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AUTHOR(S) John K. Dienes

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Los Alamos Los Alamos National Laboratory
Los Alamos, New Mexico 87545

A STATISTICAL THEORY OF FRAGMENTATION PROCESSES

John K. Dienes

Theoretical Division
Los Alamos National Laboratory
Los Alamos, NM 87545 USA

The goal of the work reviewed here is a theory of material behavior accounting for the average deformation that results from the opening, shear, growth and coalescence of an ensemble of microcracks. A concomitant is the calculation of permeability from crack structure. The first part of this paper summarizes previous developments. In particular, the initial work on this problem made use of a linear Liouville equation to characterize the change in crack distribution resulting from crack growth and coalescence. Straightforward analytic solutions to this equation were possible because the mean free path of cracks was assumed constant. Though this assumption is useful for the early stages of crack growth, increasing crack size reduces the mean free path in the later stages of fragmentation. This problem is addressed in the second part of this paper. The governing (nonlinear) Liouville equation is derived therein, and it is shown that it can be reduced to an ordinary differential equation of third order involving only a single parameter, β. This equation has now been solved numerically to determine the limiting value of the mean free path as a function of β, and the results are presented in graphical form. In the third part of this paper prospects for further developments are briefly discussed.

A STATISTICAL THEORY OF FRAGMENTATION PROCESSES

1. Review of Previous Work

One of the assumptions that has made it possible to formulate constitutive relations accounting for the behavior of fragmenting solids is that the strain rate can be represented as the sum of contributions from disparate physical processes. This assumption can be considered a generalization of Reuss' (1930) concept of separating the strain rate into a sum of elastic and plastic parts. Physically, one can envisage the total strain in a plastically deforming solid as the sum of the elastic strain in a solid lattice which is determined by the prescribed stress using linear elasticity, and of nonlinear slip along planes of weakness, represented by a phenomonological plasticity law. The superposition can be represented in a number of ways, depending on the variables selected. The most natural for computational purposes is to work with the stretching D, which is defined by separating the velocity gradient $G = (u_{i,j})$ into the sum of a symmetric part D, the stretching, and an antisymmetric part W, the vorticity, so that

$$G = D + W \quad . \tag{1.1}$$

The stretching can be related to the strain

$$E = \frac{1}{2}(B^2 - I) \quad , \tag{1.2}$$

where

$$B = FF^T \tag{1.3}$$

is the left Cauchy-Green tensor and F is the deformation gradient, by

$$\hat{E} = VDV \quad . \tag{1.4}$$

Here the circumflex denotes the material rate discussed by Dienes (1979a) and the left stretch V is defined by

$$V^2 = B \quad . \tag{1.5}$$

The above definition of strain differs from those usually adopted, but is thought to be the most useful. Specifically, the strain rate defined above is the actual rate of change (accounting for rotation) of strain. (The stretching D, often taken as the strain rate, is not the rate of any history-independent variable.) The summation principle can be applied to either $\hat{E}$ or D, with equivalent results according to (1.4), but the application in terms of D is more convenient for computational purposes.

Contributions to the stretching in a fragmenting material can arise from elastic changes in the lattice structure, D^e; from the motion of slip planes and dislocations that contribute to plastic flow, D^p; from crack opening D^o; from shearing of closed cracks D^s; from unstable growth of either open or closed cracks, D^g; from nonlinear interatomic and thermal effects represented by an isotropic equation of state, D^n; and from viscous behavior, D^v. Thus, one may write

$$D = D^e + D^p + D^o + D^s + D^g + D^n + D^v \quad . \tag{1.6}$$

Since the rate at which work is done by the stress σ is the trace of σD, each term of (1.6) is associated with the energetics of a particular deformation mechanism. The elastic term can be written, in indicial notation, as

$$D^e_{ij} = C_{ijk\ell}\hat{\sigma}_{k\ell} \tag{1.7}$$

where C is the compliance and $\hat{\sigma}$ is the stress rate defined by Dienes (1979a).

To account for plastic behavior in general deformation it is necessary to allow for arbitrary states of stress and, consequently, to make use of a hardening plasticity theory. An approach to this problem has been recently developed using kinematic hardening theory (Dienes, 1984a,b) but details are omitted here since space does not allow a complete discussion, and the main purpose of this paper is to review the work on statistical crack mechanics.

As cracks open, a certain amount of material is displaced. If it is assumed that the surrounding material does not change its density, and that the rate of material displacement in each direction is the sum of displacements due to the individual cracks, then it has been shown by Dienes et al (1980, 1983a) that the strain rate induced by an ensemble of penny-shaped cracks is

$$d^{o}_{ij} = \beta^{o} Z^{o}_{ijk\ell} \hat{\sigma}_{k\ell} \tag{1.8}$$

where β^{o} is a material constant and

$$Z^{o}_{ijk\ell} = \int_{\Omega} d\Omega n_i n_j n_k n_\ell \int_{0}^{\infty} dc \left(-\frac{\partial n^{o}}{\partial c}\right) c^3 \quad . \tag{1.9}$$

Here $n^{o}(c,\Omega,t)$ denotes the number density of open cracks with orientation Ω whose radii exceed c, and n_i denotes the ith component of a crack normal. Cracks are considered open when the normal component of traction

$$\tilde{\sigma} = \sigma_{ij} n_i n_j \tag{1.10}$$

is positive (tensile).

In addition to the stretching associated with crack opening, there can be a contribution resulting from the interfacial sliding of closed cracks, for which the normal component of traction is negative (compressive). The associated stretching is

$$d^s_{ij} = \beta^s Z^s_{ijk\ell}\hat{\sigma}_{k\ell} \tag{1.11}$$

where β^s is another material constant and

$$Z^s_{ijk\ell} = \int_\Omega d\Omega(1 - \alpha)b_{ijk\ell} \int_0^\infty dc \left(- \frac{\partial n^s}{\partial c}\right) c^3 \quad . \tag{1.12}$$

The quantity α, arising from the effect of interfacial friction between closed cracks, is given by

$$\alpha = \bar{\mu}|\tilde{\sigma}|(\Lambda - \tilde{\sigma}^2)^{-\frac{1}{2}} \tag{1.13}$$

where $\bar{\mu}$ is the coefficient of friction and Λ represents the magnitude squared of traction:

$$\Lambda = \sigma_{ij}n_j\sigma_{ik}n_k \quad . \tag{1.14}$$

Only the distribution of closed cracks $n^s(c,\Omega,t)$ enters into this calculation, and their effect is to modify only the shear strain, and not the dilatation. Thus, the operator

$$b_{ijk\ell} = \delta_{i\ell}n_jn_k + \delta_{j\ell}n_in_k - 2n_in_jn_kn_\ell \quad , \tag{1.15}$$

defined in the process of deriving (1.12), has a property such that d^s_{ii} vanishes, enforcing the shear constraint mentioned above. If α exceeds unity, the normal stress is great enough to lock cracks with the corresponding orientation, and their contribution to (1.12) is made to vanish in detailed calculations. This introduces the hysteretic effect of solid friction.

If the stress is high enough so that cracks are unstable, they grow at a rate which can approach half the wave speed, but the rate may be much lower at modest stress levels. The strain rate due to crack growth is given by

$$d^{g}_{ij} = \left(\beta^{o}\dot{Z}^{o}_{ijk\ell} + \beta^{s}\dot{Z}^{s}_{ijk\ell}\right)\sigma_{k\ell} \tag{1.16}$$

where the dots indicate that in (1.9) and (1.12), n^{o} and n^{s} are to be replaced by their time derivatives. The criteria for crack instability are discussed in a separate paragraph below.

At high pressure, when the state of stress is nearly isotropic, material behavior is dominated by an isotropic equation of state relating pressure p, to the density ρ and internal energy I, of the form (McQueen et al, 1970)

$$p = G(\rho)\rho I + f(\rho) \quad . \tag{1.17}$$

It is convenient and reasonably accurate for many materials to take $G\rho$ as a constant, $G_o\rho_o$, and to assume a linear relation

$$u_s = c_o + Su_p \tag{1.18}$$

between shock velocity u_s and particle velocity u_p. Then it is straightforward to show that

$$f(\rho) = \frac{K\theta(1 - G_o\theta/2)}{(1 - S\theta)^2} \tag{1.19}$$

where k denotes the bulk modulus, equal to $\rho_o c_o^2$, and θ denotes the compression

$$\theta = 1 - \rho_o/\rho \quad . \tag{1.20}$$

Only the nonlinear part of the equation of state (1.17) is incorporated into the term denoted by D^n of (1.6).

The overall compliance of the fractured material C can be represented as the sum

$$C = C^e + \beta^o Z^o + \beta^s Z^s \quad . \tag{1.21}$$

It is convenient for specific calculations to write

$$\underset{\sim}{D}_I^{el} = C_{IJ} \hat{\underset{\sim}{\Sigma}}_J \tag{1.22}$$

where $\underset{\sim}{D}^{el}$ and $\underset{\sim}{\Sigma}$ denote 6-index stretching and stress vectors and $\underset{\sim}{D}^{el}$ combines the stretching due to lattice strain, crack opening and crack shearing. Then (1.6) can be solved for the stress increment

$$\Delta \underset{\sim}{\Sigma} = C^{-1}(\underset{\sim}{D} - \underset{\sim}{D}^p - \underset{\sim}{D}^n - \underset{\sim}{D}^g - \underset{\sim}{D}^v - \underset{\sim}{D}^r)\Delta t \quad . \tag{1.23}$$

This formula includes $\underset{\sim}{D}^r$, an effective strain rate that arises from the calculation of stress rate, in addition to terms previously described.

When the far-field stress on a crack exceeds a critical value, a mechanical instability is induced and cracks begin to grow. This instability occurs when the work done by external tractions as a result of an increment in crack size exceeds the increase in internal energy. This internal energy is the sum of an elastic strain energy, surface energy and, for closed cracks, heating due to frictional sliding. For penny-shaped cracks in tension the strain energy is the sum of the tensile contribution determined by Sack (1946) and a shear contribution found by Segedin (1950) and is given by

$$W_o = \frac{4}{3}\frac{c^3}{\mu}\frac{1-\nu}{2-\nu}(2A - \nu\tilde{\sigma}^2) \tag{1.24}$$

as shown by Keer (1966). Here ν denotes the Poisson ratio, μ the shear modules and c, A, and $\tilde{\sigma}$ are defined above. Then it can be shown that open cracks are unstable if

$$A < \nu\tilde{\sigma}^2/2 + \pi\gamma\mu(2 - \nu)/2(1 - \nu)c \quad , \tag{1.25}$$

where γ denotes the specific surface energy. If plastic effects contribute significantly, then γ can be increased to account for enhanced toughness in an approximate way. For closed cracks, only the strain energy associated with shear is involved. This can be written

$$W_s = \frac{8}{3\mu}\frac{1-\nu}{2-\nu}(\sigma - \tau)^2 c^3 \quad , \tag{1.26}$$

where σ and τ denote the tangential components of traction associated with the far-field stress and interfacial friction. A proof that strain energy depends on stress difference in this manner, with

$$\sigma = \sqrt{\Lambda - \tilde{\sigma}^2} \quad , \quad \tau = -\bar{\mu}\tilde{\sigma} \tag{1.27}$$

is given by Dienes (1983b). Rice (1984) corrects the stability criterion derived by Dienes and shows that an instability occurs if

$$(\sigma - \tau)^2 c > \frac{\pi}{2} \frac{2 - \nu}{1 - \nu} \gamma\mu \quad . \tag{1.28}$$

The growth and coalescence of cracks is characterized by two distribution functions. The distribution of isolated cracks is characterized by $\ell(c,\Omega,t)$, the number of isolated, penny-shaped cracks per unit volume with orientation Ω at time t whose radii exceed c. The distribution of connected cracks is given by a corresponding function $m(c, \Omega,t)$. The sum of these distributions is $n(c,\Omega,t)$. It is shown by Dienes (1978) that these distributions are related by a Liouville equation

$$\ell_c \dot{c} + \dot{\ell} = - \dot{m} \quad . \tag{1.29}$$

At early times, while the mean free path is essentially constant,

$$\dot{m} = k\ell \tag{1.30}$$

where k is a constant related to the mean free path. The linear equation obtained by combining (1.29) and (1.30) is readily solvable when the initial distribution is exponential in crack size, with the result

$$\ell = \ell_o e^{-c/\bar{c}+(\dot{c}/\bar{c}-k)t} \tag{1.31}$$

valid for $c \geq \dot{c}t$; otherwise, ℓ vanishes. This solution can represent anisotropic materials such as oil shale, since ℓ_o, $\bar{c}$ and k can depend on orientation. Results showing new classes of phenomena appearing in an anisotropic material have been described by Dienes (1979b, 1981). The assumption of constant k may not be useful when the total crack area increases significantly as the result of growth. This is the subject of the second part of this paper.

The permeability tensor for fractured materials has been computed by Dienes (1982) as

$$K_{jk} = \frac{64}{45}\,\theta \int d\Omega A^3(\Omega) r_{ik} G(\Omega) \quad . \tag{1.32}$$

Here θ denotes a crack shape factor near unity, A is the aspect ratio of the cracks,

$$r_{ik} = \delta_{ik} - n_i n_k \tag{1.33}$$

and

$$G(\Omega) = -\int dc\, \frac{\partial m}{\partial c}\, c^5 \tag{1.34}$$

is the fifth moment of crack radius. If the cracks are isotropically distributed

$$K_{11} = \frac{8\pi^2}{15}\,\theta A^3 m_o \, \langle c^5 \rangle \tag{1.35}$$

and the tensor is diagonal and isotropic. The complete theory accounts for the probability of crack intersections and their coalescence into connected paths, and is discussed in the derivation. It also accounts for the possibility that there are not enough cracks to form connected paths by using percolation theory, not discussed herein.

2. Variable Mean Free Path

Consider a homogeneous, isotropic medium filled with circular cracks that are randomly, homogeneously, and isotropically distributed. If subjected to a stress field of sufficient intensity, cracks in a real material are unstable, and their growth can cause important changes in the macroscopic properties of the medium. This behavior is complicated by crack intersections, whose frequency is governed by the area per unit volume of cracks, a quantity that changes as the cracks increase in size. Thus, the mean free path of cracks is much more difficult to deal with than the mean free path o gases, which remains constant. The current approach idealizes material behavior by dividing the cracks into just two categories, those that are essentially isolated and those that have intersected a number, α, of other cracks and, as a result, are no longer capable of growth. The isolated cracks are considered active, that is, capable of unstable growth if the stress level exceeds a critical value, whereas the other cracks are considered inactive, and will not exhibit growth even at high stress levels because their edges have intersected the planes of other cracks, and there is no longer a region of stress concentration or an excess of strain energy to drive crack growth. This is of course, a great idealization, and the reader will quickly be able to demonstrate a number of geometric and physical effects which this idealization fails to account for. Still, even this relatively simple approach leads to considerable mathematical complexity. The consequences of this approach are examined in some detail here. The goal is to obtain an approximate, albeit crude, treatment of crack coalescence that is usable in real calculations of permeability, fragmentation and related effects. In addition, it is possible that the conclusions may suggest promising approximations.

The number density of isolated (active) cracks per unit volume whose radii exceed c is denoted by $L(c,t)$, and the number density of connected (inactive) cracks per unit volume whose radii exceed c is denoted by $M(c,t)$, where t denotes the duration of crack growth. The number of active cracks at time $t + \Delta t$ larger than c is equal to the number at time t larger than $c - \dot{c}\Delta t$, less the number that have been converted into inactive status, $M_t\Delta t$, in the interval Δt. Here $\dot{c}$ denotes the speed of crack growth, assumed uniform around the crack edges, but not necessarily constant in time. Subscripts denote differentiation. Symbolically,

$$L(c,t + \Delta t) = L(c - \dot{c}\Delta t,t) - M_t(c,t)\Delta t$$

or, passing to the limit of small Δt,

$$L_c\dot{c} + L_t = - M_t \quad . \tag{2.1}$$

The velocity dependence can be eliminated by introducing a new variable

$$\gamma = \int_0^t \dot{c}dt \tag{2.2}$$

representing the extent of crack growth. Then the governing equation simplifies to

$$L_c + L_\gamma = - M_\gamma \quad . \tag{2.3}$$

To make further progress it is necessary to characterize the rate of formation of inactive cracks. To this end, let L denote the length of edges of all the active cracks with radius exceeding c, per unit volume. Then, assuming circular cracks,

$$L = \int_c^\infty 2\pi c \left(-\frac{\partial L}{\partial c}\right) dc \quad . \tag{2.4}$$

The area swept out by cracks of radius exceeding c in time Δt is $L\dot{c}\Delta t$. Now, let η denote the number of crack intercepts per unit area on an arbitrary plane, a quantity independent of the plane selected in view of the statistical assumptions. (Later, it will be important to separate out the contributions from active and inactive cracks, but that is unnecessary at this point.) The variable η depends on γ, but not on c, an important point in the solution procedure that follows. Now, the rate at which cracks intersect (per unit volume) is equal to the product of two terms, the rate at which area is being swept out by growing cracks (per unit volume) and the number of cracks per unit area. The rate at which cracks become inactive is $1/\alpha$ times the rate at which intersections form. Then, using (2.2),

$$M_\gamma = L(c,\gamma)\eta(\gamma)/\alpha \quad . \tag{2.5}$$

It is convenient to define

$$z = L_c \quad , \; k = 2\pi/\alpha \tag{2.6}$$

in order to obtain a simple form for the governing equation, which then reduces to

$$z_c + z_\gamma + kc\eta z = 0 \quad . \tag{2.7}$$

The solution procedure falls naturally into two parts. In the first, a form for the solution involving only a function of one variable, $J(\gamma)$, is determined. Then, an expression for $\eta(\gamma)$ in terms of $L(\gamma,c)$ is formed. By combining these results a third-order ordinary differential equation is obtained whose solution determines J. The mean free path can then be obtained in terms of $L(\gamma,c)$.

A solution of (2.7) can be obtained by the transformation

$$u = c - \gamma \quad , v = c + \gamma \tag{2.8}$$

which results in

$$z_v + (k/4)(u + v)\eta\left(\frac{v - u}{2}\right) z = 0 \quad . \tag{2.9}$$

This has the integral

$$z = h(u)e^{k[cJ'(\gamma)-J(\gamma)]} \tag{2.10}$$

in terms of the original variables, where

$$J'' = -\eta(\gamma) \quad , J(0) = J'(0) = 0 \quad . \tag{2.11}$$

It follows that

$$L(\gamma,c) = -e^{k[\gamma J'(\gamma)-J(\gamma)]} \int_{c-\gamma}^{\infty} du\, h(u)e^{kuJ'(\gamma)} \quad . \tag{2.12}$$

If $c < \gamma$, the integrand in (2.12) has to be set to zero wherever u is negative, and it follows that L is independent of c. This represents the simple fact that when the growth is γ, no active crack can have a radius less than γ.

In Appendix A it is shown that the number of crack segments per unit area for cracks with a size distribution P(c) is

$$\eta_p = \frac{\pi}{2} \int_0^\infty P(c)\, dc \quad . \tag{2.13}$$

Then, the number of active cracks per unit area is

$$\eta_a = \frac{\pi}{2} \int_\gamma^\infty L(\gamma,c)\, dc \quad . \tag{2.14}$$

A similar relation can be written for the connected (inactive) cracks, though in this case it cannot be expected to be very accurate because the initially circular shape will not be retained in real materials after intersection. Thus, when cracks are assumed to be active until α intersections have occurred, their growth is somewhat overestimated. This estimate,

$$\eta_i = \frac{\pi}{2} \int_0^\infty dc\, M(c) \tag{2.15}$$

really represents an upper bound on η_i. The integration is carried out in Appendix B. When the result is combined with the preceding one, the total number of active and inactive cracks per unit area is

$$\eta = \eta_a + \eta_i = \frac{\pi}{2} \left\{ \int_0^\infty L(o,c)\, dc + \int_0^\gamma L(c,c)\, dc - \gamma L(\gamma,\gamma) \right\} \quad . \tag{2.16}$$

Hence,

$$\frac{d\eta}{d\gamma} = -\frac{\pi}{2}\frac{d}{d\gamma}L(\gamma,\gamma) \quad . \tag{2.17}$$

Now, though the theory of Laplace transforms is not employed here, it is convenient to use its notation, viz

$$\overline{h}(s) = \int_0^\infty e^{-su}\, h(u)\, du \quad . \tag{2.18}$$

Then, from (2.12)

$$L(\gamma,\gamma) = -e^{-k(J-\gamma J')}\overline{h}(s) \tag{2.19}$$

where

$$s = -kJ'(\gamma) \quad . \tag{2.20}$$

Combining (2.11), (2.17), and (2.19), we are led to the governing equation

$$J''' = \frac{\pi}{2}k\gamma\left(\frac{d\overline{h}}{ds} - \gamma\overline{h}\right)J''e^{-k(J-\gamma J')} \tag{2.21}$$

for $J(\gamma)$.

If the cracks have an initially exponential distribution, so that

$$h(c) = -(L_0/\overline{c})e^{-c/\overline{c}} \tag{2.22}$$

then

$$\overline{h} = -L_0/(1+\overline{c}s) \quad . \tag{2.23}$$

In this case the governing equation becomes

$$J''' = \frac{\pi k}{2} \gamma L_o \frac{\bar{c} + \gamma(1 + \bar{c}s)}{(1 + \bar{c}s)^2} J'' e^{-k(J-\gamma J')} \quad . \tag{2.24}$$

This can be put into a canonical form by the scaling transformations

$$W = -kJ \quad , \quad \theta = \gamma/\bar{c} \quad , \quad \beta = \pi k \bar{c}^{-3} L_o/2 \quad , \tag{2.25}$$

with the result

$$W_{\theta\theta\theta} = \beta\theta \frac{1 + \theta + \theta W_\theta}{(1 + W_\theta)^2} W_{\theta\theta} e^{W-\theta W_\theta} \quad . \tag{2.26}$$

Since the initial conditions for W are

$$W_{\theta\theta} = \beta \quad , \quad W = W_\theta = 0 \tag{2.27}$$

it follows that W is a function of θ with only one parameter, β. The variation of W with θ for various β is illustrated in Fig. 1.

The mean free path of cracks can be obtained by considering the projected crack area (on some arbitrary plane) per unit volume. For an isotropic distribution the projected area per unit volume is given by

$$A = \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \int_0^{2\pi} d\phi \int_0^{\infty} \pi c^2 \left(-\frac{\partial n}{\partial c}\right) dc \tag{2.28}$$

where n is the distribution of cracks per 2π. For an isotropic distribution

$$A = \pi \int_0^\infty cN \, dc \quad . \tag{2.29}$$

where $N = 2\pi n$ is the density of cracks with all orientations. If we consider a cylinder of length S and unit cross-sectional area filled with this isotropic distribution, the number of intersections that a generator of the cylinder will have with the contained cracks is AS. The length for which the number of intersections is unity is the mean free path

$$\lambda = 1/A \quad . \tag{2.30}$$

For an initially exponential distribution

$$\lambda = \lambda_o = 1/\pi L_o \bar{c}^2 \quad . \tag{2.31}$$

To determine the mean free path for the more general case it is necessary to evaluate the integral

$$A = \pi \int_0^\infty c(L + M) \, dc \tag{2.32}$$

where Eq. B.6 is used for $c < \gamma$ and Eq. B.5 for $c > \gamma$. In both cases $\bar{\gamma} < \gamma$, so for an exponential distribution the integrand is

$$L_c(\gamma,c) = -(L_o/\bar{c})e^{-k[J(\gamma)-cJ'(\gamma)]+(\gamma-c)/\bar{c}} \quad . \tag{2.33}$$

For $c < \gamma$

$$L = \frac{L_o e^{-k(J-\gamma J')}}{1 - k\bar{c}J'} \tag{2.34}$$

is independent of c, and for $c > \gamma$

$$L = \frac{L_o e^{-k(J-cJ')+(\gamma-c)/c}}{1 - k\bar{c}J'} \quad . \tag{2.35}$$

With these relations it is possible to determine the mean free path as a function of the growth γ, though it is necessary to carry out the integrations numerically.

The results can be put in dimensionless form by defining the dimensionless mean free path

$$\mu = \pi\bar{c}^2 L_o \lambda = [\int_0^\infty \theta d\theta(\tilde{\ell} + \tilde{m})]^{-1} \tag{2.36}$$

where

$$\tilde{\ell} = L/L_o \quad , \quad \tilde{m} = M/L_o \quad , \quad \theta = c/\bar{c} \quad . \tag{2.37}$$

The quantity μ is plotted as a function of β in fig. 1. A natural approximation to the solution curve is to take $\mu = \frac{1}{2}\beta^{2/3}$, drawn as a dashed line in the figure. This is equivalent to

$$\lambda_a = (\pi/8L_o\alpha^2)^{1/3} \quad , \tag{2.38}$$

a mean free path independent of the initial crack size, involving only the number density of cracks and the parameter α defining the number of intersections that terminates growth. The dimensionless crack distribution $\tilde{\ell} + \tilde{m}$ is plotted as a function of dimensionless crack radius in fig. 2.

3. Prospects for Further Development

The theoretical work, called statistical crack mechanics, outlined in this paper is intended for use in finite difference codes. It makes possible the calculation of dynamic problems involving crack growth and coalescence that arise, typically, in impact and explosion processes. It has been used to investigate blasting in oil shale and the sensitivity of propellants to impact. The approach has been to use analytic solutions to crack problems, to assume the simplest growth law, which involves self-similar growth of penny-shaped cracks, and to consider the statistics of crack coalescence using the simplest Liouville equation. This has made it possible to develop a computing scheme which works fairly well, though the complexity of the processes involved and uncertainties in the microstructure makes it very difficult to compare calculations with experiment. The calculations assume 9 possible crack orientations, thereby allowing for highly anisotropic behavior, with cracks in one orientation typically dominating. It is also assumed that the initial distribution of cracks is exponential in crack radius, with different distributions for each orientation being allowed. The strongly anisotropic behavior of oil shale resulting from its sedimentary formation was represented by allowing for an extra set of large cracks in the bedding planes. The number of crack sets (orientation and size distribution) can be readily modified in numerical schemes.

Numerous physical effects were ignored in developing the computational framework, in order to get a working computer code as soon as possible. This made it possible to examine many computational problems, including computer time and storage requirements, in the course of code development. It appears that computing speed is about a third of that for simple strength theories (plastic flow) and the memory requirement is roughly doubled. Some of the studies needed to improve the existing code are discussed in the next paragraph.

It would be useful to account for crack interactions, which are currently neglected, in some approximate way, especially since compression failures seem to involve coalescence of cracks into complex patterns. It would also be useful to account for out-of-plane growth of cracks, especially in compression where such effects seem to be very important. However, the statistical problem is greatly complicated when cracks assume complex shapes, and approximation methods would have to be devised. The effect of varying mean-free-path has been discussed at length in this paper. It seems probable that means for accounting for this effect within the computer code can be found, but work has not begun on this problem. However, the method described in the preceding section assumes an isotropic distribution. If used in its current state, it would be necessary to give up the anisotropy that is allowed in the linear, small-growth approximation. We hope to develop an approximate theory that combines aspects of both approaches. In the original work it was thought feasible to account for plastic behavior in rocks and other materials by suitably modifying the rules for interfacial friction of closed cracks. This effort is in limbo, and we have instead incorporated a phenomenological law (kinematic hardening) to account for plastic flow. Currently, interfacial friction is represented with a static value for locked cracks and a dynamic value for sliding cracks. It has been shown (Dienes 1983b) that softening and melting can occur very quickly on sliding cracks, and this may play an important role in high-speed processes. In fact, it may be that shear cracks behave more like shear bands under some conditions. A careful treatment of this problem would be of great interest. Crack speed is currently treated as a constant (somewhat below elastic wave speeds) at high stresses, but decreasing with stress to a high power at stresses below a critical value, in accord with propellant data. Different speeds are assumed for tension and shear cracks. The program also computes permeability, which can be highly anisotropic, but no allowance is made for crack opening or propagation due to internal gas pressure. Whether this effect is important is an open question in rock blasting. Another area of concern is the effect of inhomogenities, such as the aggregate in concrete, aluminum in propellants, or inclusions in rock, which can profoundly affect crack stability and the pattern of crack growth, but no means for accounting for those effects have yet been devised.

Thus, with statistical crack mechanics it is possible to account for failure in a somewhat detailed way, and the possibility of understanding processes such as fragmentation, spall, explosive and propellant sensitivity, blasting, and projectile penetration are greatly improved. On the other hand, many new questions about failure mechanisms are raised. More correlation with experiments and development of suitable diagnostics is needed to test the proposed approach to failure. On the other hand, it seems likely that we will be able to improve predictions and interpretations of failure more reliably with statistical crack mechanics and, in particular to understand better size and rate effects, which have not been amenable to detailed theoretical analysis in the past. These effects are now known to be of great importance, and better methods to relate laboratory and field behavior as well as static and dynamic behavior are needed.

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I am indebted to R. Mjolsness for going over the analysis of variable mean free path, pointing out an error, and for useful discussions.

APPENDIX A

Consider a space filled with penny-shaped cracks. It is useful to compute the distribution of segments formed by their intersections with a plane, S. If the cracks are randomly, isotropically, and homogeneously distributed, then the segment distribution is independent of S. Now, let $n(\Omega,c)$ denote the number of cracks per unit volume and per sterradian with orientation Ω whose radii exceed c. Take the plane S as the x-y plane, and let the distance of the center of a crack from the x-y plane be denoted by z. Let θ denote the angle of the crack normal with the z axis, so that

$$z = \sqrt{c^2 - \ell^2/4}\ \sin\theta \qquad (A1)$$

where ℓ is the length of the intercept, as illustrated in fig. A. Then, the number of cracks whose radii lie in the range $(c, c + \Delta c)$ having centers in the infinite slab $(z, z + \Delta z)$ is

$$-\frac{\partial n}{\partial c}\,\Delta c\,\Delta z \quad .$$

It follows that the number of intercepts with length exceeding ℓ by cracks with orientation Ω is

$$\tilde{n}(\ell,\Omega) = -\int_{\ell/2}^{\infty} dc \int_{-z(\ell)}^{z(\ell)} \frac{\partial n}{\partial c}\, d\zeta = -2\int_{\ell/2}^{\infty} \sqrt{c^2 - \ell^2/4}\ \frac{\partial n}{\partial c}\,\sin\theta\, dc \quad . \qquad (A2)$$

The number of intercepts of all orientations is, then,

$$P(\ell) = \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} \tilde{n}(\ell,\Omega) \, d\phi \quad . \tag{A3}$$

Carrying out the integrations over ϕ and θ on the assumption of random orientation one finds

$$P(o) = \pi^2 \int_0^{\infty} n \, dc \quad . \tag{A4}$$

The more general case of the relation between $P(\ell)$ and $n(\ell)$, which can be reduced to an Abelian integral equation, is discussed by Dienes (1979c), but here only the result given above is necessary. Since

$$n = N/2\pi \tag{A5}$$

the number of crack intercepts per unit area an also be written

$$P(o) = \frac{\pi}{2} \int_0^{\infty} N \, dc \quad . \tag{A6}$$

APPENDIX B

The number of crack intercepts per unit area with an arbitrary plane S, counting only those cracks that are inactive, is given by

$$\eta_i = \frac{\pi}{2} \int_0^\infty M(c)\, dc \tag{B1}$$

using the result of Appendix A. This relation can be expressed in terms of the density L in the following manner. Let

$$m = -M_{c\gamma} = L_{c\gamma} + L_{cc} \tag{B2}$$

denote the rate of production of connected cracks. In fig. B the γ-c plane is divided into regions G where growth occurs and NG where it does not. In G

$$M = \int_0^\gamma d\overline{\gamma} \int_0^\infty m\, d\overline{c} \tag{B3}$$

and in NG

$$M = \int_0^c d\overline{\gamma} \int_c^\infty m\, d\overline{c} + \int_c^\gamma d\overline{\gamma} \int_{\overline{\gamma}}^\infty m\, d\overline{c} \quad . \tag{B4}$$

Then, it is straighforward to show that in G ($\gamma < c$)

$$M = L(o,c) - L(\gamma,c) - \int_0^\gamma L_c(\overline{\gamma},c)\, d\overline{\gamma} \tag{B5}$$

and in NG ($\gamma > c$)

$$M = L(o,c) - L(\gamma,\gamma) - \int_{o}^{c} L_c(\overline{\gamma},c)\, d\overline{\gamma} \quad . \tag{B6}$$

Using these results to evaluate (B1) it can be shown that

$$\eta_1 = \frac{\pi}{2} \left\{ \int_{o}^{\infty} L(o,c)\, dc - \int_{\gamma}^{\infty} L(\gamma,c)\, dc + \int_{o}^{\gamma} L(c,c)\, dc - \gamma L(\gamma,\gamma) \right\} \quad . \tag{B7}$$

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Fig. 1. The dimensionless mean free path $\mu = \pi \bar{c}^{2} L_o \lambda$ as a function of $\beta = \pi k \bar{c}^{3} L_o/2$, in the limit of large crack growth γ. This limit depends only on β, which can be considered a dimensionless crack density. The approximation $u = \frac{1}{2} \beta^{2/3}$ may be useful in formulating algorithms for computer analysis.

Fig. 2. The dimensionless crack distribution function $\tilde{n} = \tilde{\ell} + \tilde{m}$ as a function of dimensionless crack size, $\theta = c/\bar{c}$, for various values of the dimensionless parameter $\beta = \pi k \bar{c}^{3} L_o/2$. This distribution is the limit at late times, large γ.

Fig. A. Intersection of a circular crack of radius c with the x - y phase as viewed horizontally from the edge (left) and from a direction at 90° to the edge (right). The figure illustrates the intercepts length ℓ, the inclination θ, and is the basis for Eq. A1.

Fig. B. Illustration of the γ - c phase showing regions where crack growth is possible ($c > \gamma$) and impossible ($\gamma > c$), for use in computing B3 and B4.

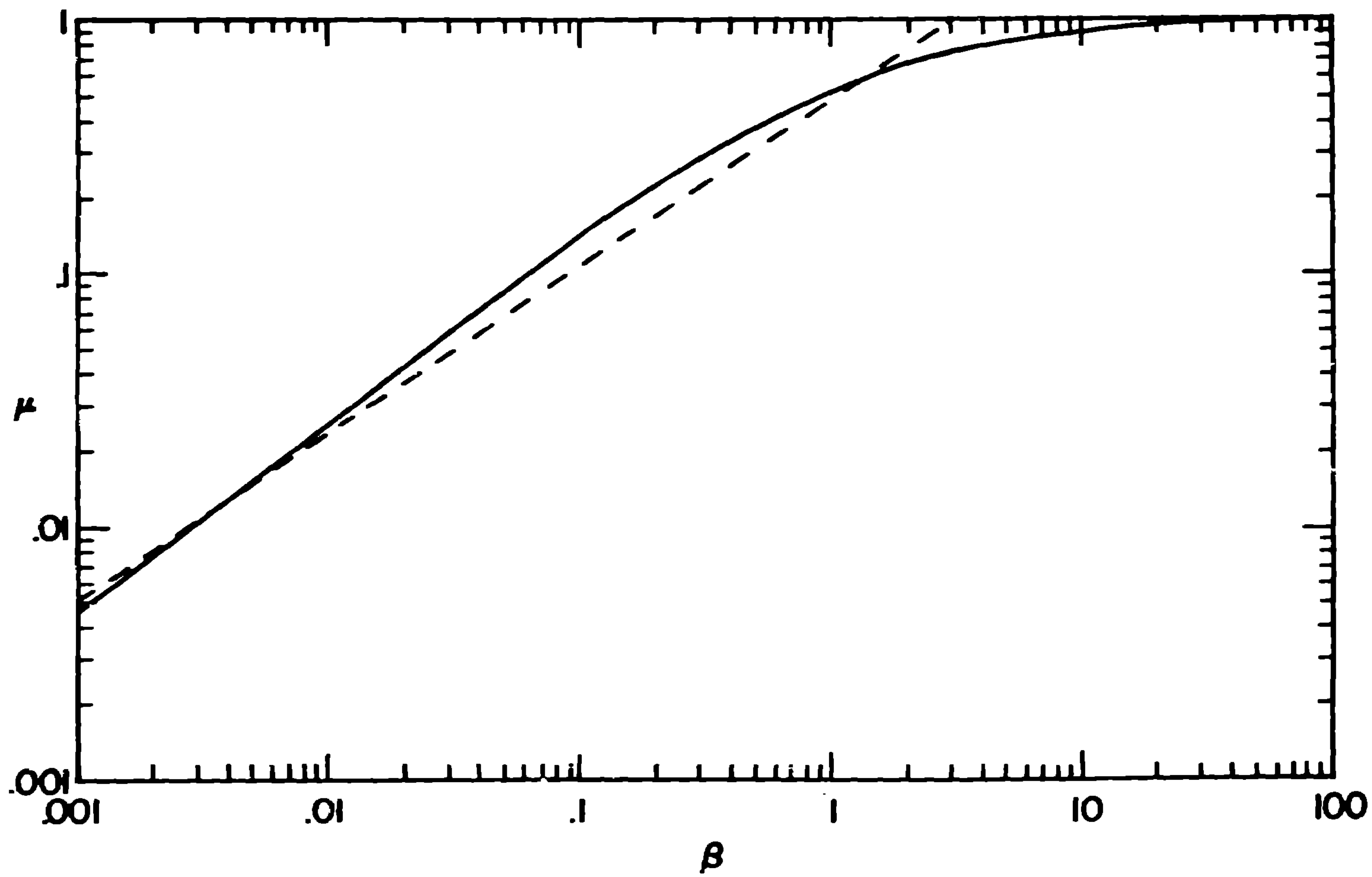

Fig. 1

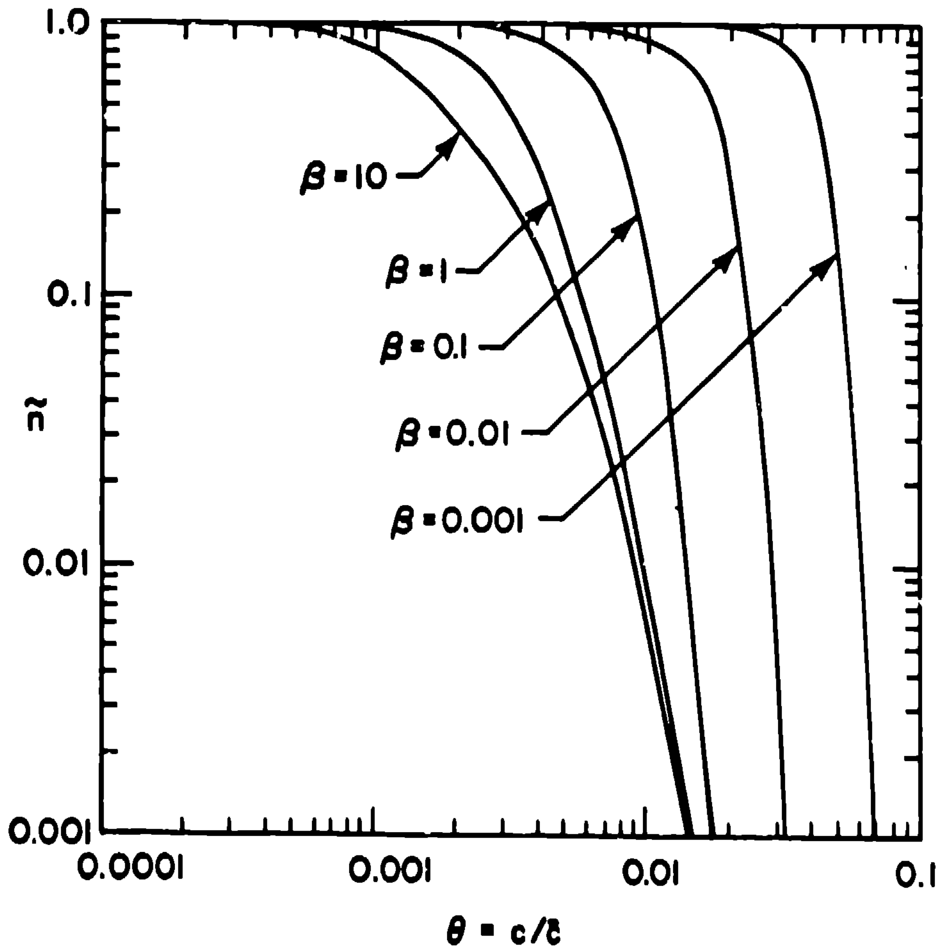

Fig. 2

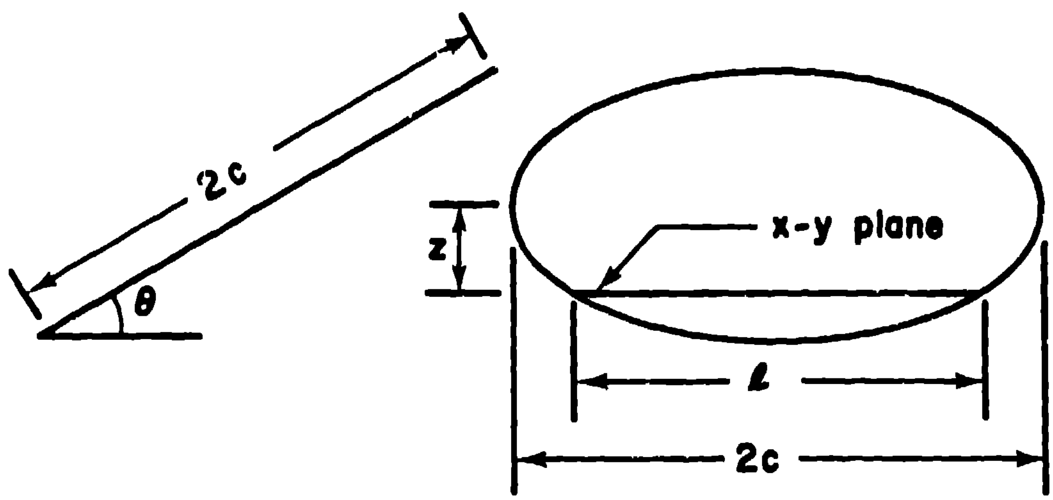

Fig. A

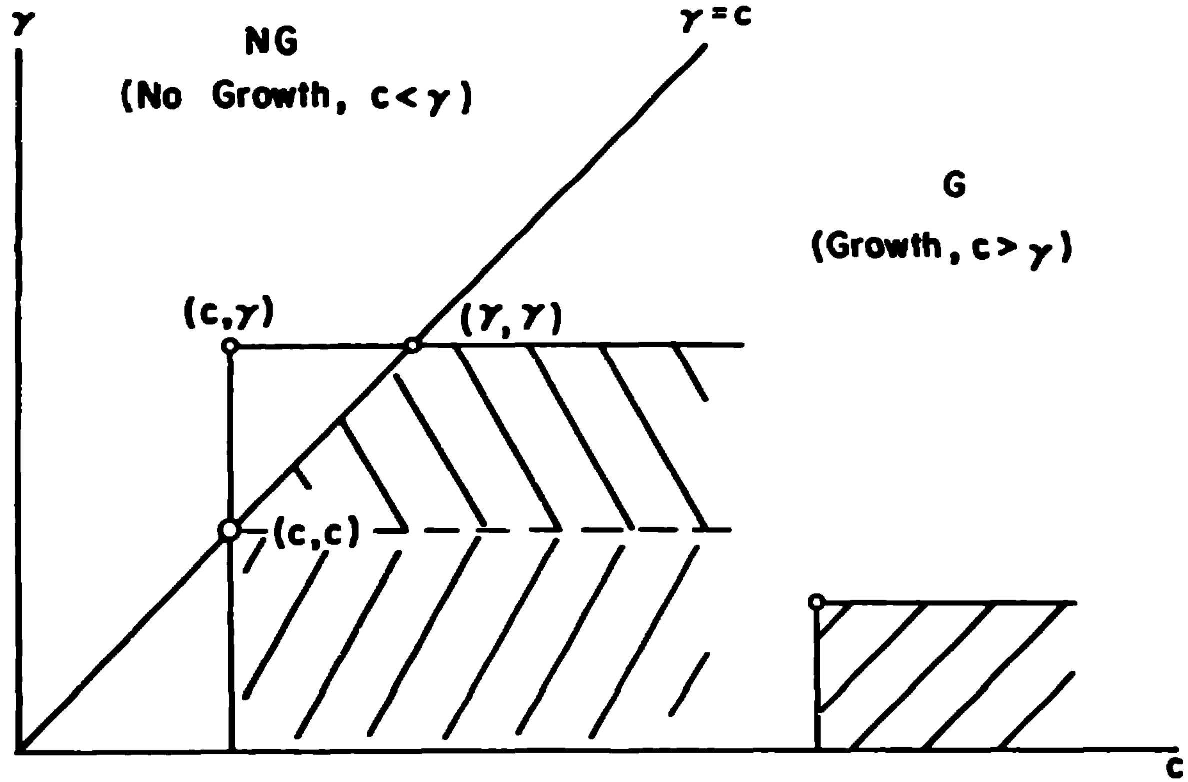

Fig. B