(\rho_k\)'s are known for \(-M \leq k \leq M\), expression (2.10) will be maximized with respect to the \(\rho_k\)'s, with \(|k| > M\). Setting the appropriate derivatives to zero gives
\[ (2.11) \quad \frac{\partial I}{\partial \rho} = \int_{-f_M}^{f_M} \frac{1}{2f_M} \frac{e^{-i2\pi f\Delta t}}{\rho(f)} \, df \]

\[ = \frac{1}{2f_M} \int_{-f_M}^{f_M} \frac{e^{-i2\pi f\Delta t}}{\rho(f)} \, df \]

\[ = 0, \quad |\ell| > M. \]

But \([P(f)]^{-1}\) has a Fourier expansion so that

\[ (2.12) \quad [P(f)]^{-1} = \sum_{m=-\infty}^{\infty} \lambda_m e^{-i2\pi fm\Delta t}. \]

Thus (2.11) becomes

\[ (2.13) \quad \frac{\partial I}{\partial \rho} = \frac{1}{2f_M} \int_{-f_M}^{f_M} \left( \sum_{m=-\infty}^{\infty} \lambda_m e^{-i2\pi fm\Delta t} \right) e^{-i2\pi f\ell\Delta t} \, df \]

\[ = \frac{1}{2f_M} \int_{-f_M}^{f_M} \left( \sum_{m=-\infty}^{\infty} \lambda_m (\cos 2\pi f\Delta tm - i \sin 2\pi f\Delta tm) \right) \times \]

\[ (\cos 2\pi f\Delta t\ell - i \sin 2\pi f\Delta t\ell) \, df \]

\[ = 0 \quad \text{if} \quad |\ell| > M. \]
Because of the orthogonal properties of sines and cosines, (2.13) implies that $\lambda_m = 0$ for $|m| > M$.

From (2.12) and the preceding, we get

\begin{equation}
(2.14) \quad P(f) = \frac{1}{\sum_{m=-M}^{M} \lambda_m e^{-i2\pi fm\Delta t}}.
\end{equation}

Thus (2.8) becomes

\begin{equation}
(2.15) \quad c_k = \int_{-f}^{f} \sum_{m=-M}^{M} \lambda_m e^{-i2\pi fm\Delta t} df.
\end{equation}

Let $z = e^{-i2\pi f\Delta t}$. Under this transformation,

\begin{equation}
\frac{dz}{df} = -i2\pi \Delta t e^{-i2\pi f\Delta t} df, \quad df = -dz/(i2\pi \Delta t).
\end{equation}

and (2.15) changes to

\begin{equation}
(2.16) \quad c_k = \frac{1}{2\pi i \Delta t} \int_{C} \frac{z^{-k-1}}{\sum_{m=-M}^{M} \lambda_m z^m} dz,
\end{equation}

where the contour integral around the unit circle $|z| = 1$ is computed in the counterclockwise direction.

Now, $\lambda_m z^m$ must be real and positive for $|z| = 1$ since we require $P(f)$ to be real and positive. So

\begin{equation}
(2.17) \quad \sum_{m=-M}^{M} \lambda_m z^m = \left( \sum_{m=-M}^{M} \lambda_m z^m \right)^* = \sum_{m=-M}^{M} \lambda_m z^m = \sum_{m=-M}^{M} \lambda_m (1/z)^m.
\end{equation}
where * denotes complex conjugation and \( z z^* = 1 \).

The expanded form of the left hand side of (2.17) is

\[ \lambda_0 + \lambda_1 z + \ldots + \lambda_M z^M + \lambda_{-1} z^{-1} + \ldots + \lambda_{-M} z^{-M} \]

whereas the right hand side is

\[ \lambda_0^* + \lambda_1^* z^{-1} + \ldots + \lambda_M^* z^{-M} + \lambda_{-1}^* z + \ldots + \lambda_{-M}^* z^M \]

By equating the coefficients in (2.17), we have the relationship

\[ \lambda_j = \lambda_{-j}^* \quad j = 1, 2, \ldots, M \]

and

\[ \sum_{m=-M}^{M} \lambda_m z^m = \lambda_0 + \lambda_1 z + \ldots + \lambda_M z^M + \lambda_{-1} z^{-1} + \ldots + \lambda_{-M} z^{-M} \]

By the fundamental theorem of Algebra, \( \sum_{m=-M}^{M} z^m \) has 2M roots and furthermore is factorable in the form

\[ \sum_{m=-M}^{M} \lambda_m z^m = (b_0 + b_1 z + \ldots + b_M z^M) \times (b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}) \]

\[ = (b_0 + b_1 z + \ldots + b_M z^M) \times (b_0 + b_1^* z^{-1} + \ldots + b_M^* z^{-M}) \]
Factoring \( b_0 \) from (2.20) leads to

\[
\sum_{m=-M}^{M} \lambda_m z^m = b_0^2 \left( 1 + a_1 z + \ldots + a_M z^M \right) x (1 + a_1^* z^{-1} + \ldots + a_M^* z^{-M})
\]

\[
= \left( b_0^2 / 2 f_M \Delta t \right) \left( \sum_{m=0}^{M} a_m z^m \right) \left( \sum_{m=0}^{M} a_m^* z^{-m} \right),
\]

where \( b_0 > 0 \), \( a_0 = 1 \), \( a_i = b_i / b_0 \), \( a_i^* = b_i^* / b_0 \), \( i = 1, 2, \ldots, M \)
and \( 2 f_M \Delta t = 1 \).

Let \( P_M = 2 f_M / b_0^2 \). Then \( P_M > 0 \) and Equation (2.14) becomes

\[
P(f) = \frac{1}{\sum_{m=-M}^{M} \lambda_m z^m}
\]

\[
= \frac{P_M \Delta t}{\left( \sum_{m=0}^{M} a_m z^m \right) \left( \sum_{m=0}^{M} a_m^* z^{-m} \right)}.
\]

The autocorrelation values in (2.16) simplify into

\[
\rho_k = \frac{P_M}{2 \pi i} \int z^{-k-1} \left( \sum_{m=0}^{M} a_m z^m \right) \left( \sum_{m=0}^{M} a_m^* z^{-m} \right) dz,
\]
-M \leq k \leq M. Replacing k by k-j with j \geq 0 in (2.23) gives

\begin{equation}
(2.24) \quad p_{k-j} = \frac{P}{2\pi i} \oint \frac{z^{-k+j-1}}{\left(\sum_{m=0}^{M} a_m z^m\right) \left(\sum_{m=0}^{M} a_m^* z^{-m}\right)} dz
\end{equation}

Multiplying (2.24) by a_k^* and forming summations result in

\begin{equation}
(2.25) \quad \sum_{k=0}^{M} a_k^* p_{k-j} = \frac{P}{2\pi i} \oint \frac{z^{j-1}}{\left(\sum_{m=0}^{M} a_m z^m\right) \left(\sum_{m=0}^{M} a_m^* z^{-m}\right)} dz
\end{equation}

\begin{align*}
&= \frac{P}{2\pi i} \oint \frac{z^{j-1}}{\sum_{m=0}^{M} a_m z^m} \prod_{j > 0} dz , \quad j \geq 0 .
\end{align*}

What needs to be done at this point is to simplify the contour integral on the right hand side of (2.25). To do so requires Cauchy's famous theorems in complex variables theory (see Nehari 1968).

Theorem A: Let f(z) be analytic in a region R and on its boundary C.

Then:

\begin{equation}
(2.26) \quad \oint_C f(z) dz = 0 .
\end{equation}
Theorem B: Cauchy's Integral Formula. If \( f(z) \) is analytic inside and on the boundary \( C \) of a simply-connected region \( R \) and \( a \) any point inside \( C \), then

\[
(2.27) \quad f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} \, dz .
\]

For \( j = 0 \), Equation (2.25) becomes

\[
(2.28) \quad \sum_{k=0}^{M} a_k^* \rho_k = P_M \, \oint \frac{1}{z} \, \sum_{m=0}^{M} a_m z^m \, dz .
\]

Let \( f(z) = \frac{1}{\sum_{m=0}^{M} a_m z^m} \). Then \( f(z) \) is analytic inside and on the unit circle \( |z| = 1 \). Thus by Cauchy's integral formula,

\[
(2.29) \quad \frac{P_M}{2\pi i} \oint \frac{1}{z} \, \sum_{m=0}^{M} a_m z^m \, \frac{dz}{z} = P_M \, f(0) = P_M ,
\]

since \( f(0) = a_0 = 1 \).

For \( j \geq 1 \),

\[
(2.30) \quad \frac{P_M}{2\pi i} \oint \frac{z^{j-1}}{\sum_{m=0}^{M} a_m z^m} \, dz = \frac{P_M}{2\pi i} \oint g(z) \, dz ,
\]
where \( g(z) = z^{j-1} \sum_{m=0}^{M} a_m z^m \), a function which is analytic inside and on the unit circle \( |z| = 1 \). By Theorem A, \( \int g(z) \, dz = 0 \). So

\[
(2.31) \quad \frac{P_M}{2\pi i} \oint \frac{z^{j-1}}{\sum_{m=0}^{M} a_m z^m} \, dz = 0, \quad j \geq 1.
\]

By combining (2.25), (2.29) and (2.30), we get

\[
(2.32) \quad \sum_{k=0}^{M} a_k \rho_k^{j-k} = \left\{ \begin{array}{ll}
P_M & \text{if } j = 0, \\
0 & \text{if } j \geq 1.
\end{array} \right.
\]

Or similarly, since \( \rho_k^{j-k} = \rho_{j-k} \), (2.32) becomes

\[
(2.33) \quad \sum_{k=0}^{M} \rho_k^{j-k} a_k = \left\{ \begin{array}{ll}
P_M & \text{if } j = 0, \\
0 & \text{if } j \geq 1.
\end{array} \right.
\]

The matrix equation corresponding to (2.33) is

\[
(2.34) \quad \begin{bmatrix}
p_0 & p_1 & \cdots & \cdots & p_M \\
p_{-1} & p_0 & \cdots & \cdots & p_{M-1} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
p_{-M} & p_{-M+1} & \cdots & \cdots & p_0
\end{bmatrix} \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
\vdots \\
a_M
\end{bmatrix} = \begin{bmatrix}
P_M \\
0 \\
\vdots \\
\vdots \\
0
\end{bmatrix}
\]
The elements \(a_i, i = 0,1,2,\ldots,M\) with \(a_0 = 1\) are normally called filter or reflection coefficients and Equation (2.34) is the equation for designing an \(M + 1\) term prediction error filter with mean square error \(P_M\). The meaning of these terms will become clearer in the next section. The matrix in (2.34) is a Toeplitz matrix.

Briefly reviewing the results of this section, we see that the power spectrum which maximizes (2.7) subject to (2.8) is the one given in (2.22). We can rewrite (2.22) in the following way:

\[
P(f) = \frac{P_{M+1}}{1 + \sum_{m=1}^{M} a_m z^{-m}}
\]

\[
= \frac{P_{M+1}}{1 + \sum_{m=1}^{M} a_m e^{-i2\pi f_m \Delta t} z^{-m}}
\]

since \(z = e^{-i2\pi f \Delta t}\) and \(a_0 = 1\). The terms \(P_M\) and \(a_m\), \(m = 1,\ldots,M\) are given by the solution to the matrix equation (2.34).

Once these are known, the power spectrum \(P(f)\) can be computed for values of the frequency \(f\) in the range \(-\frac{1}{2\Delta t} < f < \frac{1}{2\Delta t}\).

E. Predictive Filtering and the Mean Square Error \(P_M\).

As in Chapter I, let the given real time series be \(x_0, x_1, \ldots, x_{n-1}\). Assume further that the mean of the process has been approximated and subtracted from the \(x_i\)'s. The mathematical derivations in the preceding sections do not directly involve the \(x_i\)'s. Instead, the autocorrelation function values if known provide the basic information
in calculating the power spectrum $P(f)$. However, by working with the $x_i$'s directly, one can arrive at the same matrix equation (2.34). The idea is to apply the filter coefficients $a_i$, $i = 1, ..., M$, with $M < n$ to the time series through a linear prediction relationship.

Let the time series $x_0, x_1, ..., x_{n-1}$ be written in the form

$$x_0, x_1, x_2, ..., x_{t-M}, x_{t-M+1}, ..., x_{t-1}, x_t, ..., x_{n-1}.$$  

Consider the expression

$$\sum_{s=0}^{M} a_s x_{t-s} = x_t - (-a_1 x_{t-1} - ... - a_M x_{t-M}),$$  

where $a_0 = 1$. Furthermore let

$$x_t = \sum_{s=1}^{M} (-a_s) x_{t-s}. $$

Then

$$e_t = x_t - \hat{x}_t = \sum_{s=0}^{M} a_s x_{t-s} $$

is actually the error in predicting $x_t$ from the previous values $x_{t-1}, ..., x_{t-M}$ by means of the prediction coefficients $a_1, a_2, ..., a_M$. The error $e_t$, therefore, has to be kept close to a minimum for a good prediction.
The mean square error is defined to be

\[ E\{e_t^2\} = E\{(x_t - \hat{x}_t)^2\} = E\left\{ \left( \sum_{s=0}^{M} a_s x_{t-s} \right)^2 \right\} \]

where \( E \) stands for the expectation operator. Now

\[
E\{e_t^2\} = E\left\{ x_t \sum_{s=1}^{M} (-a_s) x_{t-s} \right\}^2
\]

\[ = E\{x_t^2\} - 2 \sum_{s=1}^{M} (-a_s) E\{x_t x_{t-s}\} + \sum_{s=1}^{M} (-a_s)^2 E\{x_{t-s}\}^2 \]

Since the \( x_i \)'s are assumed to be given, we would like to minimize the mean square error with respect to \( a_j, j = 1, \ldots, M \).

\[
\frac{\partial E\{e_t^2\}}{\partial a_j} = 2E\{x_t x_{t-j}\} + 2 \sum_{s=1}^{M} (-a_s) E\{x_t x_{t-s} (-x_{t-j})\}
\]

\[ = 0 . \]

By definition, the autocorrelation is

\[ \rho_t = E \left\{ x_s x_{s-t} \right\} , \]
for real $x$ values. So (2.41) becomes

$$(2.43) \quad 2p_j + 2 \sum_{s=1}^{M} (a_s) p_{j-s} = 0,$$

or

$$(2.44) \quad p_j + \sum_{s=1}^{M} a_s p_{j-s} = 0, j = 1, \ldots, M.$$

We thereby have the system of equations

$$\begin{align*}
p_1 + a_1 p_0 + \ldots + a_M p_{1-M} &= 0 \\
p_2 + a_1 p_1 + \ldots + a_M p_{2-M} &= 0 \\
\vdots & \vdots \\
p_M + a_1 p_{M-1} + \ldots + a_M p_0 &= 0.
\end{align*}$$

(2.45)

The above system of equations shows that each $p_j, j = 1, \ldots, M$ is a linear combination of the others. To complete the set, one adds the equation

$$(2.46) \quad p_0 + a_1 p_1 + \ldots + a_M p_M = E \sum_{t} x_t \psi_t,$$

which follows from (2.38) since
We can actually show that $P_M$ whose value appears in Equation (2.33) can be expressed as

$$P_M = E|e_t x_t| = E|e_t|^2.$$

(2.48)

That is, $P_M$ is just the least mean square error in the linear prediction. From (2.33) and (2.47), it follows that

$$P_M = \sum_{s=0}^{M} a_s \rho_s = E|e_t x_t|^2.$$

(2.49)

To show that $P_M = E|e_t|^2$ we first note that from (2.49) it follows that (2.46) can be rewritten with the quantity $P_M$ on the right hand side and when this equation is appended to the system (2.45) the result is identical to the system (2.34). This follows from the fact that $\rho_s - \rho_s$. Equation (2.34) is pre-multiplied by the row vector $[1 \ a_1 \ a_2 \ ... \ a_M]$ to get
\[
\begin{bmatrix} a_1 & a_2 & \ldots & a_M \end{bmatrix} \begin{bmatrix} c_0 & c_1 & \cdots & c_M \\ c_{-1} & c_0 & \cdots & c_{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{-M} & c_{-M+1} & \cdots & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_M \end{bmatrix} = P_M,
\]

(2.50)

from which it follows that

\[
P_M = \sum_{v=0}^{M} \sum_{s=0}^{M} a_v c_{v-s} a_s
\]

(2.51)

\[
= \sum_{v=0}^{M} \sum_{s=0}^{M} a_v E\{x_{t-s} x_{t-v} a_s\} = E\left(\sum_{s=0}^{M} a_s x_{t-s}\right)^2.
\]

Applying (2.39) gives

\[
P_M = E\left(\sum_{s=0}^{M} a_s x_{t-s}\right)^2 = E\{e_t\}^2.
\]

(2.52)

For a fixed \(M\) and set of \(M\) filter coefficients, every \(x_t\) in the set \(\{x_0, x_1, \ldots, x_{n-1}\}\) can be predicted linearly as long as one stays within the set of \(x\) - values. We can think of this as the process in Diagram 2.1 with \(a_0 = 1\). The above process is called filtering. In particular, it is called forward filtering, \(x_t\) being predicted from its \(M\) previous values. Backward filtering, on the other hand, is described in Diagram 2.2.
Diagram 2.1. Forward filtering.

Diagram 2.2. Backward filtering.

In Diagram 2.2, $x_{t-M}$ is predicted from its $M$ future values $x_{t-M+1}$, ..., $x_t$. The error in the prediction is $h_t$ and

$$ h_t = \sum_{s=0}^{M} a_s x_{t-M+s} $$

(2.53)

$$ = a_M x_t + a_{M-1} x_{t-1} + \ldots + a_1 x_{t-M+1} + x_{t-M} $$

Let

(2.54) $$ \hat{x}_{t-M} = \sum_{s=1}^{M} (a_s) x_{t-M+s} $$
Then

\[ E \left\{ h_t^2 \right\} = E \left\{ (x_{t-M} - x_{t-M})^2 \right\} \]

(2.55)

\[ = E \left\{ x_{t-M}^2 - 2 \sum_{s=1}^{M} (-a_s) x_{t-M+s} \right\} . \]

By taking the partial derivative of \( E \{ h_t^2 \} \) with respect to \( a_j, j = 1, \ldots, M \) and equating the results to 0, we get

\[
\frac{\partial E \{ h_t^2 \}}{\partial a_j} = \frac{\partial}{\partial a_j} \left[ E \left\{ x_{t-M} x_{t-M} \right\} - 2 \sum_{s=1}^{M} (-a_s) E \left\{ x_{t-M+s} x_{t-M} \right\} \right] \\
+ E \left\{ \sum_{s=1}^{M} (-a_s) x_{t-M+s} \right\} = 0 .
\]

(2.56)

By applying the definition of the autocorrelation function to (2.56), the following formula is obtained:

\[
\gamma_j + \sum_{s=1}^{M} a_s \rho_{j-s} = 0, j = 1, \ldots, M .
\]

(2.57)
Equation (2.57) is exactly Equation (2.44). This implies that forward and backward filtering lead to the same matrix equation and that

\[(2.58) \quad P_M = E_{|t|} = \mu_t \hat{Z}_t \]

for all possible \( t \).

The following simple example will help illustrate the ideas described in this section. Suppose the time series is the set of values \( A = \{x_0, x_1, x_2, x_3\} \). Let \( M = 2 \). The set \( A \) has two \( M + 1 \) ordered subsets: \( A_1 = \{x_0, x_1, x_2\} \) and \( A_2 = \{x_1, x_2, x_3\} \).

Denote the filter coefficients by \( a_0 = 1, a_1, a_2 \). Then both \( x_0 \) and \( x_1 \) can be predicted from their future values by backward filtering and

\[(2.59) \quad \hat{x}_0 = (-a_1) x_1 + (-a_2) x_2 , \]

\[(2.59) \quad \hat{x}_1 = (-a_1) x_2 + (-a_2) x_3 , \]

where \( \hat{x}_0, \hat{x}_1 \) are the predicted values of \( x_0, x_1 \) respectively.

The backward filtering errors are:

\[(2.60) \quad h_0 = x_0 - \hat{x}_0 = x_0 - x_0 + a_1 x_1 + a_2 x_2 , \]

\[(2.60) \quad h_1 = x_1 - \hat{x}_1 = x_1 - x_1 + a_1 x_2 + a_2 x_3 . \]
On the other hand, $x_2$ and $x_3$ are predicted by forward filtering and

$$
\hat{x}_2 = (-a_1) x_1 + (-a_2) x_0,
$$
(2.61)

$$
\hat{x}_3 = (-a_1) x_2 + (-a_2) x_1,
$$

where $\hat{x}_2, \hat{x}_3$ are the predicted values of $x_2$ and $x_3$ respectively.

The forward filtering errors are:

$$
e_2 = x_2 - \hat{x}_2 = x_2 + a_1 x_1 + a_2 x_0,
$$
(2.62)

$$
e_3 = x_3 - \hat{x}_3 = x_3 + a_1 x_2 + a_2 x_1.
$$

Briefly summarizing the discussion in the preceding pages, in predicting a point in a time series, either from its future or past values, one should minimize the mean square error $P_M$ with respect to the filter coefficients $a_j$. In the next section, we discuss an algorithm (developed by J. P. Burg, 1975) which determines the values of the $a_j$'s and $P_M$ in Equation (2.34).

F. The Burg Algorithm for Estimating $P_M$ and the Filter Coefficients

J. P. Burg's algorithm for finding the filter coefficients and $P_M$ in Equation (2.34) depend on both forward and backward filtering, i.e., the points in the time series are predicted from their previous and
future values. The solution to Equation (2.34) is based on a modified version of the Levinson algorithm for solving a Toeplitz matrix equation. Since the algorithm is an iterative procedure, the filter coefficients \((1, a_1, a_2, \ldots, a_M)\) will be written as \((1, a_1^M, a_2^M, \ldots, a_M^M)\).

Starting with a one-point filter \((a_{00} = 1)\), we have

\[
(2.63) \quad \begin{bmatrix} \rho_0 \\ a_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_{11} \end{bmatrix} = \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \quad \text{or} \quad \rho_0 = P_0
\]

So the value of \(P_0\) is determined by the autocorrelation \(\rho_0\). The two-point filter has the matrix equation

\[
(2.64) \quad \begin{bmatrix} \rho_0 & \rho_1 \\ \rho_1 & \rho_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_{11} \end{bmatrix} = \begin{bmatrix} P_1 \\ 0 \end{bmatrix}
\]

Assuming that \(a_{11}\) and \(P_1\) have been determined (the actual computations are described in the next section) the three-point filter is obtained from the preceding equation by means of the following extension:

\[
(2.65) \quad \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 \\ \rho_1 & \rho_0 & \rho_1 \\ \rho_2 & \rho_1 & \rho_0 \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ a_{11} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ a_{11} \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} P_1 \\ 0 \\ \Delta_1 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ \Delta_1 \\ P_1 \end{bmatrix}
\]
where it is required that

\[(2.66) \quad \alpha_1 + c_1 p_1 = 0 .\]

We note that the vectors \(c_1 = \begin{bmatrix} 0 \\ a_{11} \\ 1 \end{bmatrix}\) and \(c_1 = \begin{bmatrix} \alpha_1 \\ 0 \\ p_1 \end{bmatrix}\) are just multiples of the reverses of \(\ell_11\) and \(\ell_0\) respectively. The reason for also applying the filter in reverse is that each point in the time series \(x_0, x_1, ..., x_{n-1}\) is predicted forward and backward. The three-point filter then is \((1, a_{12}, a_{22})\) where

\[(2.67) \quad a_{12} = a_{11} + c_1 a_{11}, \quad a_{22} = c_1, \quad p_2 = p_1 + c_1 \Delta_1 .\]

Extending the matrix equation in (2.65) leads to:

\[(2.68) \quad \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_2 & 0 \\ \alpha_2 & \alpha_1 & 0 & \alpha_1 \\ \alpha_3 & \alpha_2 & \alpha_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ a_{12} \\ a_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} p_2 \\ 0 \\ 0 \\ \Delta_2 \end{bmatrix} + \begin{bmatrix} \Delta_2 \\ 0 \\ 0 \\ p_2 \end{bmatrix}.\]
where in this case the four-point filter is \((1, a_{13}, a_{23}, a_{33})\) with

\[
a_{13} = a_{12} + c_2 a_{22},
\]

\[
a_{23} = a_{22} + c_2 a_{12}, \quad a_{33} = c_2.
\]  

(2.69)

Again it is required that \(y_{22} + c_2 P_2 = 0\).

Continuing in this manner, we get the general form:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
a_1 & a_0 & \cdots & a_0 \\
a_1 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \cdots & 1 \\
a_1 & \cdots & \cdots & c_0
\end{bmatrix}
\begin{bmatrix}
a_{0, M} \\
a_{2, M} \\
a_{2, M} \\
\vdots \\
a_{0, M}
\end{bmatrix}
= \begin{bmatrix}
p_{M} \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]  

(2.70)

valid for any \(M < n\).

Furthermore, for \(i = 1, \ldots, M-1\),

\[
a_{i, M} = a_{i, M-1} + c_{M-1} a_{M-i, M-1},
\]

(2.71)

\[
a_{M, M} = c_{M-1}, \quad p_{M} = p_{M-1} + c_{M-1} \Delta_{M-1},
\]

\[
\Delta_{M-1} + c_{M-1} p_{M-1} = 0.
\]
Now since $M = c_{M-1}^P M$, then

$$
p_M = p_{M-1} + c_{M-1}^P \Delta_{M-1}
$$  \hspace{1cm} (2.72)

$$
= p_{M-1} - c_{M-1}^2 p_{M-1}
$$

$$
= p_{M-1} (1 - c_{M-1}^2)
$$

$$
= p_{M-1} (1 - a_{MM}^2)
$$

The least mean square error $p_M$ can only be estimated in this algorithm. Let us go back to the time series $\{x_0, x_1, \ldots, x_{n-1}\}$ and assume that there are $L$ possible subsets of the form $\{x_{t_j-M}, x_{t_j-M+1}, \ldots, x_{t_j}\}$, $j = 1, \ldots, L$. By applying the filter $(1, a_1, M, \ldots, a_{M,M})$ in a forward manner to each of the $x_{t_j}$, $j = 1, \ldots, L$, we get

$$
x_{t_j} + a_{1,M} x_{t_j-1} + \ldots + a_{M,M} x_{t_j-M}
$$

$$
= x_{t_j} + (a_{1,M-1} + c_{M-1} a_{M-1,M-1}) x_{t_j-1}
$$

$$
+ (a_{2,M-1} + c_{M-1} a_{M-2,M-1}) x_{t_j-2}
$$

$$
+ \ldots + c_{M-1} x_{t_j-M},
$$  \hspace{1cm} (2.73)

$$
j = 1, \ldots, L.
Equation (2.73) gives the error in predicting $x_{t+j}$ from its previous M values. Since there are L equations of the form (2.73), the mean square error $P_M$ can be estimated by the squared averages

$$P_{1M} = \frac{1}{\gamma_j} \sum_{j=1}^{L} \gamma_j \left[ x_{t_j} + (a_{1,M-1} + c_{M-1} a_{M-1,M-1}) x_{t_j-1} + \ldots + (a_{M-1,M-1} + c_{M-1} a_{1,M-1}) x_{t_j-M+1} + (c_{M-1} x_{t_j-M})^2 \right],$$

(2.74)

where the $\gamma_j$'s are weights such that $\sum_{j=1}^{L} \gamma_j = 1$ and $\gamma_j > 0$.

The terms in (2.74) can be re-grouped using the quantities

$$x_j = a_{M-1,M-1} x_{t_j-M+1} + \ldots + a_{1,M-1} x_{t_j-1} + x_{t_j},$$

(2.75)

$$x_j = x_{t_j-M} + a_{1,M-1} x_{t_j-M+1} + \ldots + a_{M-1,M-1} x_{t_j-1},$$

which are seen to be a forward prediction error and a backward prediction error respectively. The new expression is then

$$P_{1M} = \sum_{j=1}^{L} \gamma_j (\alpha_j + c_{M-1} \rho_j)^2,$$

(2.76)
If on the other hand, the filter \((1 \ a_1 \ldots \ a_M)\) were applied backward, then

\[
x_{t_j-M} + a_1 \cdot x_{t_j-M+1} + a_2 \cdot x_{t_j-M+2} + \ldots + a_M \cdot x_{t_j}
\]

\[
= x_{t_j-M} + (a_{1,M-1} + c_{M-1} \ a_{M-1,M-1}) x_{t_j-M+1}
\]

\[
+ (a_{2,M-1} + c_{M-1} \ a_{M-2,M-1}) x_{t_j-M+2}
\]

\[
+ \ldots + (a_{M-1,M-1} + c_{M-1} \ a_{1,M-1}) x_{t_j-1}
\]

\[
+ c_{M-1} \ x_{t_j}.
\]

As was noted earlier, the backward and forward filters should produce the same mean square error \(P_M\). So \(P_M\) can also be estimated by

\[
P_{2M} = \sum_{j=1}^{L} \gamma_j \left[ x_{t_j-M} + (a_{1,M-1} + c_{M-1} \ a_{M-1,M-1}) x_{t_j-M+1}
\right.

\[
+ (a_{2,M-1} + c_{M-1} \ a_{M-2,M-1}) x_{t_j-M+2}
\]

\[
+ \ldots + c_{M-1} \ x_{t_j}\right]
\]

\[
= \sum_{j=1}^{L} \gamma_j (\beta_j + c_{M-1} \ a_j)^2.
\]
Thus one class of estimates for $P_M$ is

\[ \hat{P}_M = \frac{1}{2}(P_{1M} + P_{2M}) \]

\[ = \frac{1}{2} \sum_{j=1}^{L} \gamma_j \left[ (\alpha_j + c_{M-1} \delta_j)^2 + (\phi_j + c_{M-1} \alpha_j)^2 \right] \]

where it is required that the $\gamma_j$ be specified.

G. Andersen’s Version of the Burg Algorithm for Estimating Coefficients

In applying the above algorithm, one would start with an estimate of $P_0$. Then at each stage $M$, the quantity $\hat{P}_M / \hat{a}_M$ is set equal to zero. For $M = 0$, $P_0$ is estimated by

\[ P_0 = r_0 \]

\[ = \left( \frac{1}{n-1} \right)^n \sum_{j=0}^{n-1} x_j x_j \]

where $r_0$ is an estimate of the autocorrelation $\alpha_0$.

For $M = 1$, the set $\{x_0, x_1, ..., x_{n-1}\}$ would have $n-1$ subsets: $\{x_0, x_1\}, \{x_1, x_2\}, ..., \{x_{n-2}, x_{n-1}\}$ and the average power is:

\[ \hat{P}_1 = \frac{1}{2} \frac{1}{n-1} \sum_{j=0}^{n-2} \left[ (x_j + a_{1j} x_{j+1})^2 + (x_{j+1} + a_{1j} x_{j})^2 \right] \]
where the weights are \( \gamma_j = \frac{1}{n-1} \), \( j = 0, \ldots, n-2 \) (see N. Andersen, 1974). Equating the partial of \( \hat{P}_1 \) with respect to \( a_{11} \) to zero, we get

\[
\frac{\partial \hat{P}_1}{\partial a_{11}} = \frac{1}{2(n-1)} \sum_{j=0}^{n-2} \left[ 2(x_j + a_{11} x_{j+1}) x_{j+1} + 2(x_{j+1} + a_{11} x_j) x_j \right] = 0.
\]

(2.82)

So

\[
a_{11} = -\frac{2}{n-2} \sum_{j=0}^{n-2} x_j x_{j+1} / \sum_{j=0}^{n-2} (x_j^2 + x_{j+1}^2).
\]

(2.83)

Since \( a_{11} \) is known at this point, one solves for \( \hat{P}_1 \) by using (2.72),

\[
\hat{P}_1 = \hat{P}_0 (1 - a_{11}^2) = r_0 (1 - a_{11}^2).
\]

(2.84)

For \( M=2 \), the set \{\( x_0, x_1, \ldots, x_{n-1} \}\) has \( n-2 \) subsets with \( M + 1 = 3 \) consecutive elements in each subset: \{\( x_0, x_1, x_3 \)\}, \{\( x_1, x_2, x_3 \)\}, \ldots, \{\( x_{n-3}, x_{n-2}, x_{n-1} \)\}. The resulting average power is:
\[ p_2 = \frac{1}{2} \sum_{j=0}^{n-3} (x_j + a_{12} x_{j+1} + a_{22} x_{j+2})^2 \]

\[ + (x_{j+2} + a_{12} x_{j+1} + a_{22} x_{j})^2 \]

(2.35)

\[ = \frac{1}{2} \sum_{j=0}^{n-3} [x_j + (a_{11} + a_{22} a_{11}) x_{j+1} + a_{22} x_{j+2}]^2 \]

\[ + [x_{j+2} + (a_{11} + a_{22} a_{11}) x_{j+1} + a_{22} x_{j+2}]^2 \]

since \( a_{12} = a_{11} + a_{22} a_{11} \) from (2.67). Thus

\[ \hat{p}_2 = \frac{1}{n-2} \sum_{j=0}^{n-3} [2(x_j + a_{11} x_{j+1} + a_{22} a_{11} x_{j+2}) + a_{22} x_{j+2}) (a_{11} x_{j+1} + x_{j+2}) + 2(x_{j+2} + a_{11} x_{j+1} + a_{22} a_{11} x_{j+2})] \]

(2.86)

\[ + a_{11} x_{j+1} + a_{22} a_{11} x_{j+1} + a_{22} x_{j}) (a_{11} x_{j+1} + x_{j}) \]

\[ = 0 . \]
The value for \( a_{22} \) is the following form:

\[
a_{22} = -2 \sum_{j=0}^{n-3} (a_{11} x_j x_{j+1} + a_{11}^2 x_j^2 + a_{11} x_{j+1} x_{j+2} + x_j x_{j+2}) \\
+ x_j x_{j+2} + \sum_{j=0}^{n-3} (x_j^2 + 2a_{11} x_j x_{j+1} + 2a_{11}^2 x_{j+1}^2 + 2a_{11} x_{j+1} x_{j+2} + x_{j+2}^2).
\]

(2.87)

Since \( a_{22} \) is known, \( P_2 \) is given by the expression:

\[
\hat{P}_2 = \hat{P}_1 (1 - a_{22}^2).
\]

(2.88)

Continuing in this manner, at the \( M \)th stage, there will be \( n-M \) subsets of \( \{x_0, x_1, \ldots, x_{n-1}\} \) and they are of the form:

\( \{x_0, x_1, \ldots, x_M\} \), \( \{x_1, x_2, \ldots, x_{M+1}\} \), \ldots, \( \{x_{n-M-1}, x_{n-M}, \ldots, x_{n-1}\} \). The average output power is then

\[
P_M = \frac{1}{2} \sum_{j=0}^{n-M} \left[ \frac{(n-1)-M}{n-M} \left( x_j + \sum_{k=1}^{M} a_{k,M} x_{j+k} \right)^2 \\
+ (x_j+M + \sum_{k=1}^{M} a_{k,M} x_{j+M-k})^2 \right]
\]

(2.89)

\[
= \frac{1}{2} \sum_{j=0}^{n-M} \left[ \frac{(n-1)-M}{n-M} \left( \sum_{k=0}^{M} a_{k,M} x_{j+k} \right)^2 \\
+ \left( \sum_{k=0}^{M} a_{k,M} x_{j+M-k} \right)^2 \right].
\]
where $a_{0,M} = 1$. From (2.71), we recall that $a_{i,M} = a_{i,M-1} + a_{M-i,M-1}, i = 1, 2, \ldots, M-1$. If we impose the added condition that $a_{0,M} = 1$ and $a_{i,M} = 0$ for $i > M$, then (2.71) holds for all $M$ and (2.89) simplifies to:

$$
(2.90) \quad P_M = \frac{1}{2} \frac{1}{n-M} \sum_{j=0}^{n-M}(\sum_{k=0}^{M} (a_{k,M-1} + a_{M-k,M-1}) x_{j+k})^2
$$

$$
+ (\sum_{k=0}^{M} (a_{k,M-1} + a_{M-k,M-1}) x_{j+k})^2
$$

$$
= \frac{1}{2} \frac{1}{n-M} \sum_{j=0}^{n-M} \left[ \sum_{k=0}^{M} a_{k,M-1} x_{j+k} + a_{M-k,M-1} \sum_{k=0}^{M} a_{M-k,M-1} x_{j+k} \right]^2
$$

$$
+ (\sum_{k=0}^{M} a_{k,M-1} x_{j+k})^2 + a_{M-k,M-1} \sum_{k=0}^{M} a_{M-k,M-1} x_{j+k})^2
$$

Let

$$
(2.91) \quad p_{j,M} = \sum_{k=0}^{M} a_{k,M-1} x_{j+k} = \sum_{k=0}^{M} a_{M-k,M-1} x_{j+k},
$$

$$
q_{j,M} = \sum_{j=0}^{M} a_{M-k,M-1} x_{j+k} = \sum_{k=0}^{M} a_{k,M-1} x_{j+k},
$$

$$
j = 0, 1, \ldots, (n-1) - M.$$
Then

\[ p_M = \frac{1}{2} \frac{1}{n-M} \sum_{j=0}^{(n-1)-M} \left[ (p_{j \cdot m} + a_{MM} q_{j\cdot m})^2 + (q_{j \cdot M} + a_{\cdot M \cdot M} p_{j \cdot M})^2 \right] . \]  

By setting \( \frac{\partial p_M}{\partial a_{MM}} = 0 \), we get the value for \( a_{MM} \):

\[ a_{MM} = -2 \frac{(n-1)-M}{\sum_{j=0}^{(n-1)-M} p_{j \cdot M} q_{j \cdot M} \sum_{j=0}^{(n-1)-M} (p_{j \cdot M}^2 + q_{j \cdot M}^2)} . \]  

The recursion formulas for the \( p_{jm} \) and \( q_{jm} \) are derived from (2.71) and (2.91). They are:

\[ p_{j\cdot M} = p_{j,M-1} + a_{M-1,M-1} q_{j,M-1} , \]
\[ q_{j\cdot M} = q_{j+1,M-1} + a_{M-1,M-1} p_{j+1,M-1} . \]

The starting values for the equations in (2.94) for \( M=1 \) are:

\[ p_{j1} = x_j , q_{j1} = x_{j+1} , j = 0, 1, \ldots, n-2 . \]

The equations in (2.94) and (2.95) are implemented in the computer program which is discussed later in this report.
CHAPTER III

APPLICATIONS OF MESA AND FOURIER ANALYSIS

A. Comparative Study of MESA and Fourier Analysis.

EXAMPLE 3.1.

To study the effectiveness of the two methods, consider a mathematical test case consisting of sine functions. The equation is:

\[ x_t = 3 \sin (0.05(2\pi t)) + 10 \sin (0.1(2\pi t)) + 5 \sin (0.2(2\pi t)) + 8 \sin (0.21(2\pi t)), \]

\( t = 0, 1, ..., 99. \)

No trend in the time series is removed in the analysis. The discretized function \( x_t \) and its spectra are shown in Figures 3.1A, 3.1B, and 3.1C. By examining the equation for \( x_t \), one can tell that the true periods are at the frequencies: \( f = 0.05, f = 0.1, f = 0.2, f = 0.21 \), corresponding to periods: \( P = 20, P = 10, P = 5 \) and \( P = 4.76 \) respectively. Table 3.1 gives a listing of the periods as predicted by both forms of analysis.

As one can infer from Figures 3.1B, 3.1C and Table 3.1, MESA gives somewhat better predictions of the frequencies of the periodic components. The number of filter coefficients \( M \) used in MESA for this particular problem is 51.
Fig. 3.1. A. Mathematical test case of four sinusoids (100 data points).
B. Fourier spectrum.
C. Maximum entropy spectrum with M = 51.
TABLE 3.1.

FREQUENCY COMPONENTS OF THE SINE PROBLEM
OF EXAMPLE 3.1 (100 data points)

<table>
<thead>
<tr>
<th>Frequencies at which true periods occur</th>
<th>Frequencies predicted by MESA - 51 filter coefficients used</th>
<th>Frequencies predicted by Fourier analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) 0.05</td>
<td>0.0500</td>
<td>0.0475</td>
</tr>
<tr>
<td>(2) 0.10</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>(3) 0.20</td>
<td>0.2000</td>
<td>0.1975</td>
</tr>
<tr>
<td>(4) 0.21</td>
<td>0.2125</td>
<td>0.2125</td>
</tr>
</tbody>
</table>

EXAMPLE 3.2

What happens in the spectrum of the time series $x_t$ in Example 3.1 if the number of sampled points is reduced? Suppose only 20 points are used, say at $t = 0, 1, \ldots, 19$. The shorter time series $x_t$ is shown in Figure 3.2A and its spectra in Figures 3.2B and 3.2C. Table 3.2 lists the frequencies for both spectra.

TABLE 3.2.

FREQUENCY COMPONENTS OF THE SINE PROBLEM OF EXAMPLE 3.1 (20 data points)

<table>
<thead>
<tr>
<th>Frequencies at which true periods occur</th>
<th>Frequencies predicted by MESA - 13 filter coefficients used</th>
<th>Frequencies predicted by Fourier analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) 0.05</td>
<td>0.0475</td>
<td>0.0300</td>
</tr>
<tr>
<td>(2) 0.10</td>
<td>0.0975</td>
<td>0.0925</td>
</tr>
<tr>
<td>(3) 0.20</td>
<td>0.1925</td>
<td>0.2075</td>
</tr>
<tr>
<td>(4) 0.21</td>
<td>0.2075</td>
<td>0.2075</td>
</tr>
</tbody>
</table>
Fig. 3.2. A. Mathematical test case of four sinusoids (20 data points).
B. Fourier spectrum.
C. Maximum entropy spectrum with $M = 13$. 
Since the number of data points is relatively small, one certainly can expect shifts in the spectral peaks. Such shifts are present in both Figures 3.2B and 3.2C, but the frequencies predicted by MESA are consistently as accurate or more accurate than those predicted by Fourier analysis. The true peak at $f = 0.2$ is missing in the Fourier spectrum, but is approximated by $f = 0.1925$ in the MESA spectrum. The fact that MESA is able to distinguish between two very close frequencies indicates one strong advantage this method has over Fourier analysis. One should also take note that the longest period $P = 20$ is approximated by $P = 21.053$ by MESA, but is given as $P = 33.33$ by the Fourier spectrum.

So far the applications we have been considering involve mathematical test cases, where the periods of the functions are known in advance. In the following three examples, we present real time series.

EXAMPLE 3.3

Figure 3.3A gives a plot of the Mackenzie River region lynx fur returns of the Hudson Bay Company from 1821-1934 (Elton and Nicholson 1942), a period covering 114 years. It is evident that a definite pattern exists. In fact, as one closely examines the points, a period of roughly 10 years emerges. Indeed, both Fourier and MESA spectra bring out this dominant period as can be seen in Figures 3.3B and 3.3C. Both lynx spectra agree quite closely in their estimates of the period of the major peak and of the two more important minor peaks, corresponding to periods of roughly 38 and 11 years. The cycle of about 38 years seems
Fig. 3.3. A. Lynx fur returns (1821 - 1934), 
(Hudson Bay Company data: Straight 
line represents the linear trend). 
B. Maximum entropy spectrum with 
M = 69.
to be a harmonic of 9.66 years, the major period, for $9.66 \times 4 = 38.64$ years.

TABLE 3.3

FREQUENCY COMPONENTS OF LYNX DATA

<table>
<thead>
<tr>
<th>Major frequencies predicted by MESA - 69 filter coefficients used</th>
<th>Corresponding periods</th>
<th>Major frequencies predicted by Fourier analysis</th>
<th>Corresponding periods</th>
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<tbody>
<tr>
<td>(1) 0.0258</td>
<td>38.65</td>
<td>0.0258</td>
<td>38.65</td>
</tr>
<tr>
<td>(2) 0.0959</td>
<td>10.42</td>
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<td>10.89</td>
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<tr>
<td>(3) 0.1035</td>
<td>9.66</td>
<td>0.1035</td>
<td>9.66</td>
</tr>
<tr>
<td>(4) 0.1310</td>
<td>7.63</td>
<td>0.1327</td>
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<tr>
<td>(5) 0.2086</td>
<td>4.79</td>
<td>0.2078</td>
<td>4.31</td>
</tr>
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</table>

EXAMPLE 3.4

Another well-studied time series is the Zurich sunspot record from 1700 to 1976 (see Figure 3.4A) as given by M. Waldemeier. The sunspot time series shows a strong periodicity of 11 years. There also seems to be a period of about 10 years if one divides the interval 1700-1976 into roughly 1700-1800, 1800-1900, 1900-1976. Table 3.4 gives a listing of the more dominant peaks in the spectra (Figures 3.4B and 3.4C).

The 99.83 year - period appears to be a harmonic of 9.983, since $9.983 \times 10 = 99.83$. Several papers have been written [see for example (Currie 1972), (Cole 1973)] in an attempt to explain the different periods (other than those appearing in Table 3.4) in the sunspot spectrum. Some authors have claimed that certain periods are the results of beat
Fig. 3.4. A. Waldenmayer Sunspot data (1700 - 1976) (Straight line represents the linear trend).
B. Fourier spectrum.
C. Maximum entropy spectrum with $M = 160$. 
TABLE 3.4
FREQUENCY COMPONENTS OF SUNSPOT DATA

<table>
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<tr>
<th>Major frequencies predicted by MESA - 160 filter coefficients used</th>
<th>Corresponding periods</th>
<th>Major frequencies predicted by Fourier analysis</th>
<th>Corresponding periods</th>
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</thead>
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<td>99.83</td>
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<td>(2) 0.0175</td>
<td>57.05</td>
<td>0.0175</td>
<td>57.05</td>
</tr>
<tr>
<td>(3) 0.0834</td>
<td>11.98</td>
<td>0.0826</td>
<td>12.10</td>
</tr>
<tr>
<td>(4) 0.0909</td>
<td>10.99</td>
<td>0.0901</td>
<td>11.09</td>
</tr>
<tr>
<td>(5) 0.0943</td>
<td>10.60</td>
<td>0.0952</td>
<td>10.51</td>
</tr>
<tr>
<td>(6) 0.1001</td>
<td>9.98</td>
<td>0.1001</td>
<td>9.98</td>
</tr>
</tbody>
</table>

frequencies (Wolff 1975). It is commonly accepted that the cycle of about 11 years is a real one.

In the sunspot and lynx records, the number of available data points is more than 10 times the length of the most dominant periods (the MESA estimates of which are 9.66 years in the lynx and 10.99 in the sunspots). Both MESA and Fourier analyses are in agreement with regard to the frequency estimates in the lynx spectrum and the sunspot spectrum. The performance of Fourier analysis is comparable to that of MESA in cases where there are long records available for the time series under study.
EXAMPLE 3.5

We now consider a shorter time series consisting of 42 data points - the annual commercial landings of the striped bass from the Middle Atlantic region in the years 1930 - 1971 (Van Winkle et al., in press) (see Figure 3.5A). A summary of the frequency components is listed in Table 3.5.

TABLE 3.5
FREQUENCY COMPONENTS OF MIDDLE ATLANTIC STRIPED BASS COMMERCIAL LANDINGS

<table>
<thead>
<tr>
<th>Major frequencies predicted by MESA - 26 filter coefficients used</th>
<th>Corresponding periods</th>
<th>Major frequencies predicted by Fourier analysis</th>
<th>Corresponding periods</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>(2) 0.0550</td>
<td>18.15</td>
<td>0.0509</td>
<td>19.64</td>
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<tr>
<td>(3) 0.0793</td>
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</tr>
<tr>
<td>(4) 0.1302</td>
<td>7.68</td>
<td>0.1310</td>
<td>7.63</td>
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<tr>
<td>(5) 0.2671</td>
<td>3.74</td>
<td>0.2621</td>
<td>3.82</td>
</tr>
</tbody>
</table>

The spectra in Figures 3.5B and 3.5C differ in the frequency interval (0.00 - 0.10). In the Fourier spectrum, only one peak at $f = 0.0509$ appears within the interval. A lot of unresolved power is also present. In the same interval the MESA spectrum has three peaks at $f = 0.0250$, $f = 0.0550$, and $f = 0.0793$. The frequency $f = 0.2621$ with corresponding period $P = 3.815$ in Figure 2.5R [denoted peak (5)] appears to be a harmonic of $f = 0.1310$, since $2 \times 0.1310 = 0.2620$. 
Fig. 3.5. A. Middle Atlantic striped bass yearly landings (1930 - 1971) (Straight line represents the linear trend).
B. Fourier spectrum.
C. Maximum entropy spectrum with $M = 26$. 
B. The Effect of Noise

To compare the performance of the two methods when there is noise in the time series, we consider the sine function of Example 3.1, but utilize only the first 20 points (from $t = 0$ to $t = 19$). There are two major reasons for doing so. First of all, the longest period of the sine function is at frequency $f = 0.05$, corresponding to a period of $P = 20$. In Fourier analysis, the longest period predicted is the length of the time series. Secondly, we would like to see how well MESA and Fourier analysis perform whenever the length of available data is relatively short. We take note that when the number of points is 100, the predictions given by both methods are in fairly good agreement (see Table 3.1). When the number of points decreases to 20, the difference between the MESA spectrum and the Fourier spectrum can easily be detected (see Table 3.2). The equation under study here is:

$$x_t = 3 \sin (0.05 (2 \pi t)) + 10 \sin (0.1 (2 \pi t)) +$$
$$5 \sin (0.2 (2 \pi t)) + 3 \sin (0.21 (2 \pi t)) +$$
$$R(t; N(0, 6.5)),$$

$t = 0, 1, \ldots, 19$, where

$R(t; N(0, 6.5))$ = a random number drawn for each time $t$ from a normal distribution $N(0, 6.5)$;

$N(0, 6.5)$ = the normal distribution with mean 0 and standard deviation 6.5;

6.5 = $\bar{x}$ ($3 + 10 + 5 + 8$), the average of the amplitudes;

$\bar{x}$ = the noise level on a scale of 0.0 to 1.0.
### Table 3.6

**Fourier Analysis: Frequency Components of Example 3.1**  
(20 data points at different noise levels)

<table>
<thead>
<tr>
<th>Noise Level ((\lambda))</th>
<th>Frequencies ((f)) of the three highest peaks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.05</td>
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<tr>
<td>0.00</td>
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<td>0.0300</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0300</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0275</td>
</tr>
<tr>
<td>0.30</td>
<td>0.0275</td>
</tr>
</tbody>
</table>

Table 3.6 gives the estimated Fourier frequencies for various noise levels. The peak at \(f = 0.05\) is approximated by \(f = 0.03\) when \(\lambda = 0.0, 0.05\) and 0.10. As \(\lambda\) increases to 0.20 and 0.30, the peak shifts even farther away from the true one. In the sine function under study, the largest amplitude is 10 and it belongs to the sinusoid with period at \(f = 0.10\). This sinusoid is least affected by the increase in noise. The frequency estimate of \(f = 0.10\) is \(f = 0.0925\) for the first four levels of \(\lambda\) and only drops to \(f = 0.09\) when \(\lambda\) increases to 0.30. The third peak of the sine function is at \(f = 0.2000\) and does not appear in Table 3.6 at any noise level. This is because the true peak at \(f = 0.2100\) is very close to the true peak \(f = 0.09\). As a result, there is a lot of unresolved power at \(f = 0.2075\), at all levels of \(\lambda\) in Table 3.6 (see Figure 3.28 for \(\lambda = 0.00\)).
In Table 3.7, the results from MESA are given at different values of the number of filter coefficients $M$. For $M = 3$ and $\alpha = 0.0, 0.05, 0.10, 0.20, 0.30$, one obtains a very poor spectral representation indicating only one of the four frequencies. As $M$ takes the values 4, 5, 6, 7, at least two of the periods are resolved at $\alpha = 0.0, 0.05$ and 0.10. As the level of noise goes up, the frequencies undergo greater shifts for these values of $M$. Moreover, when $\alpha = 0.30$, spurious peaks appear at $f = 0.4100$ (for $M = 6$) and $f = 0.3525$ (for $M = 7$). Using $M = 8, 9, 10, 11$ results in a spectrum with three of the four frequencies of $x_t$ at low levels of noise. As $M$ takes the values 12 - 19 and when the noise level stays relatively low, all four frequencies are estimated but only with the addition of spurious peaks as well.

We notice that using too small a value of $M$ results in low resolution, and using too high a value of $M$ results not only in high resolution but also in the appearance of several spurious peaks. Furthermore, for both high levels of noise and large values of $M$, the number of spurious peaks tends to be high.

Relatively good spectra are obtained for values of $M$ in the range 8 - 14. Although spurious peaks do appear, MESA still gives better estimates than Fourier analysis. Also the problem of sidebands in Fourier analysis is absent from the MESA spectrum (see Figures 3.1B, 3.1C, 3.2B and 3.2C).

Figures 3.6A, 3.6B and 3.6C illustrate the MESA spectrum of $x_t$ at $\alpha = 0.0, \alpha = 0.10$, and $\alpha = 0.30$ respectively. The value of $M$ for these plots is 10. Because of the linear plot scaling, the peaks at
### TABLE 3.7

**MESA: FREQUENCY COMPONENTS OF EXAMPLE 3.1**  
(20 data points at different noise levels)

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<tr>
<th>Number of filter coefficients (M)</th>
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<th>10%</th>
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<td>0.1900</td>
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*Denotes frequencies at which spurious peaks occur.*
TABLE 3.7: Cont'd.

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*Denotes frequencies at which spurious peaks occur.
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<th>Noise Level (α)</th>
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*Denotes frequencies at which spurious peaks occur.
TABLE 3.7: Cont'd.

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*Denotes frequencies at which spurious peaks occur.
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<td>0.0850*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
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<td>0.0975</td>
<td>0.0950</td>
<td>0.0950</td>
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</tr>
<tr>
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<tr>
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</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td>0.2725*</td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.3150*</td>
<td>0.3150*</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>0.4125*</td>
<td>0.4000*</td>
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<td>0.0975</td>
<td>0.0975</td>
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<td>0.0950</td>
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<tr>
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<td></td>
<td></td>
<td></td>
<td>0.1300*</td>
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<tr>
<td></td>
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<td></td>
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<td>0.1800</td>
<td>0.1600</td>
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<tr>
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<td>0.2050</td>
<td>0.2050</td>
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</tr>
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<td>0.2675*</td>
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<td></td>
<td></td>
<td></td>
<td>0.3250*</td>
<td>0.3100*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.3150*</td>
<td>0.3150*</td>
</tr>
</tbody>
</table>

*Denotes frequencies at which spurious peaks occur.
<table>
<thead>
<tr>
<th>Number of filter coefficients (M)</th>
<th>Noise Level (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>0.3350*</td>
</tr>
<tr>
<td></td>
<td>0.4150*</td>
</tr>
<tr>
<td>19</td>
<td>0.0325*</td>
</tr>
<tr>
<td></td>
<td>0.0500</td>
</tr>
<tr>
<td></td>
<td>0.0875*</td>
</tr>
<tr>
<td></td>
<td>0.0975</td>
</tr>
<tr>
<td></td>
<td>0.1350*</td>
</tr>
<tr>
<td></td>
<td>0.1800</td>
</tr>
<tr>
<td></td>
<td>0.2025*</td>
</tr>
<tr>
<td></td>
<td>0.2075</td>
</tr>
<tr>
<td></td>
<td>0.2250*</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Denotes frequencies at which spurious peaks occur.
Fig. 3.6. Maximum entropy spectrum of four sinusoids

A: M = 10^3, α = 0.01
B: M = 10^1, α = 0.10
C: M = 10^1, α = 0.30
0.3075 and 0.4025 in Figure 3.6C cannot be seen, but do show up when a logarithmic scale is used.

The preceding studies gives some indications of the advantages of MESA over Fourier analysis. Below are the major points to be considered in choosing between the two methods.

<table>
<thead>
<tr>
<th>Fourier Analysis</th>
<th>MESA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Assumes a zero extension of the data and uses data windows.</td>
<td>1. Makes no assumptions outside the given data and no data windows are used.</td>
</tr>
<tr>
<td>2. The longest period predicted is the length of available data.</td>
<td>2. Can predict periods longer than the length of available data.</td>
</tr>
<tr>
<td>3. Sideband effects are produced resulting from power leakage.</td>
<td>3. No sidebands are produced.</td>
</tr>
<tr>
<td>4. Frequency components very close to each other cannot be resolved.</td>
<td>4. Can resolve frequency components very close to each other.</td>
</tr>
</tbody>
</table>

Another basic difference between the MESA spectrum and the Fourier spectrum is that in the former, the power in the spectral peak is proportional to the area under the curve, whereas in the latter, the power is proportional to the height of the peak.
CHAPTER IV

GUIDELINES IN SPECTRAL STUDIES

A. The Akaike Final Prediction Error (FPE).

The determination of the number of filter coefficients (M) which produces the best spectrum is one of the major unsolved problems in MESA. As a general rule, using a very small value for M results in a spectrum with low resolution; on the other hand, a very large value of M, which is close to the total number of data points n results in spurious peaks in the spectrum. A common practice is to use values of M close to half the total number of data points.

T. J. Ulrych (1975) suggested the use of the Akaike final prediction error criterion. This criterion is expressed in terms of the mean square error $P_M$ of the prediction filter used to estimate the MESA spectrum. The idea is to get the least value for $P_M$ at which point the error in the prediction is minimum.

Again our time series will be $x_0, x_1, ..., x_{n-1}$. The final prediction error is defined by:

$$FPE(M) = Q \cdot \hat{P}_M,$$

where $\hat{P}_M$ is the expression in Equation (2.92) and

$$Q = \begin{cases} \frac{n + M}{n - M} & \text{if the time series is not detrended.} \\ \frac{n + M + 1}{n - (M + 1)} & \text{if the mean is removed.} \\ \frac{n + M + 2}{n - (M + 2)} & \text{if the series is linearly detrended.} \end{cases}$$
To implement the FPE criterion, one computes Equation (4.1) for all $M \leq n-1$. The value of $M$ which gives the smallest $\text{FPE}(M)$ determines the number of filter coefficients to use. Table 4.1 lists the FPE values of the sine function in Example 3.1. The FPE values are given for $V' = 1, \ldots, 97$. They indicate a final prediction error function which attains a minimum at $M = 77$.

### Table 4.1

FPE VALUES FOR EXAMPLE 3.1 (100 data points)

<table>
<thead>
<tr>
<th>Number of filter coefficients $M$</th>
<th>FPE</th>
<th>Number of filter coefficients $M$</th>
<th>FPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.67667D-01</td>
<td>21</td>
<td>4.69691D-12</td>
</tr>
<tr>
<td>2</td>
<td>2.93505D-01</td>
<td>22</td>
<td>8.43425D-13</td>
</tr>
<tr>
<td>3</td>
<td>1.65996D-01</td>
<td>23</td>
<td>7.30391D-13</td>
</tr>
<tr>
<td>4</td>
<td>8.86949D-01</td>
<td>24</td>
<td>2.63503D-14</td>
</tr>
<tr>
<td>5</td>
<td>2.94535D-01</td>
<td>25</td>
<td>9.82543D-15</td>
</tr>
<tr>
<td>6</td>
<td>1.92259D-01</td>
<td>26</td>
<td>6.92126D-15</td>
</tr>
<tr>
<td>7</td>
<td>9.21715D-02</td>
<td>27</td>
<td>4.30912D-15</td>
</tr>
<tr>
<td>8</td>
<td>1.60844D-03</td>
<td>28</td>
<td>3.76044D-15</td>
</tr>
<tr>
<td>9</td>
<td>7.56116D-04</td>
<td>29</td>
<td>1.37626D-15</td>
</tr>
<tr>
<td>10</td>
<td>2.98365D-04</td>
<td>30</td>
<td>9.00214D-16</td>
</tr>
<tr>
<td>11</td>
<td>2.36250D-04</td>
<td>31</td>
<td>9.20203D-16</td>
</tr>
<tr>
<td>12</td>
<td>4.74525D-06</td>
<td>32</td>
<td>4.59301D-16</td>
</tr>
<tr>
<td>13</td>
<td>1.09780D-06</td>
<td>33</td>
<td>2.44462D-16</td>
</tr>
<tr>
<td>14</td>
<td>5.55949D-07</td>
<td>34</td>
<td>2.46991D-16</td>
</tr>
<tr>
<td>15</td>
<td>3.85160D-07</td>
<td>35</td>
<td>2.12235D-16</td>
</tr>
<tr>
<td>16</td>
<td>6.04607D-08</td>
<td>36</td>
<td>1.97891D-16</td>
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<tr>
<td>17</td>
<td>2.05121D-08</td>
<td>37</td>
<td>1.08859D-16</td>
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<td>18</td>
<td>3.47970D-09</td>
<td>38</td>
<td>1.06943D-16</td>
</tr>
<tr>
<td>19</td>
<td>3.10793D-09</td>
<td>39</td>
<td>8.10464D-17</td>
</tr>
<tr>
<td>20</td>
<td>1.06768D-11</td>
<td>40</td>
<td>7.90130D-17</td>
</tr>
</tbody>
</table>
The spectrum for \( M = 77 \) (Figure 4.1) shows the peaks at \( f = 0.05, 0.10, 0.20, \) and 0.2125. These are the same frequencies predicted with \( M = 51 \) (see Figure 3.1C). Although a minimum FPE is obtained for \( M = 77 \), we studied several spectra for \( M \) in the interval \((50, 80)\). The FPE
assumes a relatively flat valley in this interval. Hence, very good estimates of the true peaks are produced. In this case the FPE does not seem to be a very discriminating criterion.

Fig. 4.1. Maximum Entropy Spectrum of Example 3.1 with M=77.

As another example of the unreliability of the minimum FPE criterion, we have plotted the FPE for the Middle Atlantic striped bass landings (Example 3.5) in Figure 4.2A. The FPE minimum is at M = 1. We have already noted that using very small values of M results in a poorly resolved spectrum. However, as one examines the plot for increasing values of M, a major numerical difference is noted between M = 25 and M = 26. The FPE function increases at almost every step before M = 25 and then takes a sudden drop at M = 26. It is interesting to see what happens to the spectrum as M goes from 25 to 26.
Fig. 4.2. A. FPE of the Middle Atlantic striped bass landings.
B. Maximum entropy spectrum with $M = 25$.
C. Maximum entropy spectrum with $M = 26$. 
Figure 4.28 shows the spectrum for M = 25 and Figure 4.2C for M = 26. A major change occurs in the frequency interval (0.0, 0.1) as M changes from 25 to 26. Instead of just one major peak within this interval when M = 25, three appear when M = 26. It appears that the three peaks are brought into 'focus'.

For many of the other time series that we have analyzed, the same kind of behavior occurs - that is, when the FPE takes a sudden drop in value, the spectrum produces sharper peaks. Whether, of course, the other two peaks that come into 'focus' are significant is another question.

It is also good practice, when deciding what value of M to choose, to refer to the Fourier spectrum of the time series as a guide in determining what value of M would give a good MESA spectrum.

3. Fisher's Test Statistic - The Periodogram Test

The periodogram is one of the earliest forms of spectral estimates. It can actually be derived from the sample spectrum in Equations (1.11), (1.12), (1.13). Using the same notation as in Chapter I, the periodogram values FF(f) are given by the formula:

\[
(4.3) \quad FF(f) = \frac{n}{2} \left[ a(f)^2 + b(f)^2 \right]
\]
where \( f = k/n, \ k = 1, \ldots, m \) if \( n = 2m \). If \( n = 2m + 1 \), Equation (4.3) applies for \( f = k/n, \ k = 0, 1, \ldots, m \). Since we are using discrete values for \( f \), (4.3) can be simplified into:

\[
(4.4) \quad FF_k = \frac{n}{2} \left[ a_k^2 + b_k^2 \right].
\]

One of the questions often considered is whether the maximum value of \( FF_k \) is significantly above the noise level. With Fisher's test, we proceed the following way to answer the question. Let

\[
FF_L = \text{the largest periodogram ordinate}, \quad \text{and}
\]

\[
\sum_{k=1}^{m} FF_k = \text{the sum of the periodogram ordinates}.
\]

The test statistic used is

\[
(4.5) \quad z = \frac{FF_L}{\left[ (1/m) \sum_{k=1}^{m} FF_k \right]}.
\]

Fisher (1929) showed that for \( g > 0 \), the probability that the largest of \( m \) normalized terms \( (FF_i/ \sum_{k=1}^{m} FF_k, \ i = 1, \ldots, m) \) should exceed \( g \) is

\[
(4.6) \quad P_r \left( m^{-1} z > g \right) = m (1-g)^{m-1} - \frac{m-1}{2} \left( 1 - 2g \right)^{m-1} + \ldots + \frac{m!}{s! (m-s)!} \left( 1-sg \right)^{m-1}
\]
where $s$ is the largest integer less than $1/g$. Table 4.2 gives the 1 and 5 percentage points for the distribution of $\xi$ as given by W. Fuller (1976).

**TABLE 4.2**

PERCENTAGE POINTS FOR THE RATIO OF LARGEST PERIODOGRAM ORDINATE TO THE AVERAGE

<table>
<thead>
<tr>
<th>Number of ordinates</th>
<th>Probability of larger value</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>1.950</td>
</tr>
<tr>
<td>3</td>
<td>2.613</td>
</tr>
<tr>
<td>4</td>
<td>3.072</td>
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<tr>
<td>5</td>
<td>3.419</td>
</tr>
<tr>
<td>6</td>
<td>3.697</td>
</tr>
<tr>
<td>7</td>
<td>3.928</td>
</tr>
<tr>
<td>8</td>
<td>4.125</td>
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<tr>
<td>9</td>
<td>4.297</td>
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<tr>
<td>10</td>
<td>4.450</td>
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<tr>
<td>15</td>
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<td>5.701</td>
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<td>5.935</td>
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<tr>
<td>40</td>
<td>6.295</td>
</tr>
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<td>50</td>
<td>6.567</td>
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<td>60</td>
<td>6.785</td>
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<td>70</td>
<td>6.967</td>
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<tr>
<td>80</td>
<td>7.122</td>
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<td>90</td>
<td>7.258</td>
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<td>100</td>
<td>7.378</td>
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<td>350</td>
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<td>8.889</td>
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<td>500</td>
<td>9.123</td>
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<td>9.313</td>
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<td>700</td>
<td>9.473</td>
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<td>800</td>
<td>9.612</td>
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<tr>
<td>900</td>
<td>9.733</td>
</tr>
<tr>
<td>1000</td>
<td>9.642</td>
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</table>
To illustrate Fisher's test, the lynx data will be used. The sample spectrum (Figure 3.3B) exhibits the tallest peak corresponding to a period of 9.66 years. Recall that the straight line trend was subtracted from the data before calculating the spectrum. The null hypothesis is

\[ H_0: x_t = \mu + \beta t + e_t. \]

The alternative hypothesis is

\[ H_A: x_t = \mu + \beta t + A \sin \left( \frac{2\pi t}{9.66} + \phi \right) + e_t, \]

where \( e_t \) is the stochastic error term.

The frequencies used for the sample spectrum are \( f = i/114, i = 1,2, \ldots, 57 \). Then

\[
\frac{\sum_{k=1}^{57} \text{FF}_k}{(1/57) \sum_{k=1}^{57} \text{FF}_k} = \frac{(1.42571 \times 10^3)}{(1/57)(2.79734 \times 10^3)}
\]

\[ = 29.0511 \]

From Table 4.2, the 1% point for this value of \( \xi \) is about 8.20. So 29.055 > 8.20 and \( H_0 \) is rejected. One concludes that the 9.66 period is statistically significant.
C. The Cumulative Periodogram Test

In analyzing a real time series like the lynx data, our normal course is to remove the linear trend in the data before analysis. That is, we really are fitting the model

\[ x_t = \mu + \beta t + e_t. \]

The series that then undergoes analysis is \( e_t \), the estimated error. Thus a question that one might ask is whether the \( e_t \)'s under study comprise just white noise.

It can be shown (Box and Jenkins 1970) that the power spectrum \( P(f) \) for white noise has a constant value \( 2\sigma_e^2 \) (\( \sigma_e^2 \) is the variance for \( e \)) over the frequency interval \([0.0, 0.5]\). Thus the integrated spectrum for white noise is

\[
(4.7) \quad PP(f) = \int_0^f P(v) \, dv = 2 \sigma_e^2 f,
\]

with \( f \) defined in the interval \([0.0, 0.5]\).

Dividing both sides of (4.7) by \( \sigma_e^2 \) gives the equation

\[
(4.8) \quad PP(f) / \sigma_e^2 = 2 f.
\]
Thus the plot of $PP(f)/\sigma_e^2$ against $f$ is a straight line joining $(0,0)$ to $(0.5, 1)$. For test data, the integrated spectrum can be approximated by the cumulative periodogram $CP_k$:

$$CP_k = \sum_{i=1}^{k} FF_i / \sum_{j=1}^{m} FF_j, \; k = 1, \ldots, m,$$

which is compared with the white noise straight line.

To determine the significance of the deviations of the cumulative periodogram of the test data from the theoretical straight line joining $(0,0)$ to $(0.5,1)$, limit lines can be drawn on each side of the theoretical line, using the Kolmogorov–Smirnov test (Box and Jenkins 1970). If indeed the series $e_t$ were to comprise white noise, then the cumulative periodogram would cross these limit lines with a stated probability. It should be pointed out however, that the probabilities are only approximated. The limit lines in the frequency interval $[0.0, 0.5]$ are given by the equations:

$$y = 2 f + K_x / \sqrt{q}$$

where

$$q = \begin{cases} 
(n-2) / 2, & \text{if } n \text{ is even} \\
(n-1) / 2, & \text{if } n \text{ is odd},
\end{cases}$$

$f$ is the frequency in the interval $[0.0, 0.5]$, $n$ is the total number of data points, $\alpha$ the stated probability.
The cumulative periodogram of a truly random series would then lie outside the limit lines of Equation (4.10) for an approximate fraction \( \alpha \) of the time. Table 4.3 lists \( K_\alpha \) versus \( \alpha \). This table is adapted from Box and Jenkins (1970).

<table>
<thead>
<tr>
<th>Probability ( \alpha )</th>
<th>( K_\alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.63</td>
</tr>
<tr>
<td>0.05</td>
<td>1.36</td>
</tr>
<tr>
<td>0.10</td>
<td>1.22</td>
</tr>
<tr>
<td>0.25</td>
<td>1.02</td>
</tr>
</tbody>
</table>

Figure 4.3 is the cumulative periodogram of the lynx data after the linear trend is removed. In this case \( n = 114 \). One can see that the periodogram has certainly crossed the limit lines, \( y = 2f \pm K_{0.05} \sqrt{q} \), where \( q = \sqrt{(114-2)/2} = 7.48 \). Our conclusion therefore is that \( e_t \) is not white noise and the errors are not independent. The model \( x_t = \mu + \beta t + e_t \) is thus inadequate.

D. The Integrated Maximum Entropy Spectrum

In the sample spectrum in Fourier analysis, the power density in a peak (i.e., the strength of contribution of the frequency \( f \)) is directly related to the peak amplitude. In the maximum entropy spectrum, the power density is directly related to the area under the spectral peak [see Lacoss (1971)]. So in order to judge the contribution of a spectral peak in MESA, the area under the peak has to be computed. This
Fig. 4.3. Cumulative periodogram of the Lynx Data.

- Solid Curve - Cumulative periodogram.
- Dashed lines - Limit lines \( y = 2f \pm k_{0.05} / \sqrt{q} \).

suggests the use of an integrated spectrum obtained from the convolution of the MESA spectrum with a rectangular window of height unity and width equal to the effective width of the peaks in the MESA spectrum. The heights of the peaks in the integrated spectrum will then be proportional to the corresponding power densities.

The maximum entropy spectrum as given by Equation (2.35) is

\[
(4.12) \quad P(f) = \frac{P_M^2}{1 + \left| \sum_{m=1}^{M} a_m e^{-i2\pi fm\lambda t} \right|^2}
\]
The integrated spectrum IP(v) will then be of the form

\[ IP(v_i) = \int_{f_i}^{f_i+1} P(f) \, df, \]

where \( f_{i+1} - f_i \) is the bandwidth of integration and
\[ v_i = (f_{i+1} + f_i)/2. \]

To illustrate the above ideas, consider the spectrum \( P(f) \) in Figure 4.4A. By choosing an appropriate width of integration, one can find the areas under the two spectral peaks. Figure 4.4B shows the estimated integral of \( P(f) \) where

\[ IP(v_j) = \int_{f_j}^{f_j+1} P(f) \, df. \]

For an \( M \)-length prediction error filter, only \( M/2 + 1 \) discrete frequency spectral components can be obtained (Jensen and Ulrych 1973). Hence to obtain an estimate of the bandwidth of integration we take the Nyquist frequency \( f_M \) and divide it by \( M/2 + 1 \). Once the integrated spectrum is computed, the height of a peak would then be proportional to the power density.
Fig. 4.4. (A) A sample maximum entropy spectrum. (B) The integrated maximum entropy spectrum.

The integrated MESA spectrum for a given time series is very similar in appearance to the Fourier spectrum for the same series, but it is not possible to easily judge whether or not a peak in the integrated MESA spectrum is significantly above the noise level because there does not yet exist a MESA analogue of Fisher's test for the periodogram. The reason is that at the present time no one knows the statistical distribution of integrated spectral estimates for random time series. It is, however, possible to obtain a significance test by a rather cumbersome Monte-Carlo technique which has been described by Jensen and Ulrych (1973). Essentially what one does is to generate a large number
of random time series having the same number of sample points and the
same sample spacing as the series being studied. One then calculates
the integrated MESA spectrum of each of these random series using the
same number of filter coefficients and the same integration window width
used to construct the integrated spectrum of the original sample. One
can then use this collection of integrated spectra to empirically con­
struct a table giving, as a function of peak height, the probability of
obtaining that peak height by chance simply by counting the number of
spectra having peaks that high or higher and dividing by the total num­
ber of spectra used in the p.cess. The quality of the approximate
probabilities obtained in this manner will, of course, improve as more
random time series are used in the procedure.
CHAPTER V
THE COMPUTER PROGRAM AND ITS USE

A. Explanation of the Subroutines

Essentially, the program consists of the following routines:

(1) Main program - which initializes the data values and other parameters needed in spectral analysis.

(2) Subroutine TRANSl - which provides different calls to SPECTR.

(3) Subroutine SPECTR - which does spectral analysis using both Fourier and MESA methods. In turn, SPECTR calls:

   (a) Subroutine DETRND - a routine to be provided by the user if detrending is done.

   (b) Subroutine SFT - computes the Fourier sine and cosine terms needed to produce the sample spectrum.

   (c) DCADRE - a built-in IMSL subroutine used in computing the integrated spectrum if it is desired.

   (d) Double Precision Function SPCVAL - computes the function used in integrating the spectrum.

   (e) Subroutines DOPLT, DOPLT1, DOPLT2, DOPLT3, DOPLT4, DOPLT5, DOPLT6, DOPLT7, DOPLT8, DOPLT9, SINGLE, GPHBGN, GPHBG1, GPHEND, ANGTIC - perform the necessary plotting using the DISSPLA routines.

1. The Main Program

   The parameters which are given initial values are read in this section of the program and are transferred to TRANSl and SPECTR by the common statement:
Table 5.1 lists in alphabetical order the variables which are read, their dimensions, types and what they represent. Currently, the dimensions allow a maximum of 600 data points to be analyzed by SPECTR.

**TABLE 5.1**

**LIST OF VARIABLES READ IN THE MAIN PROGRAM**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>DATE</td>
<td>600</td>
<td>REAL*8</td>
<td>time array (times at which sample values occur).</td>
</tr>
<tr>
<td>DELTA</td>
<td></td>
<td>REAL*8</td>
<td>sampling interval.</td>
</tr>
<tr>
<td>IAUTCV</td>
<td>I*4</td>
<td></td>
<td>indicator for option to plot MESA estimate of autocovariance function.</td>
</tr>
<tr>
<td>IFPEPL</td>
<td>I*4</td>
<td></td>
<td>indicator for option to print/plot final prediction error.</td>
</tr>
<tr>
<td>IINTSP</td>
<td>I*4</td>
<td></td>
<td>indicator for option to compute integrated MESA spectrum.</td>
</tr>
<tr>
<td>IPRINT</td>
<td>I*4</td>
<td></td>
<td>indicator for option to plot spectra.</td>
</tr>
<tr>
<td>Variable</td>
<td>Dimension</td>
<td>Type</td>
<td>Explanation</td>
</tr>
<tr>
<td>----------</td>
<td>-----------</td>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>IPRINT</td>
<td>I*4</td>
<td></td>
<td>indicator for option to print output.</td>
</tr>
<tr>
<td>ITREND</td>
<td>I*4</td>
<td></td>
<td>indicator for option to detrend data.</td>
</tr>
<tr>
<td>IYWACV</td>
<td>I*4</td>
<td></td>
<td>indicator for option to compute and plot Yule-Walker estimate of the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>autocovariance function.</td>
</tr>
<tr>
<td>MCOUNT</td>
<td>600</td>
<td>I*4</td>
<td>contains the array of numbers which correspond to the number of filter</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>coefficients desired in each MESA spectrum.</td>
</tr>
<tr>
<td>NAM</td>
<td>14</td>
<td>Real*4</td>
<td>title for the data set.</td>
</tr>
<tr>
<td>NFILT</td>
<td>I*4</td>
<td></td>
<td>total number of maximum entropy spectra to be calculated with different</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>values of the number of filter coefficients.</td>
</tr>
<tr>
<td>NPTS</td>
<td>I*4</td>
<td></td>
<td>number of data points in time series.</td>
</tr>
<tr>
<td>NV</td>
<td>I*4</td>
<td></td>
<td>number of frequencies desired in output; should not exceed a maximum value</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>of 599.</td>
</tr>
</tbody>
</table>
TABLE 5.1: Cont'd.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>VMAX</td>
<td></td>
<td>Real*8</td>
<td>maximum frequency for the output spectrum; has a maximum value of 0.5, the Nyquist frequency.</td>
</tr>
<tr>
<td>XLBL</td>
<td>2</td>
<td>Real*4</td>
<td>label for time axis (x-axis) used in plotting.</td>
</tr>
<tr>
<td>Y</td>
<td>600</td>
<td>Real*8</td>
<td>time series to be analyzed.</td>
</tr>
<tr>
<td>YLBL</td>
<td>2</td>
<td>Real*4</td>
<td>label for time series data (y-axis) used in plotting.</td>
</tr>
</tbody>
</table>

The indicators IAUTCV, IFPEPL, IINTSP, IPRINT, ITREND, IYWACV should be initialized to values of 0 or 1. In each case, 1 is a positive response and 0 is a negative response.

The variables NAM, XLBL, YLBL, which are read in the main program, appear in the common block BLKLBL. These are mainly used for identification of data and plots. BLKLBL is shared by the Main program, TRANSL, SPECTR and DOPLT. The form of the statement is

```
COMMON/BLKLBL/NAM(14), XLBL(2), YLBL(2) .
```

The input cards, all of which are read in the main program, must have the following formats. The formats appear in the order the cards are read.
(1) Input card on first READ statement.

```
XLBL  YLBL  DELTA  NPTS  NFILT  NV  VMAX
  $   $
  2A4  2X  2A4  2X  E10.5  I5  I5  I5  E10.0
```

XLBL and YLBL must terminate with a dollar sign.

(2) Input card on second READ statement.

```
NAM
  $
  14A4
```

NAM must also end in a dollar sign.

(3) Input cards for third READ statement.

```
MCOUNT(1)  MCOUNT(2)  ...  MCOUNT(NFILT)
  I5  I5  I5
```

(4) Input cards for fourth READ statement. This set contains the time series to be analyzed.

```
DATE(1)  Y(1)
DATE(2)  Y(2)
  .  .
  .  .
  .  .
DATE(NPTS)  Y(NPTS)
  F4.0  F10.0
```
(5) Input card for the fifth READ statement.

```
IPRINT  IPLOT  ITREND  IFPEPL  IINTSP  IAUTCV  IYWACV
```

```
15  15  15  15  15  15  15  15
```

A flowchart of the main program is given in Figure 5.1.

```
Fig. 5.1. Flowchart of the Main Program. Numbers (e.g., 1 and 500) refer to statement numbers in the program.
```
2. **SUBROUTINE TRANS1**

The variables in this subroutine are those that appear in the common area SPELBL. This subroutine allows the user to perform transformations on the raw data (e.g., take logarithms) before doing spectral analysis. Up to five different spectra with five different transformations are allowed.

The basic statements in TRANS1 are the following:

1. **SUBROUTINE TRANS1**
   
   Optional user supplied statements for desired transformation on data.
   
   ```
   CALL SPECTR
   .
   .
   .
   RETURN
   ```

2. **ENTRY TRANS2**
   
   Optional user supplied statements for second desired transformation.
   
   ```
   CALL SPECTR
   .
   .
   .
   RETURN
   ```
(3) ENTRY TRANS3
   Optional user supplied statements.
   CALL SPECTR
   .
   .
   .
   RETURN

(4) ENTRY TRANS4
   Optional user supplied statements.
   CALL SPECTR
   .
   .
   .
   RETURN

(5) ENTRY TRANS5
   Optional user supplied statements.
   CALL SPECTR
   .
   .
   .
   RETURN

If the user wants to analyze only the input data, he need not supply additional statements and any of the 4 additional entries can be effectively deactivated by omitting CALL SPECTR.
An example of data transformation using the second Entry call
(i.e., ENTRY TRANS2) is the following:

ENTRY TRANS2
DO 20 I = 1, NPTS
  Y(I) = DLOG(Y(I))
20 CONTINUE
CALL SPECTR
RETURN

In the preceding statements, the natural logarithm of the input data
Y(I), I=1, ..., NPTS is taken and the result is stored in Y(I) again.
Thus when SPECTR is called, the logarithmically transformed data are
analyzed. The program listing for SUBROUTINE TRANS1 given in Appendix
III gives the configuration of statements needed to first analyze the
original data and then its logarithm.

3. SUBROUTINE SPECTR

The main function of SPECTR is to perform the following:

(1) Detrend the input data by calling DETRND if desired;

(2) Compute:
    (a) the Yule-Walker estimates of the autocovariance function if
desired.
    (b) the Fourier spectrum.
    (c) the periodogram ordinates and their sum.
    (d) the cumulative periodogram.
    (e) the maximum entropy spectrum.
(f) the autocorrelation function resulting from the maximum entropy computations.

(g) the integrated maximum entropy spectrum if desired.

(h) the final prediction error if desired.

(3) Print and plot computed values.

A flowchart of subroutine SPECTR appears in Figure 5.2. Table 5.2 lists the parameters in subroutine SPECTR which appear in common blocks on DIMENSION statements. This list includes the variables which must appear in the calling program and the arrays whose dimension the user might want to alter if he desires to make changes in the programs. Variables that are only used internally in SPECTR or in subroutines called by SPECTR are not included in the list.

4. SUBROUTINE DETRND

DETRND is called by SPECTR with the statement:

CALL DETRND (NPTS, DATE, YEAR, Y, XDIFF, X, SFPE)

where

NPTS - number of data points, i.e., length of time series.
DATE - time array that must not be destroyed.
YEAR - user-evaluated time values; these are to be initialized in DETRND.
Y - input time series values; these values must not be destroyed.
XDIFF - array containing the detrended data; computed in DETRND.
X - array containing the trend of the data; computed in DETRND.
Fig. 5.2. Flowchart of Subroutine SPECTR. Numbers (e.g., 100 and 300) correspond to statement numbers in the program.
## TABLE 5.2
LIST OF PARAMETERS IN SUBROUTINE SPECTR

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>600</td>
<td>Real*8</td>
<td>Vector of working storage.</td>
</tr>
<tr>
<td>ACFN</td>
<td>600</td>
<td>Real*8</td>
<td>Vector of working storage.</td>
</tr>
<tr>
<td>ACS</td>
<td>600</td>
<td>Real*8</td>
<td>Vector of working storage.</td>
</tr>
<tr>
<td>AMP</td>
<td>600</td>
<td>Real*8</td>
<td>Vector of working storage.</td>
</tr>
<tr>
<td>ASF</td>
<td>600</td>
<td>Real*8</td>
<td>Vector containing Fourier power spectra values.</td>
</tr>
<tr>
<td>B</td>
<td>600</td>
<td>Real*8</td>
<td>Vector of working storage.</td>
</tr>
<tr>
<td>BSN</td>
<td>600</td>
<td>Real*8</td>
<td>Vector of working storage.</td>
</tr>
<tr>
<td>CUMPER</td>
<td>300</td>
<td>Real*8</td>
<td>Array of cumulative periodogram values.</td>
</tr>
<tr>
<td>DATE</td>
<td>600</td>
<td>Real*8</td>
<td>Time values for input series.</td>
</tr>
<tr>
<td>DELTA</td>
<td></td>
<td>Real*8</td>
<td>Sampling interval ('delta T').</td>
</tr>
<tr>
<td>FPE</td>
<td>600</td>
<td>Real*8</td>
<td>Array containing final prediction error.</td>
</tr>
<tr>
<td>FPER</td>
<td>300</td>
<td>Real*8</td>
<td>Array of periodogram frequencies.</td>
</tr>
<tr>
<td>IAUTCV</td>
<td></td>
<td>I*4</td>
<td>Autocovariance plotting flag (Burg estimate).</td>
</tr>
<tr>
<td>IFPEPL</td>
<td></td>
<td>I*4</td>
<td>Final prediction error flag.</td>
</tr>
<tr>
<td>Variable</td>
<td>Dimension</td>
<td>Type</td>
<td>Explanation</td>
</tr>
<tr>
<td>-----------</td>
<td>-----------</td>
<td>-----------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>IPLOT</td>
<td>I*4</td>
<td></td>
<td>Plotting flag.</td>
</tr>
<tr>
<td>IPRINT</td>
<td>I*4</td>
<td></td>
<td>Printing flag.</td>
</tr>
<tr>
<td>ITREND</td>
<td>I*4</td>
<td></td>
<td>Detrending flag.</td>
</tr>
<tr>
<td>IYWACV</td>
<td>I*4</td>
<td></td>
<td>Yule-Walker autocovariance flag.</td>
</tr>
<tr>
<td>MCOUN</td>
<td>600</td>
<td>I*4</td>
<td>Vector of prediction error filter lengths to use.</td>
</tr>
<tr>
<td>NAM</td>
<td>14</td>
<td>I*4</td>
<td>Title for the data sec.</td>
</tr>
<tr>
<td>NFILT</td>
<td>I*4</td>
<td></td>
<td>Number of prediction error filter lengths to try.</td>
</tr>
<tr>
<td>NPTS</td>
<td>I*4</td>
<td></td>
<td>Number of data points in time series.</td>
</tr>
<tr>
<td>NV</td>
<td>I*4</td>
<td></td>
<td>Number of frequencies at which spectrum is calculated.</td>
</tr>
<tr>
<td>PERDG</td>
<td>300</td>
<td>Real*8</td>
<td>Array of periodogram values.</td>
</tr>
<tr>
<td>PERIOD</td>
<td>600</td>
<td>Real*8</td>
<td>Vector of working storage. Upon return to the main program, this vector contains the periods corresponding to the frequencies in V.</td>
</tr>
<tr>
<td>PPERK</td>
<td>300</td>
<td>Real*8</td>
<td>Array of periodogram periods corresponding to FPER.</td>
</tr>
</tbody>
</table>
TABLE 5.2: Cont'd

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>600</td>
<td>Real*8</td>
<td>Maximum entropy spectral values are stored in this array.</td>
</tr>
<tr>
<td>STATE</td>
<td>2</td>
<td>Real*4</td>
<td>Label for dependent variable (y-axis)</td>
</tr>
<tr>
<td>V</td>
<td>600</td>
<td>Real*8</td>
<td>Output vector of frequencies at which spectrum is calculated.</td>
</tr>
<tr>
<td>VMAX</td>
<td></td>
<td>Real*8</td>
<td>Maximum frequency at which spectrum is to be calculated.</td>
</tr>
<tr>
<td>X</td>
<td>600</td>
<td>Real*8</td>
<td>Upon returning from DETRND (if detrending is done), it should contain the trend of the input data. It is later used as a vector of working storage.</td>
</tr>
<tr>
<td>XD</td>
<td>600</td>
<td>Real*8</td>
<td>If detrending is done, it should contain the detrended data; if not, it is the original data. Spectral analysis is applied on XD.</td>
</tr>
<tr>
<td>XDIFF</td>
<td>600</td>
<td>Real*8</td>
<td>Upon returning from DETRND, it is the detrended data. It is later used as a working vector.</td>
</tr>
</tbody>
</table>
TABLE 5.2: Cont'd

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>600</td>
<td>Real*8</td>
<td>Input time series values.</td>
</tr>
<tr>
<td>YEAR</td>
<td>600</td>
<td>Real*8</td>
<td>Vector of working storage (can be used as time variable in DETRND).</td>
</tr>
<tr>
<td>YTIME</td>
<td>2</td>
<td>Real*4</td>
<td>Label for the time axis.</td>
</tr>
<tr>
<td>Z</td>
<td>600</td>
<td>Real*8</td>
<td>Prediction error filter coefficients.</td>
</tr>
</tbody>
</table>

SFPE - number of components in the trend of the time series to be removed. For example, if the mean is removed, SFPE = 1; if the intercept and slope are to be removed, SFPE = 2, and so on. SFPE is given the value 0 in SPECTR if detrending is not done.

Should the user choose to analyze the original data without detrending, the dummy subroutine in Table 5.3 may be used.

If the mean of the input data is to be removed, i.e., the model \( x_t = \mu + e_t \) is to be fitted, then the subroutine in Table 5.4 will be appropriate to use.

Table 5.5 contains a detrending subroutine in which the linear trend of a time series is eliminated. A least-squares routine BLSQ is called to perform the fitting of the straight line to the data. A write-up of BLSQ appears in Appendix II. SUBROUTINE BLSQ is available from disk at ORNL (see, for example, control card for "BROOKS.LOAD. MODULES" in Tables 6.3 and 6.4).
### Table 5.3. Dummy subroutine DETRND

<table>
<thead>
<tr>
<th>Statement number columns 1-5</th>
<th>Continuation column 6</th>
<th>Fortran statement columns 7-72</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SUBROUTINE DETRND(NPTS,DATE,YEAR,Y,XDIFF,X,SFPE)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IMPLICIT REAL*(A-H,O-Z)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>DIMENSION DATE(1), YEAR(1), Y(1), XDIFF(1), X(1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RETURN</td>
</tr>
<tr>
<td></td>
<td></td>
<td>END</td>
</tr>
</tbody>
</table>

```fortran
SUBROUTINE DETRND(NPTS,DATE,YEAR,Y,XDIFF,X,SFPE)
IMPLICIT REAL*(A-H,O-Z)
DIMENSION DATE(1), YEAR(1), Y(1), XDIFF(1), X(1)
RETURN
END
```
A detrending subroutine for removal of the mean of a time series

**Table 5.4. A detrending subroutine for removal of the mean of a time series**

<table>
<thead>
<tr>
<th>Statement number</th>
<th>Continuation column 6</th>
<th>Fortran statement columns 7-72</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SUBROUTINE DETRND(NPTS,DATE,YEAR,Y,YDIFF,X,SFPE)</td>
<td>Since only the mean is removed, SFPL = 1.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IMPLICIT REAL*8(A-H,O-Z)</td>
<td>Compute the mean of the Y array.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>DIMENSION DATE(1), YEAR(1), Y(1), YDIFF(1), X(1)</td>
<td>Store the trend of the data in Y array.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SFPE = 1.0</td>
<td>Store the detrended data in XDIFF array.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>AVERAGE = 0.0</td>
<td></td>
</tr>
<tr>
<td>DO 100 I = 1, NPTS</td>
<td></td>
<td>AVERAGE = AVERAGE + Y(I)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>AVERAGE = AVERAGE / NPTS</td>
<td></td>
</tr>
<tr>
<td>DO 300 I = 1, NPTS</td>
<td></td>
<td>X(I) = AVERAGE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>XDIFF(I) = Y(I) - X(I)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td>RETURN</td>
<td></td>
<td>END</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.5. A detrending subroutine for the removal of the linear trend in a time series

<table>
<thead>
<tr>
<th>Statement number</th>
<th>Continuation column 6</th>
<th>Fortran statement columns 7-72</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SUBROUTINE DETRND(NPTS,DATE,YEAR,Y,XDIFF,X,SFPE)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>IMPLICIT REAL*8(A-H,O-Z)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DIMENSION DATE(1),YEAR(1),Y(1),XDIFF(1),X(1),</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>A(601,3),IRUN(601),COEF(2),COLF(2),T(2,2),T1(2,2),</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>VAR(2,2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>NROW = 601</td>
<td>NROW is the row dimension of the least squares matrix A.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>NLSCOL = 2</td>
<td>NLSCOL is the number of coefficients of the linear fit.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>NT = 2</td>
<td>NT is the row dimension of the matrices T and T1, which are computed in the least squares routine BLSQ.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SFPE = NLSCOL</td>
<td>SFPE = NLSCOL since there are two linear coefficients - the intercept and slope - to be removed.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 50 I = 1,NPTS</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>YEAR(I) = 1 - 0.5</td>
<td>Initialize the time variable</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A(I,1) = 1.0</td>
<td>Set up the least squares matrix.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A(I,2) = YEAR(I)</td>
<td></td>
</tr>
</tbody>
</table>

NROW is the row dimension of the least squares matrix A. NLSCOL is the number of coefficients of the linear fit. NT is the row dimension of the matrices T and T1, which are computed in the least squares routine BLSQ. SFPE = NLSCOL since there are two linear coefficients - the intercept and slope - to be removed. Initialize the time variable. Set up the least squares matrix.
<table>
<thead>
<tr>
<th>Statement number columns 1-5</th>
<th>Continuation column 6</th>
<th>Fortran statement columns 7-72</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>CONTINUE</td>
<td>CALL BLSQ(A,Y,COEF,RESID,IRUN,T,T1,NPTS,NLSCOL,</td>
<td>Call the least squares routine BLSQ to perform the linear fit.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>NROW,NLSCOL)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 80 I = 1,NLSCOL</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SUM = 0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 70 J = 1,NLSCOL</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SUM = SUM + T1(I,J) * COEF(J)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>COEF2(I) = SUM</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 80 I = 1,NLSCOL</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>COEF(1) = COEF2(I)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 100 I = 1,NPTS</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>X(I) = A(I,NLSCOL + 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CALL BLSQ(A,Y,COEF,RESID,IRUN,T,T1,NPTS,NLSCOL,</td>
<td>Compute the coefficients of the linear fit using values returned from BLSQ.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>NROW,NLSCOL)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 80 I = 1,NLSCOL</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SUM = 0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 70 J = 1,NLSCOL</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SUM = SUM + T1(I,J) * COEF(J)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>COEF2(I) = SUM</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 80 I = 1,NLSCOL</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>COEF(1) = COEF2(I)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>DO 100 I = 1,NPTS</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>X(I) = A(I,NLSCOL + 1)</td>
<td></td>
</tr>
</tbody>
</table>

Call the least squares routine BLSQ to perform the linear fit.
Compute the coefficients of the linear fit using values returned from BLSQ.

COEF(1) contains the intercept of the time series.
COEF(2) contains the slope of the trend line.
The trend of the data comes back in A(I,NLSCOL + 1) and is stored in X array.
Table 5.5. (continued)

<table>
<thead>
<tr>
<th>Statement number columns 1-5</th>
<th>Continuation column 6</th>
<th>Fortran statement columns 7-72</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td></td>
<td>XOIFF(I) = Y(I) - X(I)</td>
<td>The detrended data is stored in XOIFF array.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CONTINUE</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>RETURN</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>END</td>
<td></td>
</tr>
</tbody>
</table>
SUBROUTINE BLSQ returns (in the array T1) the information needed to compute the variance - covariance matrix of the coefficients (i.e. the slope and intercept) and these values could be stored in the array VAR which is not used in the preceding program (Table 5.5). A SUBROUTINE DETRND which does add this final step is given in the program listings in Appendix IV.

It should be obvious from studying Table 5.5 and Appendix II how to remove more complicated detrending functions using BLSQ or even a non-linear least squares program if needed.

5. SUBROUTINE SFT

The computation of the factors

\[ \sum_{j=0}^{n-1} x_j \cos 2\pi jf , \sum_{j=0}^{n-1} x_j \sin 2\pi jf , \]

from Equations (1.12) and (1.13) respectively, which are used in the evaluation of the sample Fourier spectrum, is done in SFT. Subroutine SFT is called by SPECTR with:

CALL SFT(XD, ACS, BSM, V, KF1, WFR, PI) .

The corresponding arguments in SFT are given by:

SUBROUTINE SFT(X, A, B, V, K, N, PI)

These arguments are described in Table 5.6.
6. DOUBLE PRECISION FUNCTION SPCVAL

The double precision function SPCVAL computes for any frequency the maximum entropy spectral value $P(f)$ as given by Equation (2.35). SPCVAL is used in the integrated maximum entropy spectrum calculations. It is the function which is integrated by the IMSL subroutine DCADRE, which is described in Appendix I. The IMSL subroutine DCADRE is available from disk at ORNL (see, for example, control card for "JDAIMSL1" in Tables 6.3 and 6.4). Table 5.7 lists the important parameters in SPCVAL.

**TABLE 5.6**  
**A LIST OF THE PARAMETERS IN SUBROUTINE SFT**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>Real*8</td>
<td>The array containing the factors $\sum_{j=0}^{n-1} x_j \cos 2\pi j f$ upon return. It has a maximum dimension of 600.</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>Real*8</td>
<td>The array containing the factors $\sum_{j=1}^{n-1} x_j \sin 2\pi j f$ upon return. It has a maximum dimension of 600.</td>
</tr>
<tr>
<td>K</td>
<td>I*4</td>
<td></td>
<td>Number of frequencies at which spectrum is calculated.</td>
</tr>
<tr>
<td>N</td>
<td>I*4</td>
<td></td>
<td>Number of points in the series under analysis.</td>
</tr>
<tr>
<td>PI</td>
<td>Real*8</td>
<td></td>
<td>The real number $\pi$.</td>
</tr>
</tbody>
</table>
### TABLE 5.6: Cont'd

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>1</td>
<td>Real*8</td>
<td>Output vector of frequencies at which spectrum is calculated.</td>
</tr>
<tr>
<td>X</td>
<td>1</td>
<td>Real*8</td>
<td>Time series under analysis.</td>
</tr>
</tbody>
</table>

### TABLE 5.7

A list of parameters in the function subroutine SPCVAL

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMP</td>
<td>600</td>
<td>Real*8</td>
<td>The array containing $P_M$.</td>
</tr>
<tr>
<td>DTIME</td>
<td></td>
<td>Real*8</td>
<td>The time interval $\Delta t$.</td>
</tr>
<tr>
<td>FREQ</td>
<td></td>
<td>Real*8</td>
<td>The frequency $f$.</td>
</tr>
<tr>
<td>M</td>
<td>4</td>
<td></td>
<td>The number of filter coefficients $M$.</td>
</tr>
<tr>
<td>PI</td>
<td></td>
<td>Real*8</td>
<td>The real number $\pi$.</td>
</tr>
<tr>
<td>SPCVAL</td>
<td></td>
<td>Real*8</td>
<td>The spectral value $P(f)$.</td>
</tr>
<tr>
<td>Z</td>
<td>600</td>
<td>Real*8</td>
<td>The array of filter coefficients $a_i,M$.</td>
</tr>
</tbody>
</table>
7. **SUBROUTINE DOPLT**

Subroutine DOPLT is a plotting routine which calls other subrou­tines primarily from the plotting package DISSPLA (DISSPLA is a univer­sal package that performs the necessary plotting). DISSPLA is available from disk at ORNL (see, for example, control card for "DISSPLA.LOAD" in Tables 6.3 and 6.4). There are nine ENTRY statements, the function of each are listed as follows:

1. **ENTRY DOPLT1** - plots the original data against the time values; if the data are detrended, the trend is also plotted on the same set of axes.

2. **ENTRY DOPLT2** - if detrending is performed, the detrended data are plotted against the time values; the line $y = 0$ is also included in the plot and serves as a reference line for the residuals. Furthermore, a plot of the residuals against the fitted values or trend is done.

3. **ENTRY DOPLT3** - plots the Yule-Walker autocorrelation function against the lag values if IYWACV = 1.

4. **ENTRY DOPLT4** - plots the Fourier sample spectrum against frequency; it also plots the cumulative Fourier power spectrum against frequency. The 95% Kolmogorov-Smirnov bounds are also plotted on the latter graph.

5. **ENTRY DOPLT5** - plots the Burg autocorrelation function against the number of filter coefficients if IAUTCV = 1.

6. **ENTRY DOPLT6** - plots the maximum entropy spectrum against frequency.

7. **ENTRY DOPLT7** - plots the integrated maximum entropy spectrum against frequency if IINTSP = 1.
(6) ENTRY DOPLT8 - plots the final prediction error against the number of filter coefficients if IFPEPL = 1.

(9) ENTRY DOPLT9 - plots the periodogram and cumulative periodogram as a function of frequency. The 95% Kolmogorov-Smirnov bounds are included in the latter plot.

The other subroutines called by DOPLT after each ENTRY statement are mainly for setting up the axes for plotting. They are:

(1) SUBROUTINE SINGLE - transforms double precision variables into single precision. This is necessary since DISSPLA routines handle only Real*4 information.

(2) SUBROUTINE ANGTIC - controls the number of tick marks and angular position of the numbers on the axes.

(3) SUBROUTINE GPHBGN - sets up the initial x and y axes for plotting according to the range of the data pairs.

(4) SUBROUTINE GPHBG1 - sets up the initial x and y axes for plotting the FPE. This uses logarithmic scaling of the axes, as contrasted with GPHBGN which makes use of linear scaling.

(5) SUBROUTINE GPHEND - plots the specific curve and draws a frame about the picture.

All the necessary parameters for plotting are transferred to Subroutine DOPLT by the two common blocks SPECBL and BLKLBL.
CHAPTER VI

ILLUSTRATION OF THE USE OF THE PROGRAM

A. An Ecological Example

The time series we will consider is the striped bass catch per unit effort data for the commercial fishery in the Hudson River from 1955 - 1975, a series of length 21 years. The set of input cards appears in Table 6.1. The first card contains the labels for the x and y axes, the time interval, the number of points in the time series, the number of times MESA is to be performed, the number of frequencies and the maximum frequency. The second card has the label for the time series data. The third card is the number of filter coefficients used. The time series follows on cards 4 - 24. The last card has all the options to print, to plot, to detrend, to compute the FPE, to compute the integrated spectrum to compute the autocorrelation function from MESA, and to compute the Yule - Walker estimates of the autocorrelation. All these option indicators are given the value 1.

The data were detrended using a SUBROUTINE DETRND which removed a linear trend by calling BLSQ, the least squares procedure (DETRND is in Appendix IV and BLSQ in Appendix II). Spectral analysis was done on both the raw data and the natural logarithm of the raw data. However, only output for the raw data will be presented. The printed output appears in Figure 6.1 and the plots in Figure 6.2A thru Figure 6.2L. The printout on the residual sum of squares, coefficients of the linear fit, and variance-covariance matrix come from DETRND. DISSPLA plot messages are also inserted.
Table 6.1. Input cards for the Hudson River catch per unit effort data

<table>
<thead>
<tr>
<th>Card</th>
<th>YEAR</th>
<th>$</th>
<th>CATCH</th>
<th>$</th>
<th>1.0</th>
<th>21</th>
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<th>199</th>
<th>0.5</th>
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</tr>
<tr>
<td>4</td>
<td>1955</td>
<td>19</td>
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Table 6.1. (continued)

<table>
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</tbody>
</table>

**NOTE:** '_' indicates a space.
### Analysis of the Original Input Data Without Any Prior Transformations

**Hudson River Catch Data Only**

<table>
<thead>
<tr>
<th>Year</th>
<th>C/F</th>
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</tr>
</thead>
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<td>-1.3488000 02</td>
</tr>
<tr>
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<tr>
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</tr>
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<td>1.96000</td>
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<td>1.96400</td>
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<td>-1.9500000 03</td>
</tr>
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<td>0.3</td>
<td>-2.8400000 01</td>
</tr>
<tr>
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<td>0.3</td>
<td>-1.9500000 03</td>
</tr>
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<td>0.3</td>
<td>-3.9700000 01</td>
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<tr>
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<td>0.3</td>
<td>-3.8500000 01</td>
</tr>
<tr>
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<td>0.3</td>
<td>-2.5080000 01</td>
</tr>
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</table>

### Coefficients of the Linear Fit

**Intercept** = 2.7104260 03  **Slope** = -1.24641820 01

### Variance-Covariance Matrix

<table>
<thead>
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<th></th>
<th>Column 1</th>
<th>Column 2</th>
</tr>
</thead>
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<tr>
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<td>2.2617597777777830 00</td>
<td>-1.6506777777777777 00</td>
</tr>
<tr>
<td>2</td>
<td>-1.6506777777777777 00</td>
<td>1.5845500000000000 00</td>
</tr>
</tbody>
</table>

### The Spectral Analysis Will Be Applied to the Detrended Data

<table>
<thead>
<tr>
<th>Year</th>
<th>C/F</th>
<th>Detrended C/F</th>
</tr>
</thead>
<tbody>
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<td>1.45500</td>
<td>0.3</td>
<td>-1.3488000 02</td>
</tr>
<tr>
<td>1.95600</td>
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<td>-1.8901000 02</td>
</tr>
<tr>
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<td>0.3</td>
<td>-2.5500000 03</td>
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<tr>
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<td>0.3</td>
<td>-2.4700000 03</td>
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<tr>
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<td>-4.0900000 01</td>
</tr>
<tr>
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<td>0.3</td>
<td>-2.5720000 03</td>
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<tr>
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<td>0.3</td>
<td>-1.6700000 01</td>
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<tr>
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<td>0.3</td>
<td>-2.0290000 01</td>
</tr>
<tr>
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<td>0.3</td>
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<tr>
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<td>0.3</td>
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</tr>
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<td>-3.9700000 01</td>
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</tr>
<tr>
<td>1.96900</td>
<td>0.3</td>
<td>-2.5080000 01</td>
</tr>
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</table>

*Fig. 6.1: Standard computer output.*
TITLE-FASTER ESTIMATE OF AUTO-CORRELATION PLOT

<table>
<thead>
<tr>
<th>LAG</th>
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</tr>
<tr>
<td>1.000 00</td>
<td>-0.010000 01</td>
</tr>
<tr>
<td>2.000 00</td>
<td>-0.010000 01</td>
</tr>
<tr>
<td>3.000 00</td>
<td>-0.010000 01</td>
</tr>
<tr>
<td>4.000 00</td>
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<td>-0.010000 01</td>
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<tr>
<td>6.000 00</td>
<td>-0.010000 01</td>
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<tr>
<td>7.000 00</td>
<td>-0.010000 01</td>
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<td>9.000 00</td>
<td>-0.010000 01</td>
</tr>
<tr>
<td>10.000 00</td>
<td>-0.010000 01</td>
</tr>
</tbody>
</table>

PERIODICITY

<table>
<thead>
<tr>
<th>X</th>
<th>FREQ.</th>
<th>PERIOD</th>
<th>PERIODICITY</th>
<th>CUMULATIVE ORIGIATE</th>
<th>PERIODICITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>201000000</td>
<td>0.08705</td>
<td>0.00500</td>
<td>0.00500</td>
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<td>0.01500</td>
<td>201000000</td>
<td>0.08705</td>
<td>0.00500</td>
<td>0.00500</td>
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<td>0.02000</td>
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<td>0.00500</td>
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<td>0.08705</td>
<td>0.00500</td>
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<td>0.08705</td>
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<td>0.00500</td>
<td>0.00500</td>
</tr>
<tr>
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<td>0.00500</td>
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<tr>
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<td>201000000</td>
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<td>0.00500</td>
</tr>
</tbody>
</table>

SUM OF PERIODICITY ORDINATES = 2.252230 07
MAXIMUM ORIGIATE = 7.577570 00
ACCORDING AT FREQENT = 4.527510-02
PERIOD = 1.540000 01
FISHER STATISTIC = 3.342090 00
NUMBER OF ORDINATES = 10

Fig. 6.1. (continued)
<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0.0904070</td>
<td>12.418</td>
</tr>
<tr>
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<td>0.0804146</td>
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</tr>
<tr>
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</tr>
<tr>
<td>12</td>
<td>0.0704221</td>
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</tr>
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<td>13</td>
<td>0.0604297</td>
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<tr>
<td>14</td>
<td>0.0504373</td>
<td>12.418</td>
</tr>
<tr>
<td>15</td>
<td>0.0404449</td>
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</tr>
<tr>
<td>16</td>
<td>0.0304524</td>
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</tr>
<tr>
<td>17</td>
<td>0.0204600</td>
<td>12.418</td>
</tr>
<tr>
<td>18</td>
<td>0.0104676</td>
<td>12.418</td>
</tr>
<tr>
<td>19</td>
<td>0.0004752</td>
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</tr>
<tr>
<td>20</td>
<td>0.1004828</td>
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</tr>
<tr>
<td>21</td>
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</tr>
<tr>
<td>22</td>
<td>0.0805000</td>
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<td>23</td>
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</tr>
<tr>
<td>27</td>
<td>0.0305380</td>
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<td>0.0205456</td>
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<td>29</td>
<td>0.0105532</td>
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</tr>
<tr>
<td>30</td>
<td>0.0005608</td>
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</tr>
</tbody>
</table>

Fig. 6. (continued)
### Final Prediction Error

<table>
<thead>
<tr>
<th>Number of Filler Coefficients</th>
<th>Final Prediction Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0457250 05</td>
</tr>
<tr>
<td>2</td>
<td>2.2659280 05</td>
</tr>
<tr>
<td>3</td>
<td>0.2860890 00</td>
</tr>
<tr>
<td>4</td>
<td>8.161970-01</td>
</tr>
<tr>
<td>5</td>
<td>0.031723-01</td>
</tr>
<tr>
<td>6</td>
<td>0.0279460 00</td>
</tr>
<tr>
<td>7</td>
<td>0.0291119 07</td>
</tr>
<tr>
<td>8</td>
<td>0.010772-01</td>
</tr>
<tr>
<td>9</td>
<td>0.040212 05</td>
</tr>
<tr>
<td>10</td>
<td>0.0029260 00</td>
</tr>
<tr>
<td>11</td>
<td>0.000000 00</td>
</tr>
</tbody>
</table>

Fig. 6.1. (continued).
A. Hudson River catch per unit effort data; dashed line represents the linear trend (Source for the time series: Texas Instruments, Inc., 1977, p. 76, Table 7.2-1).

B. Detrended Hudson River catch per unit effort; solid line represents the axis $y = 0$.

Fig. 6.2. Analysis of the Hudson River catch per unit effort data.
C. Residuals versus fitted values.

D. Yule - Walker autocorrelation function.

Fig. 6.2. (continued).
E. Periodogram.

F. Cumulative periodogram.

Fig. 6.2. (continued).
G. Fourier amplitude spectrum.

H. Cumulative Fourier power spectrum.

Fig. 6.2. (continued).
I. Burg autocorrelation estimate.

J. Maximum entropy spectrum with $M = 18$.

Fig. 6.2. (continued).
K. Integrated maximum entropy spectrum.

L. Final prediction error

Fig. 6.2. (continued).
B. Setting up the Program for Execution

The time it takes to run the program depends on many factors, primarily the length of the time series, the number of frequencies used, the number of times MESA is performed and the use of other options. Table 6.2 lists a few examples indicating the time and core used. Although the list is short, it is hoped that it will serve as a guide in the classification of a program run. All the data were linearly detrended.

For users at ORNL, Table 6.3 is a typical setup of the job control cards for a Class C job with disks used for plotting. Similarly, Table 6.4 is a setup for a class E job with the use of tapes in plotting. The job initials BLKL (wherever they appear) should be changed to that of the user's.

The inclusion of the following cards is explained:

1. "/DD_DSN = SYS2.DISPLA, DISP = SHR ............... "gives access to the plotting routines.

2. "/DD_DSNAME = BROOKS.LOAD.MODULES .............." gives access to the BLSQ.

3. "/DD_DSNAME = JDAIMSL1.REF 10276 .............." gives access to DCADRE.
### Table 6.2. List of programs runs; time and core used.

<table>
<thead>
<tr>
<th>Number of points in time series</th>
<th>IPRINT</th>
<th>ILOT</th>
<th>ITREND</th>
<th>IFPEPL</th>
<th>IYACV</th>
<th>INTSP</th>
<th>IAUTCV</th>
<th>Number of frequencies</th>
<th>Mount values</th>
<th>Logarithmic transform analyzed</th>
<th>Computer time and core used</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
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<td>yes</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>600</td>
<td>8-17</td>
<td>yes</td>
<td>1.04 minutes 286 K</td>
</tr>
<tr>
<td>21</td>
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<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>600</td>
<td>19</td>
<td>yes</td>
<td>23.90 seconds 286 K</td>
</tr>
<tr>
<td>31</td>
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<td>yes</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>501</td>
<td>15,16</td>
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<td>41.49 seconds 280 K</td>
</tr>
<tr>
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<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>600</td>
<td>40</td>
<td>yes</td>
<td>28.37 seconds 320 K</td>
</tr>
<tr>
<td>70</td>
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<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>201</td>
<td>30</td>
<td>no</td>
<td>11.35 seconds 280 K</td>
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<td>no</td>
<td>no</td>
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<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>600</td>
<td>109</td>
<td>no</td>
<td>35.46 seconds 290 K</td>
</tr>
<tr>
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<td>yes</td>
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<td>no</td>
<td>no</td>
<td>no</td>
<td>600</td>
<td>160</td>
<td>yes</td>
<td>49.70 seconds 320 K</td>
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<tr>
<td>277</td>
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<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>buu</td>
<td>275</td>
<td>yes</td>
<td>1.06 minutes 320 K</td>
</tr>
</tbody>
</table>
Table 6.3. Job setup for disk plotting

//BLKL JOB (16064), 'ENV_2001_X-10_KIRK'
//*CLASS_CPU91=5M,REGION=540
//_EXEC_FORTHCLG, PARM.FORT='XREF',
//_PARM.GO='TIME=5.0, EU=-1, DUMP=-1, SO=51', REGION.GO=540K
//FORT.SYSIN_DD_*
.
.
.
.

source deck
.
.
.
.
*/
//LKED.SYSLIB_DD
//_DD
//_DD
//_DD
//_DD_DSN = SYS2.DISPLA, DISP=SHR
//_DD_DSNAME=BUOOS.LOAD.MODULES,UNIT=2314,
//_VOLUME=SER=ZZZZZZ, DISP=SHR
//_DD_DSNAME=JDAIMSL1.REF10276, UNIT=2314, VOL=SER=CADPK1, DISP=SHR
//LKED.PLOTUBS_DD_DSN=JGSPLTH, DISP=SHR
//LKED.SYSIN_DD_*
  INCLUDE_PLOTUBS
/*
Table 6.3. (continued)

```
//GO.FT49F001 DD UNIT=IN20U2,DISP=(NEW,KEEP),
//_SPACE=(3208,40,RLSE),DSN=PLOTO0.BLKL,
//_DCB=(RECFM=VS,LRECL=3204,BLKSIZE=3208)
//GO.FT05F001 DD *
.
.
.
.
data cards
.
.
.
.

/*

```
Table 6.4. Job setup for tape plotting

```plaintext
/*NOSEQCARD

//BLK_IJOB_(16064),'ENV_2001_X-10_KIRK'
/*CLASS_CPJ91=5M,REGION=540,LINES=4U,CARDS=10,SPECIAL=TAPE
/*PLOT_NUMBER=30
/_EXEC_FOR'HCLG,PARM.FORT='XREF',
/_PARM.GO='TIME=5.0,CU=-1,DUMP=1,SO=51',REGION.GO=540K
//FOR':.SYSIN_DD *
.
.
.
source deck
.
.
.
/*
//LKED.SYSLIB_DD
// DD
// DD
// DD DSN=SYS2.DISPLAYA,DISP=SHR
// DD DSNAME=DOOKS.LOAD.MODULES,UNIT=2314,
// VOLUME=SER=ZZZ,DISP=SHR
// DD DSNAME=JDAIMSL1.REF10276,UNIT=2314,VOL=SER=CADPK1,DISP=SHR
//GO.PLOTTAPE_DD.JIN=TAPE7,LABEL=(,NL),DISP=OLD, VOLUME=SER=15
//GO.FT05F001_DD~
```
Table 6.4. (continued)

| data cards |
|------------|---|
|            |
|            |
|            |
|            |
| /*         |
| //         |
|            |


Texas Instruments, Inc., "Influence of the Proposed Cornwall Pumped Storage Project and Steam Electric Generating Plants on the Hudson River Estuary with Empahsis on Striped Bass and Other Fish Populations", February 1977, Table 7.2-1.


APPENDIX I

DESCRIPTION OF FUNCTION DCADRE*

FUNCTION DCADRE (F,A,B,AERR,RERR,ERROR,IER)

DCADRE-------S/O------LIBRARY 1-------------------------------------

FUNCTION - INTEGRATE F(X) FROM A TO B, USING CAUTIOUS
            ADAPTIVE ROMBERG EXTRAPOLATION.

USAGE - FUNCTION DCADRE (F,A,B,AERR,RERR,ERROR,IER)

PARAMETERS DCADRE - ESTIMATE OF THE INTEGRAL OF F(X) FROM A TO B.

F - A SINGLE-ARGUMENT REAL FUNCTION SUBPROGRAM
    SUPPLIED BY THE USER. F MUST BE DECLARED
    EXTERNAL IN THE CALLING PROGRAM.

A,B - THE TWO ENDPOINTS OF THE INTERVAL OF
      INTEGRATION. (INPUT)

AERR - DESIRED ABSOLUTE ERROR IN THE ANSWER. (INPUT)

RERR - DESIRED RELATIVE ERROR IN THE ANSWER. (INPUT)

ERROR - ESTIMATED BOUND ON THE ABSOLUTE ERROR OF THE
        OUTPUT NUMBER, DCADRE.

IER - ERROR PARAMETER

WARNING ERROR (WITH FIX) = 64 + N
    N = 1 IMPLIES THAT ONE OR MORE SINGULARITIES
      WERE SUCCESSFULLY HANDLED.
    N = 2 IMPLIES THAT, IN SOME SUBINTERVAL(S),
      THE ESTIMATE OF THE INTEGRAL WAS ACCEPTED
      MERELY BECAUSE THE ESTIMATED ERROR WAS
      SMALL, EVEN THOUGH NO REGULAR BEHAVIOR
      WAS RECOGNIZED.

TERMINAL ERROR = 128 + N
    N = 3 -- FAILURE DUE TO INSUFFICIENT
         INTERNAL WORKING STORAGE.
    N = 4 -- FAILURE. THIS MAY BE DUE TO TOO
         MUCH NOISE IN THE FUNCTION (RELATIVE
         TO THE GIVEN ERROR REQUIREMENTS) OR
         DUE TO AN ILL-BEHAVED INTEGRAND.
    N = 5 INDICATES THAT RERR IS GREATER THAN
      0.1, OR RERR IS LESS THAN 0.0, OR RERR
      IS TOO SMALL FOR THE PRECISION OF THE
      MACHINE.

PRECISION - SINGLE/DUAL

REQD. IMSL ROUTINES - UERTST

LANGUAGE - FORTRAN

FUNCTION DCADRE(F,A,B,AERR,RERR,ERROR,IER)

Purpose

DCADRE attempts to solve the following problem: Given the name F of a
real function subprogram, two real numbers A and B, and two non-negative
numbers AERR and RERR, find a number DCADRE such that

\[ \left| \int_{A}^{B} F(x) \, dx - DCADRE \right| \leq \max \left( AERR, RERR \cdot \left| \int_{A}^{B} F(x) \, dx \right| \right) \]

Algorithm

This routine uses a scheme whereby DCADRE is computed as the sum of
estimates for the integral of F(x) over suitably chosen subintervals of
the given interval of integration. Starting with the interval of
integration itself as the first such subinterval, cautious Romberg
extrapolation is used to find an acceptable estimate on a given
subinterval. If this attempt fails, the subinterval is divided into two
subintervals of equal length, each of which is considered separately.

See reference: de Boor, Carl, "CADRE: An algorithm for numerical
quadrature", Mathematical Software (John R. Rice, Ed.), New York,

Programming Notes

1. DCADRE can, in many cases, handle jump discontinuities and certain
algebraic discontinuities. See reference for full details.

2. The relative error parameter RERR must be in the interval \([0,0.1]\).
   For example, RERR=0.1 indicates that the estimate of the integral
   is to be correct to one digit, whereas RERR=10 calls for four
digits of accuracy. If DCADRE determines that the relative
   accuracy requirement cannot be satisfied, IER is set to 133 (RERR
   should be large enough that, when added to 100.0, the result is a
   number greater than 100.0).

3. The absolute error parameter, AERR, should be nonnegative. In
   order to give a reasonable value for AERR, the user must know the
   approximate magnitude of the integral being computed. In many
   cases it is satisfactory to use AERR=0. In this case, only the
   relative error requirement is satisfied in the computation.

4. We quote from the reference, "A very cautious man would accept
   DCADRE only if IER is 0 or 65. The merely reasonable man would
   keep the faith even if IER is 66. The adventurous man is quite
   often right in accepting DCADRE even if IER is 131 or 132". Even
   when IER=0, DCADRE returns the best estimate that has been computed.
APPENDIX II

DESCRIPTION OF SUBROUTINE BLSQ*

Identification

Linear least squares solution.

Purpose

ALSQ and BLSQ are double precision FORTRAN IV subroutines to solve
the linear least squares problem by applying the Householder
reduction to the least squares matrix \[ A \]. Both programs use the
same technique in solving the least squares problem; however, BLSQ
returns additional information useful in statistical
applications. Both programs have subordinate entries, ALSQ1,
BLSQ1, which enable the user to perform additional fits at very
little cost once the least squares matrix has been reduced.

Method

The linear least squares problem to be solved may be formulated as
follows: given \( n \times m \) (\( m < n \)) matrix \( A \) of rank \( m \) and an \( n \)-vector \( y \)
find an \( m \)-vector \( b \) such that

\[
||Ab - y||^2 = \text{min.} \quad (1)
\]

The programs, in effect, use the Householder reduction of the matrix \( A \)
to triangular form to find an \( n \times n \) orthogonal matrix \( H \) such that

*Reprinted from Westley and Watts, The Computing Technology Center
where $T$ is an $m \times m$ upper triangular matrix and "0" is an $(n - m) \times m$ null matrix. If we let $U$ be the $n \times m$ matrix consisting of the first $m$ columns of $H^T$ (in order) and $V$ be the $n \times (n - m)$ matrix consisting of the remaining columns and consider the new least squares problem of minimizing $\| Uc - y \|_2^2$, then

$$c = U^T y$$

(3)

and

$$b = T^{-1} c.$$  

(4)

ALSQ performs the Householder reduction (2) and uses equations (3) and (4) to calculate $b$. BLSQ calculates and returns the vector $c$ and the matrices $T$ and $T^{-1}$ which are useful in statistical applications. The vector $b$ may subsequently be calculated by the user from equation (4). Both programs return the approximating vector $Ab = Uc$ and the residual sum of squares $\| Ab - y \|_2^2$.

Under the usual statistical assumptions of regression analyses, it can be shown that the number of runs in $VT_y$ minus one has the binomial distribution with parameters $n - m - 1$ and $.5$. If the problem being solved involves fitting continuous functions of a real variable, a statistical test based on the above may be expected to be powerful when the
fit inadequately represents the data. Because of this, BLSQ returns the number of runs in $V^Ty$.

Usage

The calling sequence for ALSQ is:

CALL ALSQ(A, Y, B, R2, N, M, NA),

where

A is a doubly subscripted double precision array containing the least squares matrix. The elements of A are altered by the program.

Y is a singly subscripted, double precision array containing the vector to be fit. The elements of Y are unchanged by the program.

B is a singly subscripted, double precision array which upon return contains the coefficients of the fit.

R2 contains, upon return, the double precision residual sum of squares of the fit.

N is the integer number of rows in the least squares matrix.

M is the integer number of columns in the least squares matrix.

NA is the integer first dimension of the array A.

The program requires that the array A be dimensioned so that it will hold one extra row and one extra column in addition to the least squares matrix, i.e., the dimension of A must be at least $N + 1$ by
M + 1. Upon return from the program, the locations A(I, M + 1) contain the components of the approximating vector Ab (see Method).

A second entry, ALSQ1, allows the user to obtain additional fits without repeating the reduction of the least squares matrix. ALSQ must be called before calling ALSQ1, and the contents of the array A must not be changed between these calls.

The calling sequence is:

\[
\text{CALL ALSQ1}(Y, B, R2, K)
\]

where Y, B, and R2 are the same as above and

\[ K \text{ is the number of columns of the least squares matrix to fit to the vector } Y; \text{ e.g., if } K = 4, \text{ the first four columns of } A \text{ will be fit to } Y \text{ returning four coefficients in the array } B. \text{ } K \text{ must be less than or equal to } M \text{ in the call ALSQ.}
\]

This entry also returns the approximating vector in the locations A(I, M + 1).

The calling sequence for BLSQ is:

\[
\text{CALL BLSQ}(A, Y, C, R2, IRUN, T, T1, N, M, NA, NT),
\]

where A, Y, R2, N, M, and NA are the same as in the calling sequence for ALSQ and

\[ C \text{ is a singly subscripted, double precision array which upon return contains the coefficients of the transformed problem given by (3).}
\]

\[ IRUN \text{ is an integer array whose } i^{\text{th}} \text{ element contains upon return the number of runs in } V^Ty \text{ after } i \text{ columns of the least squares matrix have been fitted.} \]
T is a doubly subscripted, double precision array which upon return contains the upper triangular transformation matrix of equations (2).

Tl is a doubly subscripted, double precision array which upon return contains the inverse of the transformation matrix.

NT is the first dimension of the arrays T and Tl.

BLSQ also has a second entry, BLSQl, to calculate new fits. The same restrictions apply to the use of BLSQl as apply to ALSQl. The calling sequence is:

CALL BLSQl(Y, C, R2, IRUN, K),

where Y, C, R2, IRUN are the same as in the calling sequence of BLSQ and K is the number of columns of the least squares matrix to fit to the data. Both BLSQ and BLSQl return the components of the approximating vector in the locations A(I, M + 1).

Coding Information

Although these programs are coded in double precision arithmetic, they may be easily converted to single precision. Additional accuracy may be obtained in the single precision program by accumulating inner products in double precision. The places where this should be done are signaled by the appearance of the variables SS and PP.

Reference


Author

G. W. Stewart, III, formerly with the Computing Technology Center, Union Carbide Corporation, Nuclear Division, Oak Ridge, Tennessee. [Editor's Note: For an evaluation of ALSQ in single and double precision see (2). Unfortunately, the single precision version reported in (2) did not accumulate inner products in double precision.]
APPENDIX III

PROGRAM LISTING
LEVEL 21.9 (JUN 74)

COMPILED OPTIONS - NAME= NAME.OPT=2,LICENSE=50,XSIZE=000000,
SOURCE,EXCUT,FOLIST,MODEC,LOAD,MAP,RODIT,ID,EREF

C

IEN 0002
IEN 0003
IEN 0004
IEN 0005

COMMON /SPECIAL/ DELTA,Year (600), T (600), ID (600),
1 YVAR (600), T (600), S (600), Fiq (600), Spiff (600),
2 DATA (600), T (600), ASA (600), ASA (600), ASA (600),
3 PEP (600), HPTS, Spilt, N, NCOUNT (600), PINT, PLOT, TEND,
4 EXPN, XINT, JINTC, ITLACT
COMMON /XILBL/XILBL (14), XILBL (2), YILBL (2)

C

INITIALISE VALUES.

XILABEL = IDENTIFIER (LABEL) FOR TIME AXIS (AS)
YLABEL = IDENTIFIER FOR TIME SERIES DATA (AB)
DELA = SAMPLING TIME INTERVAL
HPTS = NUMBER OF DATA POINTS IN TIME SERIES
SPILT = TOTAL NUMBER OF TIMES MAXIMUM ENTROPY IS
TO BE DONE WITH DIFFERENT VALUES FOR
THE NUMBER OF FILTER COEFFICIENTS
N = NUMBER OF FREQUENCIES NEEDED IN OUTPUT
THAX = MAXIMUM FREQUENCY FOR OUTPUT SPECTRUM
(Should not exceed the Nyquist frequency).
BAR = TITLE FOR THE DATA SET (AS)
(TITLE PRINTED AT TOP OF THE PLOTS.

C

1 CONTINUE

IEN 0007
IEN 0008
IEN 0009
IEN 0010

READ (5,20, M=N=500) XILBL, YILBL, DELTA, HPTS, Spilt, N, THAX
20 FORMAT (24X, 2I2, 2X, 2H10.0, 3H15, 2H10.0)
READ (5,21, M=N=500) BAR
21 FORMAT (24X)

READ COUNT, NCOUNT IS THE ARRAY OF NUMBERS WHICH CORRESPOND TO THE
NUMBER OF FILTER COEFFICIENTS DESIGNED IN THE MAXIMUM ENTROPY.

IEN 0011
IEN 0012

READ 60, (NCOUNT (I), I=1, Nfilt)
60 FORMAT (14I5)
READ THE DATA TO BE ANALYZED

DATA = TIME ARRAY
T  = TIME SERIES TO BE ANALYZED

THE DATA CONSISTS OF A SERIES OF POINTS
{ YEAR(I), T(I) }, I=1,NPTS.

DO 50 I=1,NPTS
READ 60,DATE(I),T(I)
50 FORMAT (1X,F8.0,F10.0)
90 CONTINUE

READ VALUES WHICH WILL GIVE CERTAIN OPTIONS.

1 MEANS 'YES'.
0 MEANS 'NO'.

IPRINT = OPTION TO PRINT.
IPLDT = OPTION TO PLOT.
ITREND = OPTION TO PLOT TRENDS.
IPPL = OPTION TO PLOT FINAL PREDICTION ERROR.
IINTS = OPTION TO COMPUTE INTEGRATED SPECTRUM.
IACV = OPTION TO PLOT AUTOCOVARIANCE FUNCTION.
IACV = OPTION TO PLOT THE WALKER ESTIMATE OF AUTOCOVARIANCE FUNCTION.

READ 100, IPRINT,IPLDT,ITREND,IPPL,IINTS,IACV
100 FORMAT (7I5)

CALL TRANS1
CALL TRANS2
CALL TRANS3
CALL TRANS4
CALL TRANS5

GO TO 1
500 CALL DOPEPL
STOP
END
LEVEL 21.6 (JUN 74)

OS/360 PORTRAIL

COMPILER OPTIONS - VAR = NAME, OPT = 02, LIST = 60, SIZE = 0000K,
SOURCE, acyclic, solist, bodeck, load, map, sorder, id, xref

C

ISS 0002
SUBROUTINE TRANS1

ISS 0003
IMPLICIT REAL*8 (A-D, O-Z)

ISS 0004
REAL*8 VAR, XBAR, YBAR

ISS 0005
COMMON / SPECIAL/ DEL (600), ID (600)
1 VAR, ID (600), YBAR (600), XBAR (600), ID (600), XBAR (600)
2 ID (600), YBAR (600), ID (600), XBAR (600), ID (600), YBAR (600)
3 PERIOD (600), NPTS, NPRINT (600), NPRINT, NPRINT, NPRINT

ISS 0006
COMMON / SPECIAL/ VAR (100), XBAR (2), YBAR (2)

ISS 0007
XLOG = 0

ISS 0008
PRINT 10

ISS 0009
10 FORMAT (1HI, 'ANALYSIS OF THE ORIGINAL INPUT',
1 'DATA WITHOUT ANY PRIOR TRANSFORMATIONS')

ISS 0010
CALL SPECTR

ISS 0011
RETURN

ISS 0012
ENTRY TRANS2

ISS 0013
XLOG = 1

ISS 0014
PRINT 30

ISS 0015
30 FORMAT (1HI, 'ANALYSIS OF THE LOGARITHM OF THE ORIGINAL DATA')

ISS 0016
DO 50 I = 1, NPTS

ISS 0017
1 IP (Y(I), XBAR, 6.) Y(I) = 1.

ISS 0018
IF (Y(I) .LT. 1.) Y(I) = 6.0

ISS 0019
50 CONTINUE

ISS 0020
CALL SPECTR

ISS 0021
ENTRY TRANS3

ISS 0022
RETURN

ISS 0023
ENTRY TRANS3

ISS 0024
RETURN

ISS 0025
ENTRY TRANS3

ISS 0026
RETURN

ISS 0027
ENTRY TRANS3

ISS 0028
RETURN

ISS 0029
END
<table>
<thead>
<tr>
<th>LABEL</th>
<th>REPRF</th>
<th>REFERENCES</th>
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</thead>
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<tr>
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<td>0009</td>
<td>0000</td>
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<tr>
<td>20</td>
<td>0015</td>
<td>0014</td>
</tr>
<tr>
<td>30</td>
<td>0020</td>
<td>0018</td>
</tr>
</tbody>
</table>
SUBROUTINE SPECTR

MAXIMUM ENTROPY SPECTRAL ANALYSIS SUBROUTINE

ACS = VECTORS OF WORKING STORAGE (SAME LENGTH AS V)
1ST USE IS TO COMPUTE INTEGRATED SPECTRUM
IF THAT OPTION IS CHOSEN. IF SO, THEN ASP
WILL CONTAIN FINAL INTEGRATED SPECTRUM UPON
RETURN TO CALLING PROGRAM.

ASP = VECTORS OF WORKING STORAGE (SAME LENGTH AS V)

ASp = FOURIER POWER SPECTRUM OF DATA (HEREIN CALculated)
(DIMENSIONED SAME LENGTH AS V)

BSN = VECTORS OF WORKING STORAGE (SAME LENGTH AS V)

CUMPER = CUMULATIVE PERIODOGRAH.

DATA = TIME VALUES FOR INPUT SERIES.

DElTA = SAMPLING INTERVAL (DELTA T)

FPE = FINAL PREDICTION ERROR (CALC. HEREIN)
THIS ARRAY SHOULD HAVE SAME DIM. AS ACHT.

FEF = PERIODOGRAH FREQUENCIES.

IAUTCH = AUTOCORRELATION PLOTTING FLAG (ZERO ESTIMATE)
IF(IAUTCH,.NE.0) PLOT THE AUTOCORRELATION FUNCTION
IF(IAUTCH,.EQ.0) DON'T PLOT

IFPPPL = PPE FLAG
IF(IFPPPL,.NE.0) PRINT AND PLOT FINAL PREDICTION
ERROR VS. NO. LAGS USED.
IF(IFPPPL,.EQ.0) DON'T PRINT AND PLOT FPE

FINISP = INTEGRATED SPECTRUM FLAG
IF(FINISP,.NE.0) CALC., PRINT, AND PLOT THE
INTEGRATED SPECTRA.
IF(FINISP,.EQ.0) DON'T CALC. INTGR. SPECTRA.

IPLOT = PLOTTING FLAG
IF(IPLOT,.NE.0) PLOT DATA AND SPECTRA
IF(IPLOT,.EQ.0) DON'T PLOT

IPRINT = PRINTING FLAG
IF(IPRINT,.NE.0) PRINT DATA AND SPECTRA
IF(IPRINT,.EQ.0) DON'T PRINT

IIPEND = DETRANED - FLAG
IF(IIPEND,.NE.0) USE DETRANED TIME SERIES
TO COMPUTE THE SPECTRA
IP(NV, TH.E. EQ. 0) USE RAW DATA.
IP(NV, TH.E. EQ. 1) USE AUTO-WALKER ESTIMATE.
IP(NV, TH.E. EQ. 0) USE FULL WALKER AUTOCOVARIANCE.
IP(NV, TH.E. EQ. 0) USE FULL WALKER ESTIMATE.
IP(NV, TH.E. EQ. 0) DON'T CALCULATE FULL WALKER ESTIMATE.
IP(NV, TH.E. EQ. 0) USE FULL WALKER AUTOCOVARIANCE.
IP(NV, TH.E. EQ. 0) USE FULL WALKER ESTIMATE.
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IP(NV, TH.E. EQ. 0) USE FULL WALKER AUTOCOVARIANCE.
IP(NV, TH.E. EQ. 0) USE FULL WALKER ESTIMATE.
IP(NV, TH.E. EQ. 0) DON'T CALCULATE FULL WALKER ESTIMATE.
IT IS LATER USED AS A WORKING VECTOR.

Y = INPUT TIME SERIES VALUES.

YEAR = WORKING VECTOR (CAN BE USED FOR THE VARIABLE IN SUBROUTINE DETREND).

YEST = LABEL FOR THE TIME AXIS

HALF OF A CHARACTER (AB) ENDING WITH $.

2 = PREP. ERROR FILTER COEFFS. (HEREIN CALL, 6 RETURNED)

** Example of FORTRAN code for detrending data **

```
IMPLICIT REAL*8(A-H,O-Z)

REAL*8 X, Y, Z

Y(1) = REAL*8(0.0)
Y(2) = REAL*8(0.0)
Y(3) = REAL*8(0.0)
Y(4) = REAL*8(0.0)

CALL DETREND(X, Y, Z)
```

** Subroutine DETREND does not destroy the arrays DATA and CATCH. **

** It should set up its own time array in YEAR. **
CALL DETAIL(NPPTS,DATA,YEAR,T,DIFF,Y,SPF)
DO 170 I=1,NPPTS
170 CONTINUE

PLOT DATA AND ITS TREND COMPONENT.

IF (FPOPL1.10.0) GO TO 300
CALL ECFT1
DO 300 I=1,NPPTS
PRINT 302
302 FORMAT (1X//,1X,'THE SPECTRAL ANALYSIS WILL BE ',
1 'APPLIED TO RAW DATA WITHOUT REMOVING THE TREND///')
SPFT = 0.0
DO 300 J=1,NPPTS
PRINT 303,YEAR,STATE
303 FORMAT (1X//,1X,YEAR,STATE)
CONTINUE
GO TO 310

30C CONTINUE
PRINT 302
302 FORMAT (1X//,1X,'THE SPECTRAL ANALYSIS WILL BE ',
1 'APPLIED TO THE DETRENDED DATA///')
304 FORMAT (1X//,1X,'APPLIED TO THE DETRENDED DATA///')
PRINT 305,NPPTS
305 FORMAT (1X//,1X,NPPTS)
CONTINUE
GO TO 310

CONTINUE
PRINT 302
302 FORMAT (1X//,1X,'THE SPECTRAL ANALYSIS WILL BE ',
1 'APPLIED TO THE DETRENDED DATA///')
304 FORMAT (1X//,1X,'APPLIED TO THE DETRENDED DATA///')
PRINT 305,NPPTS
305 FORMAT (1X//,1X,NPPTS)
CONTINUE
GO TO 310

CONTINUE
CALL ECFT2
CONTINUE
COMPUTE YOPE-WALKER ESTIMATES OF AUTOCOVARIANCE
FUNCTION IS DESIGNED. LAG VALUES STORED IF X,
ACF FUNCTION VALUES STORED IN DIFF.

IF (F1NMAY.YQ.0) GO TO 500
500 CONTINUE

CONTINUE
CALL ECFT2
CONTINUE

410 CONTINUE
C4EAP=C4E4AB/AN
ISW 0064  DO 440 I=1, NPTS
ISW 0065  ILAG=1
ISW 0066  HH1 = NPTS - ILAG
ISW 0067  DO 430 J=1, HH1
ISW 0068  ILAG=J+LAG
ISW 0069  EDIFF(J)=XDIFF(J)/4
    ! ED(J)-CHEIR)*( ED(J)-CHEIR))
ISW 0070  430 CONTINUE
ISW 0071  EDIFF(J)=EDIFF(J)/AN
ISW 0072  440 CONTINUE
ISW 0073  EDIF = EDIFF(1)
ISW 0074  DO 445 I = 1, NPTS
ISW 0075  XDIFF(J) = XDIFF(J)/EDIF
ISW 0076  445 CONTINUE

C
ISW 0077  PRINT 450
ISW 0078  450 FORMAT (11X,’DET-WALKER ESTIMATE OF AUTOCORRELATION ‘,
    1 ”FUNCTION’/100,NX,’LAG’,11%,’AUTOORB.’/)
ISW 0079  DO 460 I=1,NPTS
ISW 0080  PRINT 455, Z(I),EDIFF(I)
ISW 0081  455 Format (1H,1PE10.3,3X,1PE12.5)
ISW 0082  460 CONTINUE

C
ISW 0083  CALL COPLY3
C
    ! CONVTE PERIODOGRAM AND FOURIER SPECTRA.
    !
C
ISW 0086  500 NRT = NPTS
ISW 0087  KPI = NRT + 1
C
    ! FIRST THE FREIODOGRAM AND CUMULATIVE PERIODOGRAM,
    !
ISW 0088  PTSEP = NPTS / 2
ISW 0089  NPER = PTSEP
ISW 0090  DO 502 J=1,NPER
ISW 0091  AJ = J
ISW 0092  FPER(J) = AJ / NPTS
ISW 0093  FPER(J) = 1,000 / FPER(J)
ISW 0094  502 CONTINUE
ISW 0095  CALL SPT(ACES,B5W,$PER,$PER,NPTS,P1)
ISW 0096  SUMPER = 0,000
ISW 0097  TNCRTA = 2,000 / NPTS
ISW 0098  DO 503 I = 1,NPER
ISW 0099  PEERIC(I) = THAIRA (ACES(I)**2 + B5W(I)**2)
ISW 0100  SUMPER = SUMPER + PEERIC(I)
ISW 0101  503 CONTINUE
ISW 0102  OSUR = 0,000
ISW 0103  PERHAX = PEERIC(1)
ISW 0104  ITMAX = 1
ISW 0105  DO 505 I=1,ITMAX
ISW 0106  OSUR = OSUR + PEERIC(I)
ISW 0107  505 CONTINUE
ISW 0108  IF (PERHAX .LT. PERHAX) GO TO 505
ISW 0109  PERHAX = PEERIC(1)
ISW 0109  ITMAX = 1
ISW 0110  505 CONTINUE
ISW 0111  FISHER = FISHER / ((1,000/NPER) * SUMPER)
PRINT THE PERIODOGRAM AND CUMULATIVE PERIODOGRAM.

507 FORMAT (1H1,'PERIODOGRAM//1NO. 22.2`*`51.2PERO.+`
1 H4,'PERIODO+252,'PERIODOGRAM//53.2CUMULATIVE//1IN`
2 32X,'ORDINATE',5X,'PERIODOGRAM//')
PRINT 508, (H,APER(H),PER(H),PERD(H),CUMPER(H),K=1,2PER)

508 FORMAT (1H4,'PERIODOGRAM/O10.5,1F14.5,1D14.5,OPF10.5)
PRINT 509, (KPER,APER(H),PER(H),PERD(H),CUMPER(H),K=1,2PER)

509 FORMAT (1H4,'PERIODOGRAM/1NO. 22.2`*`51.2PERO.+`
1 H4,'PERIODO+252,'PERIODOGRAM//53.2CUMULATIVE//1IN`
2 32X,'ORDINATE',5X,'PERIODOGRAM//')
PRINT 508, (H,APER(H),PER(H),PERD(H),CUMPER(H),K=1,2PER)

508 FORMAT (1H4,'PERIODOGRAM/O10.5,1F14.5,1D14.5,OPF10.5)
PRINT 509, (KPER,APER(H),PER(H),PERD(H),CUMPER(H),K=1,2PER)

CONPUTE THE KP+1 FREQUENCIES FOR FOURIER AND NISA SPECTRA.

V(1) = 0.
V(2) = YMA / NV
KP = KP + 1
DO 510 I=1,1KP
V(I+2) = V(I) + IV(I+2)
510 CONTINUE

CONPUTE THE FOURIER SPECTRUM.

CALL SFT(ED,ACS,RSH,IV,IKF,VAR,PI)
DO 520 I=1,1KP
ASP(I) = (2.*ACSI/INR)**2 + (2.*RSH(I)/INR)**2
ASP(I) = SQRT(ASP(I))
520 CONTINUE

CONPUTE THE CUMULATIVE FOURIER SPECTRUM.

ZSUM = 0.
DO 535 I=1,1KP
YSUM = YSUM + HCPI
535 CONTINUE

DO 540 J=1,1KP
ZSUM = 0.
DO 550 J=1,1KP
ZSUM = ZSUM + YSUM
550 CONTINUE

IF (FLOT.EQ.0) GO TO 600

CALL DOPIT7(KF,KPER)
CALL DOPIT4(KF)

APPLY THE MAXIMUM ENTRAPT METHOD. REFERENCE TO
ANKENBERG'S ALGORITHM.
ISN 0200  
ISN 0201  
ISN 0202  
ISN 0203  
ISN 0204  
ISN 0205  
ISN 0206  
ISN 0207  
ISN 0208  
ISN 0209  
ISN 0210  
ISN 0211  
ISN 0212  
ISN 0213  
ISN 0214  
ISN 0215  
ISN 0216  
ISN 0217  
ISN 0218  
ISN 0219  
ISN 0220  
ISN 0221  
ISN 0222  
ISN 0223  
ISN 0224  
ISN 0225  
ISN 0226  
ISN 0227  
ISN 0228  
ISN 0229  
ISN 0230  
ISN 0231  
ISN 0232  
ISN 0233  
ISN 0234  
ISN 0235  
ISN 0236  
ISN 0237  
ISN 0238  
ISN 0239  
ISN 0240  
ISN 0241  
ISN 0242  
ISN 0243  
ISN 0244  
ISN 0245  
ISN 0246  
ISN 0247  
ISN 0248  
ISN 0249  
ISN 0250  

It seems to be a Fortran program for calculating and printing autocorrelation and entropy values. The program starts with saving the value of a variable, then continues with a DO loop to calculate the difference between elements in an array and store the results in another array. The program also includes conditions to print error messages. The program finally outputs the autocorrelation and entropy values.
DO 960 J=1,KP1
IF (V(J),EQ.0) 30 TO 920
910 PERIOD(I) = V(J)/V(I)
GO TO 930
920 PERIOD(I) = 0.
930 PRINT 940,1,V(I),PERIOD(I),S(I),ASF(I)
940 FORMAT (1X,' ',1X,1V,4,1X,9.7,6X,9.3,6X,1PE19.6,1PE19.6)
950 CONTINUE

C C PLCT MAXIMUM ENTROPY AND AUTOCORRELATION FUNCTION.
C C

980 IF (IFLOC.IEQ.0) GO TO 93
910 PERIOD(I) = S(I)+A(I)+P(I)
GO TO 930
960 DO 100 K=1,KP2
100 CONTINUE
C C COMPUTE INTEGRATED SPECTRUM IF DESIRED
C C

900 IF (INTEG.K.EQ.6) GO TO 1030

C C WINDOW = PENNY/2.000
C C PRINT 991,WINDOW,INTEGRAL(IP)
C C
991 FORMAT (1H0,' INTEGRATED MAXIMUM ENTROPY SPECTRUM',/,
1 1H0,' WINDOW = ',8H4.4,' ERROR = ',2H4.4)
992 CONTINUE

C C SPC9IN - SPCVL1 (ASP,2,DELTA,KM,*)
C C
993 CONTINUE

DO 997 I=1,KM
PRINT 99(, I,V(I),FE*10D(I|,S(1) ,A(I) .8(1)
996 CONTINUE

DO 999 I=1,KM
PRINT 991, I,A(I),8(I)
998 CONTINUE

DO 999 I=1,KM
PRINT 991, I,A(I),8(I)
998 CONTINUE

DO 999 I=1,KM
PRINT 991, I,A(I),8(I)
C'RNL-Dyn 7B-5850

999 CONTINUE
C     CALL DOPLT(XF1,MC,4)
C
1000 CONTINUE
C
ISN 0297 IS (IPNEU-LQ,0) RETURN
ISN 0299 PRINT 1010
ISN 0300 FORMAT (1H1, 'FINAL PREDICTION ERROR',/,1H1,2X,
              '12,5F,33,5H,33,5F,'FF,6,5H,33,5F,'ERROR',/)
ISN 0301 DO 1020 E=1,MC
ISN 0302 11 = E+1
ISN 0303 PRINT 1015,1,13,FFE[1]
ISN 0304 FORMAT (1H1,13,11S,12,1PD12.5)
ISN 0305 1020 CONTINUE
C
C     PLOT THE FINAL PREDICTION ERROR.
C
ISN 0306 CALL LOPLOT
ISN C007 RETURN
ISN G008 END
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DO 100 J=1,8

CT = SCBI(0.05,BF)

CSUB = CSUB - S(O)*CSUB(CV)

100 CONTINUE

CSUB = 1.000*CSUB

HT = (CSRSH(CSUS))**2

SPCV = (ASP(P-1)*DTIME)/HT

RETURN

ENTRY SPCVL1 (ASP,PTIME,AP1,N)

ENTRY SPCVL1 = 7777777.

RETURN

END
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SUBROUTINE DCST

ENTRY DOPLUS

ENTRY DOPLUS2

ENTRY DOPLUS3
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APPENDIX IV

LISTING OF SUBROUTINE DETRND
LEVEL 21.8 (JUNE 76)
05/160 PROGRAM B

COMPILE OPTIONS : HANG = BAIN, OPT = 02, LIBRARY = B0, SIZE = 0000K,
                  SOURCE, EXEC, DLIST, WORK, LOAD, EXP, RMOD, ID, TREF

SUBROUTINE DERTED(ETA, DATE, YEAR, CATCH, EDIFF, 1, SPRE)

A     = HATRIX OF DIMENSION ETA+1 BY NLSCOL+1. USED IN DLSQ.
CATCH = INPUT TIME SERIES VALUES.
CORP  = VECTOR CONTAINING THE COEFFICIENTS OF THE LINEAR FIT.
CORP2 = WORKING VECTOR OF SAME LENGTH AS CORP.
DATE  = TIME VALUES FOR INPUT SERIES.
NLSCOL = NO. OF COEFFICIENTS OF THE LINEAR FIT.
ETA   = NO. DIMENSION OF HATRIX A.
ETA   = NO. OF DATA POINTS IN TIME SERIES.
X     = LINEAR TREND OF TIME SERIES (NEITHER CALCULATED AND
       RETURNED).
EDIFF = DETERMINED TIME SERIES (NEITHER CALCULATED AND RETURNED).
YEAR  = TIME VARIABLE (NEITHER CALCULATED).

IZH 0003  IMPLICIT REAL(4)(A=0.00)  
IZH 0004  DIMENSION DATE(1), YEAR(1), CATCH(1), EDIFF(1), ETA(1)  
IZH 0005  DIMENSION A(601,3), NLS(601), CORP(10), T(2,1), T1(2,2), VAN(2,2)  
IZH 0006  DIMENSION CORP(10)

INITIALIZER VALUES.

IZH 0007  NUM = 601  
IZH 0008  NLSCOL = 2  
IZH 0009  YT = 2  
IZH 0010  EDIF = NLSCOL  
IZH 0011  DO 50 I=1, ETA  
IZH 0012  YEAR(I) = I - .5  
IZH 0013  A(I,1) = 1.0  
IZH 0014  A(I,2) = YEAR(I)  
IZH 0015  50 CONTINUE

CALL LEAST SQUARES ROUTINE.

IZH 0016  CALL ELSO(A,CATCH,CORP,RESID,IRUN,XT,ETA,NLSCOL,NUM,NLSCOL)  
IZH 0017  DO 90 I=1, NLSCOL  
IZH 0018  SUB = 0.
DO 70 J=1,NSCOL
70 CONTINUE
CORSY(J) = SUM
DO 90 I=1,NSCOL
CORSY(I) = CORSY(I) + SUM
90 CONTINUE

DO 100 I=1,NTA
100 CONTINUE
PRINT THE COEFFICIENTS OF THE THIRD LINE.

PRINT 150, NSCOL
PRINT 150, (CORSY(J), J=1,NSCOL)

PRINT 200, (CORSY(I)/NSCOL, I=1,NTA)

PRINT 200, (CORSY(I)/NSCOL, I=1,NTA)

PRINT 200, (CORSY(I)/NSCOL, I=1,NTA)

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