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L^∞ ESTIMATES AND VARIATIONAL CRIMES
IN THE FINITE ELEMENT METHOD

CHARLES I. GOLDSTEIN

November 1978

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APPLIED MATHEMATICS DEPARTMENT

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ABSTRACT

In order to numerically solve a second order elliptic boundary value problem in a bounded domain, Ω , using the finite element method, it is often necessary in practice to violate certain assumptions of the standard variational formulation. Two of these "variational crimes" will be considered here and it will be shown that optimal L^∞ estimates still hold. The first "crime" occurs when a nonconforming finite element method is employed, so that smoothness requirements are violated at interelement boundaries. In this case the norm given by $\| \cdot \|_{L^\infty(\Omega)}$ is replaced by the norm $\| \cdot \|_{L^\infty(\Omega)}^h = \max_{t \in T^h} \| \cdot \|_{L^\infty(t^h)}$, where T^h is the given partition of Ω and t^h denotes the interior of an element in this partition. The second "crime" occurs when numerical integration is employed, so that the bilinear form is perturbed.

Abstract error estimates are established. These estimates are first applied to treat a nonconforming finite element method and then to treat the case of numerical integration. In both cases, the "patch test" is crucial to the proof of L^∞ estimates, just as it was in the case of mean square estimates.

INTRODUCTION

The proof of optimal pointwise error estimates for the finite element method has been the subject of intensive research in recent years. For example, see refs. 1 to 9, where optimal (or nearly optimal) pointwise error estimates were established for second order linear elliptic boundary value problems. For a more comprehensive survey of the literature concerning this subject, see refs. 10 or 11.

For practical implementation of the finite element method it is often useful, or even necessary, to violate some of the basic assumptions of the underlying Galerkin method. These violations were referred to as variational crimes in ref. 12. It is the purpose of this paper to establish optimal L^∞ error estimates for finite element methods in which two of these variational crimes are committed.

The first crime occurs when nonconforming finite elements are employed. (This means that the trial functions fail to satisfy required smoothness conditions on interelement boundaries.) The second occurs when numerical quadrature is employed to approximately compute the coefficients of the stiffness matrix and the load vector. The third variational crime discussed in ref. 12 deals with the case in which the trial functions fail to satisfy essential boundary conditions. (This usually occurs in the presence of curved boundaries.) A number of methods for treating the last situation were considered in ref. 1, where optimal L^∞ error estimates were established.

The present paper may now be outlined as follows. In Section 2, a description is given of the second order linear elliptic boundary value problem under consideration, as well as certain fundamental properties of the finite element spaces. In addition, some of the basic results proved in refs. 1 and 2 will be recalled. In Section 3, some abstract L^∞ error estimates are established that will be applied in Sections 4 and 5. In Section 4, these estimates are applied to the case of nonconforming finite element methods. In Section 5, L^∞ estimates are obtained when numerical quadrature is employed. L^∞ estimates in the presence of numerical quadrature and isoparametric elements were obtained by Wahlbin in ref. 13.

2. PRELIMINARIES

Suppose that A denotes a second order elliptic differential operator defined in an N -dimensional domain Ω ($N \geq 2$) with boundary, $\partial\Omega$. To be specific, we consider the boundary value problem

$$Au = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + \sum_{i=1}^N b_i \frac{\partial u}{\partial x_i} + cu = f \quad (2.1)$$

in Ω , with boundary conditions given by either

$$u = 0 \quad \text{on } \partial\Omega \quad (a)$$

or

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad (b)$$

where $\partial u / \partial n$ denotes the outgoing normal derivative on $\partial\Omega$. We assume that A is elliptic and that the coefficients, a_{ij} , b_i and c , and boundary are sufficiently well behaved that assumptions (A.1) to (A.3) below are satisfied. The smoothness of f will be specified implicitly later in terms of u . The formal adjoint, A^* , is given by

$$A^*u = - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} a_{ij} \frac{\partial u}{\partial x_i} - \sum_{i=1}^N \frac{\partial}{\partial x_i} (b_i u) + cu.$$

We shall employ conventional notation for Sobolev spaces. Suppose $1 \leq p < \infty$ and M is a nonnegative integer. Set

$$|u|_{W_p^M(\Omega)} = \left(\sum_{|\alpha|=M} \|D^\alpha u\|_{L^p}^p \right)^{1/p},$$

and

$$\|u\|_{W_p^M(\Omega)} = \left(\sum_{s=0}^M |u|_{W_p^s(\Omega)}^p \right)^{1/p},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ with each $\alpha_j \geq 0$, $|\alpha| = \sum_{j=1}^N \alpha_j$, and $D^\alpha u$ denotes the weak derivative of u . For $p = \infty$, set

$$|u|_{W_\infty^M(\Omega)} = \sum_{|\alpha|=M} \|D^\alpha u\|_{L^\infty(\Omega)},$$

and

$$\|u\|_{W_\infty^M(\Omega)} = \sum_{s=0}^M |u|_{W_\infty^s(\Omega)}.$$

Define

$$W_p^M(\Omega) = \{u \mid \|u\|_{W_p^M(\Omega)} < \infty\} \text{ and } H^M(\Omega) = W_2^M(\Omega).$$

For nonintegral $s > 0$, we define $H^s(\Omega)$ in the sense of interpolation theory (see, e.g., ref. 14). $H^s(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm given by $\|\cdot\|_{H^s(\Omega)}$. The dual of $H^s(\Omega)$ will be denoted by $H^{-s}(\Omega)$. Finally, we shall denote the inner product on $L^2(\Omega)$ by $(\cdot, \cdot)$.

We are now ready to state the assumptions on A that will be required below.

(A.1) The equations $Au = f$ and $A^*u = f$ in Ω , $u = 0$ (or $\partial u / \partial n = 0$) on $\partial\Omega$ are uniquely solvable for each f in $L^2(\Omega)$.

(A.2) If $s_0 \geq 2$ and s are integers such that $0 \leq s \leq s_0 - 2$ and $f \in H^s(\Omega)$, then the solution, u , of (2.1) satisfies $\|u\|_{H^{s+2}(\Omega)} \leq C \|f\|_{H^s(\Omega)}$, with C independent of f .*

*In this paper we shall often use the same letter, C , to denote different constants when there is no danger of confusion.

For our next assumption, denote by G_{x_0} the Greens function for (2.1) with singularity at $x_0 = (x_{0_1}, \dots, x_{0_N})$ in Ω .

(A.3) Suppose that for each α : $1 \leq |\alpha| \leq d$, we have

$$|D_{x_0}^\alpha G_{x_0}(x)| \leq C |x - x_0|^{2-N-|\alpha|},$$

where d is a positive integer and C is independent of the points $x = (x_1, \dots, x_N)$ and x_0 in Ω (up to the boundary).

Remark: This result was proved in ref. 15 under appropriate smoothness assumptions on the coefficients and boundary.

A typical finite element method consists of a finite element subspace, S_K^h , and a bilinear form, $a(\cdot, \cdot)$. We assume that $a(\cdot, \cdot)$ is a symmetric, * bounded bilinear form defined on $W_1^1(\Omega) \times W_\infty^1(\Omega)$ and $H^1(\Omega) \times H^1(\Omega)$. In order to define the subspaces, S_K^h , we first define the notion of a quasi-uniform partition of Ω .

Let T^h denote a partition of Ω into a finite number of disjoint, open sets, t_i^h , $0 \leq i \leq I(h)$, such that $\bar{\Omega} = \bigcup_{i=0}^{I(h)} \bar{t}_i^h$. For example, in the plane, the t_i^h are usually triangles or rectangles. If $\partial\Omega$ is curved, then our boundary elements will have at least one curved side. We shall say that T^h is a quasi-uniform partition of Ω if each element, t_i^h contains a ball of radius $c_1 h$ and is contained in a ball of radius $c_2 h$, with $0 < c_1 < c_2$ independent of h and t_i^h .

*The symmetry of $a(\cdot, \cdot)$ is assumed in this section only for the sake of simplicity. Cases in which $a(\cdot, \cdot)$ is nonsymmetric are treated in ref. 1.

We now define our finite element spaces, S_K^h ,* to be finite dimensional subspaces of $H^1(\Omega)$ ($H^1(\Omega)$) in the case of problem (2.1)(a) ((2.1)(b)), satisfying certain conditions.

(A.4) If $v_1 \in S^h$ and $a(v_1, v) = 0$ for each $v \in S^h$, then $v_1 = 0$.

Remark: Using (A.4) and the finite dimensionality of S^h , we now define our finite element approximation, u^h , to the solution of (2.1) by means of the equation

$$a(u^h, v) = (f, v) \quad \forall v \in S^h. \quad (2.2)$$

We shall require certain approximation properties of the spaces, S^h .

(A.5) Suppose $1 < p \leq \infty$ and $\frac{N}{p} + 1 \leq s \leq K$ or $p = 1$ and $N + 1 \leq s \leq K$. Let $u \in W_p^s(t^h) \forall t^h \in T^h$. Then there exists a function $u^A \in S^h$ and a constant C such that for each $t^h \in T^h$:

$$\sum_{j=0}^1 h^{\frac{N}{p}+j} |u - u^A|_{W_\infty^j(t^h)} + \sum_{j=0}^1 h^j |u - u^A|_{W_p^j(t^h)} \leq Ch^s |u|_{W_p^s(t^h)}.$$

Remark: In practice, the function v^A is often constructed as the interpolate of v (with respect to the nodal basis specified in Condition (A.7) below) and the proof of (A.5) follows along lines given in ref. 12.

Condition (A.5) is usually needed in order to obtain mean square error estimates. The following assumption, often referred to as a "super-

*We assume that K is an integer greater than 1 and $h \in (0, h_0]$ for some $h_0 > 0$. Furthermore, from now on, we shall delete the subscript K when referring to S_K^h and i when referring to t_i^h .

"approximation" assumption is usually required (in various forms) for the proof of L^∞ error estimates.

(A.6) Let x_0 in Ω be fixed and set $r=|x-x_0|$ for each x in Ω . Then for each $v \in S^h$ and $t^h \in T^h$, we have

$$\sum_{j=0}^1 h^j |r^{\bar{N}+1} v - (r^{\bar{N}+1} v)^A|_{W_2^j(t^h)} \leq Ch^K \left(\int_{t^h} \sum_{|\alpha|=1}^K |r^{\bar{N}+1-|\alpha|}|^2 |D^{K-|\alpha|} v|^2 \right)^{1/2},$$

where $D^{K-|\alpha|} v$ denotes the derivative of v of order $K-|\alpha|$, the summation is over all multi-indices of order $|\alpha|$, $\bar{N}=N$ for $K>2$, $\bar{N}=N-1$ for $K=2$, and the constant, C , is independent of v , h , and t^h .

Remark: This version of the "superapproximation" assumption was required in refs. 1 and 2, where it followed immediately from (A.5) and the fact that v restricted to t^h was a polynomial of degree $<K \forall v \in S^h$. For an alternative version of (A.6), see ref. 8.

(A.7) The partition, T^h , is quasi-uniform. Furthermore the spaces, S^h , consist of functions, v , such that v restricted to t^h is a polynomial of degree $\leq K'$ for some $K' \geq K-1$ and each $t^h \in T^h$. Finally, S^h is generated by a nodal basis.

Remark: The following inverse inequalities are consequences of (A.7).

$$\|v\|_{W_q^s(t^h)} \leq Ch^{t-s-N(1/p-1/q)} \|v\|_{W_p^t(t^h)} \quad (2.3)$$

$\forall t^h \in T^h$ and $v \in S^h$, where $1 \leq p, q \leq \infty$ and $0 \leq t \leq s \leq K'$. For a proof of

(2.3) see, e.g., ref. 2. Another important consequence of (A.7) is the uniformity of the nodal basis for S^h (in the sense of [12], Section 3.1).

(A.8) Suppose that $-1 \leq s \leq K_0$ and $u \in H^2(\Omega)$, where $0 \leq K_0 \leq K-2$. Then

$$\|u - u^h\|_{H^{-s}(\Omega)} \leq Ch^{s+2} \|u\|_{H^2(\Omega)}.$$

We observe that (A.8) is the only assumption implying a relationship between the bilinear form $a(\cdot, \cdot)$ and Eq. (2.1). We now state three basic results that will be employed in the remainder of this paper. The proofs of these results are essentially the same as those in refs. 1 and 2 and hence will not be repeated.

Lemma 2.1: Suppose that $N \leq 2$ and conditions (A.1), (A.2), (A.4), (A.7), and (A.8) hold. Suppose also that $N/2 - 2 < s$ and $0 \leq s \leq \min(s_0 - 2, K_0)$ with s_0 and K_0 prescribed in (A.2) and (A.8), respectively. Then there exist functions $\tilde{G}_{x_0} \in H^{s_0}(\Omega) \cap H^1(\Omega)$, $G_{x_0}^h \in S^h$, and $\tilde{\delta}_{x_0} \in C_0^\infty(\Omega)$, satisfying the following conditions:

$$(i) \quad A\tilde{G}_{x_0} = \tilde{\delta}_{x_0},$$

$$(ii) \quad a(G_{x_0}^h, v) = (\tilde{\delta}_{x_0}, v) = v(x_0) \quad \forall v \in S^h,$$

$$(iii) \quad \max_{x \in \Omega} |D_x^\alpha \tilde{\delta}_{x_0}(x)| \leq C_\alpha h^{-N-|\alpha|}, \text{ where } |\alpha| \geq 0,$$

and

$$(iv) \quad \|G_{x_0} - G_{x_0}^h\|_{H^{-s}(\Omega)} \leq Ch^{s+2-N/2}.$$

In addition, the support of $\tilde{\delta}_{x_0}(x)$ is contained in the sphere of radius $C^1 h$ centered at x_0 , and the constants C_α , C^1 , and C are independent of h and the point $x_0 \in \Omega$.

The proof of Lemma 2.1 is the same as that of Lemma 1.1 in ref. 1.

Theorem 2.1: Suppose that $K \leq 3$, $N=2$ or 3 , $Au=f$, $u \in W_\infty^K(\Omega)$, and conditions (A.1), (A.2) (with $s_0=3$), (A.3) (with $d=3$), and (A.4) to (A.8) (with $K_0=1$ in (A.8)) hold.

Then

$$(a) \quad \|G_{x_0} - G_{x_0}^h\|_{W_1^1(\Omega)} \leq Ch, \text{ where } C \text{ is independent of } h \in (0, h_0]$$

and $x_0 \in \Omega$, and

$$(b) \quad \|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)}.$$

Theorem 2.2: Suppose $K=2$, $N=2$, $Au=f$, $u \in W_\infty^2(\Omega)$, and conditions (A.1), (A.2) (with $s_0=2$), (A.3) (with $d=2$), and (A.4) to (A.8) (with $K_0=0$ in (A.8)) hold. Then

$$(a) \quad \|G_{x_0} - G_{x_0}^h\|_{W_1^1(\Omega)} \leq Ch |\log h|, \text{ where } C \text{ is independent of } h \in (0, h_0]$$

and $x_0 \in \Omega$, and

$$(b) \quad \|u - u^h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}.$$

Remark: The main difficulty in the proof of these theorems lies in proving part (a). In order to see this, note the following. We may employ the properties of the Green's function, the symmetry of $a(\cdot, \cdot)$, (A.5), Lemma 2.1 and Eq. (2.2) to see that for each point $x_0 \in \Omega$

we have (with $G=G_{x_0}$, etc.):

$$\begin{aligned} |u(x_0) - u^h(x_0)| &= |a(G, u - u^h)| = |a(G - G^h, u - u^h)|, \\ &= |a(G - G^h, u - u^A)| \leq C \|G - G^h\|_{W_1^1(\Omega)} \|u - u^A\|_{W_\infty^1(\Omega)}, \\ &\leq Ch^{K-1} \|G - G^h\|_{W_1^1(\Omega)}. \end{aligned}$$

We thus see that (b) follows from (a). The technical details of the proof of (a) in these theorems may be found in refs. 1 and 2.

Corollary 2.1: Suppose that the conditions of Theorem 2.1 or 2.2 hold. Then for $2 \leq l \leq K$, we have

$$\|G^h\|_{W_1^l(\Omega)} \leq C \begin{cases} h^{-(l-2)} & \text{for } l > 2 \\ |\log h| & \text{for } l = 2 \end{cases}.$$

Proof: Set $C_h = \{x | |x - x_0| \leq h\}$, $\Omega_h = \bigcup_{t^h \in T^h} \{t^h \cap C_h \neq \emptyset\}$, and $\Omega'_h = \Omega - \Omega_h$.

Note that

$$\|G^h\|_{W_1^l(\Omega)} \leq \|G^h\|_{W_1^l(\Omega_h)} + \|G^h\|_{W_1^l(\Omega'_h)}, \quad (2.4)$$

and

$$\|G^h\|_{W_1^l(\Omega'_h)} \leq \|G - G^h\|_{W_1^l(\Omega_h)} + \|G\|_{W_1^l(\Omega'_h)}. \quad (2.5)$$

In view of condition (A.3), we have

$$\|G\|_{W_1^l(\Omega'_h)} \leq C \begin{cases} h^{-(l-2)} & \text{for } l > 2, \\ |\log h| & \text{for } l = 2. \end{cases} \quad (2.6)$$

We next consider $\|G-G^h\|_{W_1^\ell(\Omega_h')}$ and let G^A denote the interpolate of G in Ω_h' with respect to the nodal basis for S^h . Suppose $t^h \in \Omega_h'$ and apply (2.3) and (A.5). We readily obtain

$$\begin{aligned} \|G-G^h\|_{W_1^\ell(t^h)} &\leq \|G-G^A\|_{W_1^\ell(t^h)} + \|G^A-G^h\|_{W_1^\ell(t^h)} \\ &\leq c\|G\|_{W_1^\ell(t^h)} + Ch^{-(\ell-1)}\|G^A-G^h\|_{W_1^1(t^h)} \\ &\leq c\|G\|_{W_1^\ell(t^h)} + Ch^{-(\ell-1)}\|G-G^h\|_{W_1^1(t^h)}. \end{aligned} \quad (2.7)$$

We now apply (2.6), (2.7) and Theorem 2.1(a) or Theorem 2.2(a) and sum over each $t^h \in \Omega_h'$ to obtain:

$$\|G-G^h\|_{W_1^\ell(\Omega_h')} \leq c \begin{cases} h^{-(\ell-2)} & \text{for } \ell > 2 \\ |\log h| & \text{for } \ell = 2. \end{cases} \quad (2.8)$$

Finally consider

$$\|G^h\|_{W_1^\ell(\Omega_h)} \leq \|\tilde{G}-G^h\|_{W_1^\ell(\Omega_h)} + \|\tilde{G}\|_{W_1^\ell(\Omega_h)}. \quad (2.9)$$

Employing Lemma 2.1 and (A.2), we see that

$$\|\tilde{G}\|_{W_1^\ell(\Omega_h)} \leq Ch^{N/2} \|\tilde{G}\|_{H^\ell(\Omega_h)} \leq Ch^{N/2} \|\tilde{G}\|_{H^{\ell-2}(\Omega_h)} \leq Ch^{-(\ell-2)}. \quad (2.10)$$

Furthermore, note that $G^h = \tilde{G}^h$ and apply (A.2), (A.5), (A.8), Lemma 2.1, and (2.3) to obtain

$$\|\tilde{G}-G^h\|_{W_1^\ell(\Omega_h)} \leq Ch^{N/2} \left(\|\tilde{G}-\tilde{G}^A\|_{H^\ell(\Omega_h)} + h^{-\ell} \|\tilde{G}-G^h\|_{H^0(\Omega_h)} + h^{-\ell} \|\tilde{G}-\tilde{G}^A\|_{H^0(\Omega_h)} \right) \quad (2.11)$$

$$\leq Ch^{N/2} \|\tilde{G}\|_{H^\ell(\Omega_h)} \leq Ch^{N/2} \|\tilde{\delta}\|_{H^{\ell-2}(\Omega_h)} Ch^{-(\ell-2)}.$$

Combining (2.4)-(2.6) and (2.8) to (a.11), we have proved the Corollary.

Q.E.D.

In the remainder of this paper, we will be concerned with the proof of analogues of Theorems 2.1 and 2.2 when the finite element method described above is altered in either of two ways. The first possibility is that $S_h \dot{H}^1(\Omega) (H^1(\Omega))$ for Eq. (a.1)(a) ((2.1)(b)). The second possibility is that the form, $a(\cdot, \cdot)$ and/or functional $f(\cdot)$, given by $f(v) = (f, v) \forall v \in L^2(\Omega)$, is perturbed. In the next section, we shall prove abstract L^∞ error estimates for both cases. The results will then be applied to nonconforming methods in Section 4 and to numerical quadrature in Section 5. In both cases, the results will depend on estimates of the form

$$|(u(x_0) - u^h(x_0)) - a^h(G_{x_0} - G_{x_0}^h, u - u^A)| \leq Ch^K |\log h| \|\bar{K}\| u \|_{W_\infty^K(\Omega)},$$

where $\bar{K}=1$ for $K=2$ and $\bar{K}=0$ for $K>2$, and where $a^h(\cdot, \cdot)$ is a suitable perturbation of the bilinear form, $a(\cdot, \cdot)$. For the "standard" finite element method described above, $a^h(\cdot, \cdot) = a(\cdot, \cdot)$ and $C=0$.

3. ABSTRACT ERROR ESTIMATES

A. Trial Functions are not Piecewise Continuous

In this section, we shall consider a slightly more general situation than that of Section 2. We consider problem (2.1)(a), using the same notation as employed in Section 2. (The following results hold without any essential changes for problem (2.1)(b). However, we no longer assume that $S^h CH^1(\Omega) \cap W_\infty^1(\Omega)$. Instead, suppose that $S^h CH^h \cap W_\infty^h$, where H^h is a Hilbert space with norm given by $\| \cdot \|_h$, and W_1^h and W_∞^h are Banach spaces. We assume that the inclusions $H^1(\Omega) \subset H^h \subset L^2(\Omega)$ and $W_1^1(\Omega) \times W_\infty^1(\Omega) \subset W_1^h \times W_\infty^h$ are continuous. Finally, suppose that $a^h(\cdot, \cdot)$ ($a(\cdot, \cdot)$) is a bounded symmetric bilinear form defined on $H^h \times H^h$ ($H^1(\Omega) \times H^1(\Omega)$) and $W_1^h \times W_\infty^h$ ($W_1^1(\Omega) \times W_\infty^1(\Omega)$).

As in Section 2, we assume that conditions (A.1) to (A.3) hold. Furthermore, suppose that the following variations of conditions (A.4)-(A.8) hold. First of all, suppose that (A.4) (and hence (2.2)) holds with $a(\cdot, \cdot)$ replaced by $a^h(\cdot, \cdot)$ and that (A.5) and (A.7) remain unchanged. For the time being, we shall not require conditions (A.6) and (A.8).

Suppose $x_0 \in \Omega$ and observe that Lemma 2.1(i) to (iii) still holds (with $a(\cdot, \cdot)$ replaced by $a^h(\cdot, \cdot)$), provided $v(x_0)$ is defined $\forall v \in S^h$. (In

As in Section 2, we shall assume in this section that all bilinear forms are symmetric. Again, this assumption is used only for the sake of simplicity.

the case of nonconforming methods such as in Section 4, $v(x)$ may be discontinuous at interelement boundaries, so that we are forced to avoid points, x_0 , belonging to these boundaries). If we have the additional condition

$$a(u, v) = (Au, v) \quad \forall u \in \dot{H}^1(\Omega) \cap H^2(\Omega) \text{ and } v \in \dot{H}^1(\Omega) \quad (3.1)$$

and if the method is conforming (i.e. $s \in \dot{H}^1(\Omega) \cap W_\infty^1(\Omega)$), then it follows readily using (2.2) that

$$a^h(u - u^h, v^h) = 0 \quad \forall v^h \in S^h. \quad (3.2)$$

Now suppose that $x_0 \in t^h$ for some $t^h \in T^h$ and denote G_{x_0} by G , etc.

Suppose u satisfies (2.1a) and consider

$$u(x_0) - u^h(x_0) = a^h(G, u - u^h) + \begin{pmatrix} u(x_0) - u^h(x_0) \\ -a^h(G, u - u^h) \end{pmatrix}.$$

First note that

$$\begin{aligned} a^h(G, u - u^h) &= a^h(G - G^h, u - u^h) + a^h(G^h, u - u^h) \\ &= a^h(G - G^h, u - u^A) + a^h(G^h, u - u^h) + a^h(G - G^h, u^A - u^h). \end{aligned} \quad (3.3)$$

Next employ Lemma 2.1 (ii) (with $a(,)$ replaced by $a^h(,)$) to obtain

$$\begin{aligned} u(x_0) - u^h(x_0) - a^h(G, u - u^h) &= u(x_0) - a^h(G, u) + a^h(G, u^h) - u^h(x_0) \\ &= u(x_0) - a^h(G, u) + a^h(G - G^h, u^h) \\ &= u(x_0) - a^h(G, u) + a^h(G - G^h, u^A) + a^h(G - G^h, u^h - u^A). \end{aligned} \quad (3.4)$$

Finally, we add (3.3) and (3.4) and note that the last terms in each equation cancel. Hence we deduce

$$u(x_0) - u^h(x_0) = a^h(G - G^h, u - u^A) + a^h(G^h, u - u^h) + a^h(G - G^h, u^A) + (u(x_0) - a^h(G, u)). \quad (3.5)$$

Let us now consider the first term on the right side of (3.5)

and observe that

$$|a^h(G - G^h, u - u^A)| \leq \|G - G^h\|_{W_1^h} \|u - u^A\|_{W_\infty^h}. \quad (3.6)$$

In the case of conforming finite element methods, we have $S^{h,1}(\Omega)$, $a^h(\cdot, \cdot) = a(\cdot, \cdot)$, the last three terms in (3.5) vanish, $W_1^h = W_1^1(\Omega)$, and $W_\infty^h = W_\infty^1(\Omega)$. Hence for conforming finite element methods, these results reduce to those of Section 2. For nonconforming methods, we must again estimate the right side of (3.6) and also the last three terms in (3.5).

We shall see in connection with the problem of Section 4 that (3.6) may be estimated using the same arguments as those in refs. 1 and 2. For this purpose, we need conditions (A.6) and (A.8). Next, consider the third term on the right side of (3.5). We shall see in Section 4 that $a^h(G - G^h, u^A) = 0$, since we have the additional condition that u^A may be selected from $H^1(\Omega)$ for the problem treated there. (This additional assumption may be readily seen to apply in many cases of interest. However, it is not necessary in order to suitably estimate this term.) The last term in (3.5) will also be estimated in Section 4. It is often the case that this term is zero.

We are thus left with estimating the second term on the right side of (3.5). In Section 4, we accomplish this by means of the estimate

$$|a^h(v^h, u - u^h)| \leq Ch |u|_{W_\infty^2(\Omega)} \sum_{t \in T^h} |v^h|_{W_1^1(t^h)} |v^h|_{eS^h}. \quad (3.7)$$

In this case, S^h will consist of piecewise linear functions. The proof of (3.7) will be based on the use of certain polynomial invariances described in ref. 11. Combining the above estimates, we shall prove in Section 4 that

$$\|u - u^h\|_{L^\infty(\Omega)}^h \stackrel{\text{Def.}}{=} \max_{t^h \in T^h} \|u - u^h\|_{L^\infty(t^h)} \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}^2. \quad (3.8)$$

Note: The norm, $\|\cdot\|_{L^\infty(\Omega)}^h$, is used in (3.8) instead of $\|\cdot\|_{L^\infty(\Omega)}$ since the function u^h need only be defined on t^h , not necessarily on ∂t^h .

B. Perturbation of the Bilinear Form

In this subsection we assume that the finite element method is conforming, so that $S^h \subset H^1(\Omega)$. However, in this case the symmetric bilinear form, $a(\cdot, \cdot)$, and functional, $f(\cdot)$, given by $f(v) = (f, v) \forall v \in L^2(\Omega)$, are perturbed. We thus assume that there exists a second continuous symmetric bilinear form, $a^h(\cdot, \cdot)$, defined on $S^h \times S^h$, and a second continuous linear functional, $f^h(\cdot)$, defined on S^h . Note that $a(\cdot, \cdot)$ is defined on all of $H^1(\Omega) \times H^1(\Omega)$ and $f(\cdot)$ on all of $L^2(\Omega)$. We again consider Eq. (2.1)(a) and assume that conditions (A.1) to (A.3) of Section 2 hold.

We next state appropriate modifications of conditions (A.4) to (A.8) that will be required. Suppose that (A.4) holds for both $a(\cdot, \cdot)$ and $a^h(\cdot, \cdot)$. We may thus define unique functions u^h and $\tilde{u}^h$ in S^h by means

of the equations

$$\begin{cases} a^h(u^h, v^h) = f^h(v^h) & \text{and} \\ a(\tilde{u}^h, v^h) = f(v^h) \forall v^h \in S^h. \end{cases} \quad (3.9)$$

Assume that (A.5) and (A.7) remain unchanged and that (3.1) holds. We shall not require (A.6) and (A.8) at this time. Note that Lemma 2.1 (i) to (iii) follows as before, so that we may define G^h as in Lemma 2.1 (ii) (with respect to the unperturbed form, $a(\cdot, \cdot)$).

Now suppose that $x_0 \in \Omega$. We apply (3.1), (3.9), the symmetry of $a(\cdot, \cdot)$ and $a^h(\cdot, \cdot)$, and the properties of the Green's function $G_{x_0}^h = G$, to obtain

$$\begin{aligned} u(x_0) - u^h(x_0) &= a(G, u - u^h) = a(G, u - \tilde{u}^h) + a(G, \tilde{u}^h - u^h), \\ &= a(G - G^h, u - \tilde{u}^h) + a(G^h, \tilde{u}^h - u^h), \\ &= a(G - G^h, u - u^A) + (a^h - a)(G^h, u^h) + (f - f^h)(G^h). \end{aligned} \quad (3.10)$$

The first term on the right side of (3.10) is estimated by

$$\|G - G^h\|_{W_1^1(\Omega)} \|u - u^A\|_{W_\infty^1(\Omega)}. \quad \text{We may now add assumptions (A.6) and (A.8)}$$

and apply the arguments of refs. 1 and 2 to estimate this expression by

$$Ch^K \|u\|_{W_\infty^K(\Omega)}.$$

We next consider the last two terms on the right side of (3.10). In Section 5, we consider the case in which the perturbations are due to sufficiently accurate numerical quadrature schemes. We shall prove

the following estimates in Section 5:

$$|(a^h - a)(G^h, u^h)| \leq ch^{2K-3} \|u\|_{W_\infty^K(\Omega)} \|G^h\|_{W_1^{K-1}(\Omega)} \quad (3.11)$$

and

$$|(f - f^h)(G^h)| \leq ch^{2K-3} \|f\|_{W_{p_0}^{2K-3}} \|G^h\|_{W_1^{K-1}(\Omega)} \quad (3.12)$$

for $p_0 > 1$.

These estimates will be proved by again employing certain polynomial invariances described in ref. 11. The estimate for $\|G^h\|_{W_1^{K-1}(\Omega)}$ will be obtained from Corollary 2.1. Combining the above estimates, we shall prove in Section 5 that condition (A.4) holds for $a^h(\cdot, \cdot)$ and

$$\|u - u^h\|_{L^\infty(\Omega)} \leq ch^K |\log h|^{\bar{K}} (\|u\|_{W_\infty^K(\Omega)} + \|f\|_{W_{p_0}^K(\Omega)}),$$

where $\bar{K}=1$ for $K=2$ or 3 and $\bar{K}=0$ for $K>3$.

4. NONCONFORMING FINITE ELEMENT METHODS

In this section, we consider a specific nonconforming finite element method in R^2 . We shall derive L^∞ error estimates for this method by applying the general results of Section 3A. We begin by describing our boundary value problem and finite element method. Suppose that Ω is a sufficiently smooth domain in R^2 and $A = -\Delta$. The smoothness assumptions on $\partial\Omega$ are such as to guarantee conditions (A.1), (A.2) (with $s_0=2$) and (A.3) (with $d=2$). (In ref. 15, sufficient conditions are given for (A.3) to hold). We shall approximately solve problem (2.1)(a) for this domain and operator using the following nonconforming finite element method.

Suppose that Ω is replaced by an inscribed polygon, Ω^h . Consider a quasi-uniform partition, T^h , of $\overline{\Omega^h}$ into disjoint open triangles, t^h , of diameter $O(h)$. We assume that two neighboring triangles may only intersect at a vertex or along an entire side. We construct a trial space, S^h , as follows. Suppose first of all that each v^h in S^h vanishes in $\Omega - \overline{\Omega^h}$. Assume that the restriction of v^h to t^h is linear and that v^h is continuous at the midpoints of each side of t^h . Also suppose that $v^h=0$ at the midpoint of each boundary triangle of Ω^h .

Observe that functions in S^h are discontinuous in general and fail to vanish on $\partial\Omega^h$. However, S^h does contain the "conforming" subspace, S_c^h , consisting of continuous piecewise linear functions vanishing on $\partial\Omega^h$. The nonconforming space, S^h , was constructed

and applied in solving the Stokes problem in refs. 16 and 17, where this method was employed to impose the incompressibility condition, $\operatorname{div} v^h = 0$, on functions in S^h . The fact that S^h is roughly three times as large as S_c^h made it possible to impose this additional constraint.

It was shown in refs. 16 and 17 that the same L^2 and H^1 error estimates hold using the space, S^h , as with the conforming space, S_c^h . The main result of this section (Theorem 4.1 below) asserts that the same conclusion is true here with respect to the L^∞ norm. We first examine the assumptions of Sections 2 and 3 in connection with our present problem:

$$Au = -\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (4.1)$$

The bilinear form usually associated with Eq. (4.1) is given by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v. \quad (4.2)$$

This defines a bounded, symmetric bilinear form acting on $H^1(\Omega) \times H^1(\Omega)$ and $W_1^1(\Omega) \times W_\infty^1(\Omega)$. Since $S^h \not\subset W_1^1(\Omega)$, we modify the form, $a(\cdot, \cdot)$, and the spaces $W_p^\ell(\Omega)$, $\ell \geq 1$, $p \in [1, \infty]$, as follows. Note that in the following definitions, we replace Ω by Ω^h for the sake of simplicity. We shall see why this is justifiable in the proof of Theorem 4.1.

$$\text{Set } W_p^{\ell, h} = \{u \in L^p(\Omega^h) \mid \|u\|_{W_p^\ell}^h < \infty\},$$

where

$$\|u\|_{W_p^\ell}^h = \begin{cases} \left(\sum_{t^h \in \Omega^h} \|u\|_{W_p^\ell(t^h)}^p \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max_{t^h \in \Omega^h} \|u\|_{W_p^\ell(t^h)}, & p = \infty. \end{cases}$$

Denote $W_2^{\ell h}$ by $H^{\ell h}$ and $\|\cdot\|_{W_0^\ell}^h$ by $\|\cdot\|_{L^\infty}^h$. Clearly, we have

$$\|u\|_{W_p^\ell(\Omega^h)}^h = \|u\|_{W_p^\ell}^h \quad \forall u \in W_p^\ell(\Omega). \quad \text{We also define a new bilinear form,}$$

$a^h(\cdot, \cdot)$, as follows:

$$a^h(u, v) = \sum_{t^h \in \Omega^h} \int_{t^h} \nabla u \cdot \nabla v \quad \forall u, v \in H^{1h} \text{ or } u \in W_1^{1h}, v \in W_\infty^{1h}. \quad (4.3)$$

We have already noted that conditions (A.1) to (A.3) of Section 2 hold for a sufficiently smooth domain, Ω . We next consider conditions (A.4) to (A.8) with Ω replaced by Ω^h and $a(\cdot, \cdot)$ replaced by $a^h(\cdot, \cdot)$. Condition (A.4) holds using the same argument as in ref. 11 (Section 4.2) or ref. 16. As a consequence, there exists a unique function $u^h \in S^h$ satisfying the equation

$$a^h(u^h, v^h) = (f, v^h) \quad \forall v^h \in S^h. \quad (4.4)$$

Condition (A.5) holds with $K=2$ using the usual arguments based on the Bramble-Hilbert Lemma (see, e.g., ref. 12). The function, u^A , in (A.5) will be taken from now on to be the interpolate, u_c^I , of u with respect to S_c^h . Hence $u^A \in S_c^h$. Condition (A.6) follows readily from (A.5) and the fact that S^h is composed entirely of piecewise linear functions. Condition (A.7) is immediate from the definition of S^h . Condition (A.8) holds with $K_0=0$. The proof follows by

combining the same argument as in ref. 16 with (4.5) and (4.7) below.

It now follows that Lemma 2.1 holds (with $a(\cdot, \cdot)$ replaced by $a^h(\cdot, \cdot)$) provided $x_0 \in t^h$ for some $t^h \subset \Omega^h$. We shall denote the functions G_{x_0} , $\tilde{G}_{x_0}$ and $\tilde{\delta}_{x_0}$ of Lemma 2.1 by G , $\tilde{G}$ and $\tilde{\delta}$, respectively. We next state and prove the main result of this section.

Theorem 4.1: Suppose that $u \in W_\infty^2(\Omega)$. Then

$$\|u - u^h\|_{L^\infty}^h + \|u - u^h\|_{L^\infty(\Omega - \Omega^h)} \leq ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}.$$

Proof: We begin by considering $\|u - u^h\|_{L^\infty(\Omega - \Omega_h^h)}$. For this purpose, we recall that from the definition of S^h it follows that

$$\|v^h\|_{L^\infty(\Omega - \Omega_h^h)} = 0 \quad \forall v^h \in S^h. \quad (4.5)$$

Furthermore, it is readily seen that

$$\text{dist}(\partial\Omega^h, \partial\Omega) = O(h^2), \quad (4.6)$$

since $\partial\Omega$ is smooth. Hence it follows that

$$\|v\|_{L^p(\Omega - \Omega^h)} \leq ch^2 \|\nabla v\|_{L^p(\Omega)} \quad \forall v \in H^1(\Omega) \cap W_\infty^1(\Omega), \quad p \in [1, \infty]. \quad (4.7)$$

Combining (4.5) and (4.7), we see that

$$\|u - u^h\|_{L^\infty(\Omega - \Omega^h)} \leq ch^2 \|u\|_{W_\infty^1(\Omega)}. \quad (4.8)$$

We are thus left with proving

$$\|u - u^h\|_{L^\infty}^h \leq ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}. \quad (4.9)$$

Suppose that $x_0 \in t_0^h$ and recall Eq. (3.5),

$$\begin{aligned} u(x_0) - u^h(x_0) &= a^h(G - G^h, u - u^A) + a^h(G^h, u - u^h) \\ &\quad + a^h(G - G^h, u^A) + (u(x_0) - a^h(G, u)). \end{aligned} \quad (3.5)$$

Consider the last term in (3.5) and apply the properties of the Green's function, $G = G_{x_0}$, as well as (4.2) and (4.3). We thus obtain

$$u(x_0) - a^h(G, u) = \int_{\Omega - \Omega^h} \nabla G \cdot \nabla u. \quad (4.10)$$

Set $S_0^h = \Omega^h \cap C_0^h$, where C_0^h denotes a circle of radius $O(h)$ that contains the union of t_0^h and all of its neighboring triangles.*

In order to estimate the right side of (4.10), we shall need the following estimate

$$\int_{\Omega - \Omega^h} |v| \leq Ch^2 \|v\|_{W_1^1(\Omega - S_0^h)} \quad \forall v \in W_1^1(\Omega - S_0^h), \quad (4.11)$$

where C is independent of h and x_0 . The proof of (4.11) follows from (4.6) by applying the trace theorem to translates of $\partial\Omega$ and introducing a tubular neighborhood of the boundary. We now apply (4.11) to $v = \nabla G \cdot \nabla u$ to obtain

$$\int_{\Omega - \Omega^h} |\nabla G \cdot \nabla u| \leq Ch^2 \|u\|_{W_\infty^2(\Omega)} \|G\|_{W_1^2(\Omega - S_0^h)} \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}. \quad (4.12)$$

*The number of such triangles is bounded by a constant independent of h since the triangulations are quasi-uniform.

We next consider the first term on the right side of (3.5).

It follows from (A.5) and (4.3) that

$$|a^h(G-G^h, u-u^A)| \leq c \|G-G^h\|_{W_1^1}^h \|u-u^A\|_{W_\infty^1(\Omega)} \leq Ch \|u\|_{W_\infty^2(\Omega)} \|G-G^h\|_{W_1^1}^h. \quad (4.13)$$

Since we have observed above that the required conditions of Section 2 hold in this case, we may deduce from the proof of Theorem 2.2(a) that

$$\|G-G^h\|_{W_1^1}^h \leq Ch |\log h|. \quad (4.14)$$

Combining (4.13) and (4.14), we obtain

$$|a^h(G-G^h, u-u^A)| \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}. \quad (4.15)$$

We now consider the third term on the right side of (3.5) and recall that $u^A \in H^1(\Omega)$. Applying Lemma 2.1(b), the properties of G , and integration by parts over Ω , we conclude that

$$a^h(G-G^h, u^A) = 0. \quad (4.16)$$

We are thus left with proving

$$|a^h(G^h, u-u^h)| \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}. \quad (4.17)$$

We shall establish (4.17) with the aid of certain polynomial invariances. The argument is analogous to that used in ref. 11 (Section 4.2) for obtaining H^1 error estimates for a nonconforming method.

We first note that

$$a^h(u-u^h, w^h) = \sum_{t^h \subset \Omega^h} D_{t^h}(u, w^h) \quad \forall w^h \in S^h, \quad (4.18)$$

where

$$D_{t^h}(v, w^h) \equiv \sum_{\text{Def. } j=1}^3 \oint_{e_j^h} \frac{\partial v}{\partial n} (w^h - C(w^h, e_j^h)) \quad \forall v \in W_{\infty}^2(t^h), \quad w^h \in S^h, \quad (4.19)$$

with e_j^h , $j=1,2,3$, denoting the sides of t^h and $C(w^h, e_j^h)$ denoting the value of w^h at the midpoint of e_j^h . Equations (4.18) and (4.19) follow from (4.4), integration by parts and the fact that functions in S^h are continuous at the midpoint of e_j^h .

The following polynomial invariances will be crucial in obtaining the desired estimates;

$$D_{t^h}(v, p) = 0 \quad \forall v \in W_{\infty}^2(t^h) \quad \text{and} \quad p \in P_0(t^h), \quad (4.20)$$

and

$$D_{t^h}(q, w^h) = 0 \quad \forall q \in P_1(t^h) \quad \text{and} \quad w^h \in S^h, \quad (4.21)$$

where $P_{\ell}(t^h)$ denotes the space of polynomials of degree $\leq \ell$ defined on t^h and $\ell \geq 0$. The first invariance, (4.20), follows easily from (4.19). The second invariance was proved in ref. 12. It follows from the fact that the integral of a linear function, p_1 , over an interval is zero provided p_1 vanishes at the midpoint of the interval. Equation (4.21) is known as the "patch test."

We now define G^A to be the unique function in S_c^h equal to G at all vertices of the partition, T^h , that do not intersect t_0^h or any of its neighboring triangles. We set $G^A = 0$ at the remaining vertices of T^h . Since $G^A \in H^1(\Omega)$, we readily see with the aid of (4.4) and (4.18) that

$$a^h(u-u^h, G^h) = a^h(u-u^h, G^h - G^A) = \sum_{t^h \subset \Omega} D_{t^h}(u, G^h - G^A). \quad (4.22)$$

In order to estimate (4.22), we replace t^h by a reference element, $\hat{t}$, given by the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Given an arbitrary triangle, $t^h \subset \Omega$, we note that t^h is affine equivalent to $\hat{t}$. Furthermore, functions $\hat{v} \in W_\infty^2(\hat{t})$ and $\hat{p} \in \hat{P}$ correspond to analogous functions $v \in W_\infty^2(t^h)$ and $p \in P^h$, respectively, where $P^h = S^h|_{t^h}$ (i.e. S^h restricted to t^h) and $\hat{P}$ is defined as the pre-image of P^h under this affine mapping. Employing the "bilinear lemma" (see ref. 11, Section 4.2), we readily obtain the following estimate using (4.20) and (4.21):

$$|D_{\hat{t}}(\hat{v}, \hat{p})| \leq c |\hat{v}|_{W_\infty^2(\hat{t})} |\hat{p}|_{W_1^1(\hat{t})}. \quad (4.23)$$

It was shown in

$$|\hat{w}|_{W_p^j(\hat{t})} \leq ch^{\frac{j-2}{p}} |w|_{W_p^j(t^h)} \quad \forall w \in W_p^j(\hat{t}), \quad j \geq 0. \quad (4.24)$$

Applying (4.24) with $j=2$ and $p=\infty$ and 1, then substituting in (4.23), we readily deduce

$$|D_{t^h}(u, G^h - G^A)| \leq Ch |u|_{W_\infty^2(t^h)} |G^h - G^A|_{W_1^1(t^h)}. \quad (4.25)$$

Substituting (4.25) into (4.22), we obtain

$$|a^h(u-u^h, G^h)| \leq Ch|u|_{W_\infty^2(\Omega)} |G^h - G^A|_{W_1^1}^h. \quad (4.26)$$

We next observe that

$$\begin{aligned} |G^h - G^A|_{W_1^1}^h &\leq |G - G^h|_{W_1^1}^h + |G - G^A|_{W_1^1}^h \\ &\leq Ch|\log h| \|u\|_{W_\infty^2(\Omega)} + |G - G^A|_{W_1^1}^h, \end{aligned} \quad (4.27)$$

using (4.14). It follows from the definition of G^A and conditions (A.3) and (A.5) of Section 2 that

$$|G - G^A|_{W_1^1}^h \leq Ch|\log h|. \quad (4.28)$$

Combining (4.26)-(4.28), we thus conclude that

$$|a^h(u-u^h, G^h)| \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}. \quad (4.29)$$

Combining (4.10), (4.12), (4.15), (4.16), and (4.29) with (3.5),

we finally obtain

$$|u(x_0) - u^h(x_0)| \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}, \quad (4.30)$$

where C is independent of x_0 and $t^h \forall t^h \in \Omega^h$. We have thus proved (4.9) and hence Theorem 4.1, Q.E.D.

Remark: An analogue of Theorem 2.1 holds if we consider the Neumann problem, (2.1)(b), instead of (2.1)(a). In this case, the technical details are somewhat simplified since it is no longer necessary to impose the boundary condition on the trial functions. We now have $\Omega^h = \Omega$ (i.e., our boundary "triangles," t^h , may have one curved side). The last term in (3.5) is zero in this case.

NUMERICAL QUADRATURE

In this section we shall establish optimal L^∞ error estimates when the integrations necessary to obtain the finite element matrix equations are replaced by numerical quadrature. In order to isolate this particular variational crime, we shall consider the Neumann boundary condition. Hence, functions in the finite element space, S^h , need not satisfy the boundary conditions. In Remark 5.1, we shall indicate how the arguments still go through for Dirichlet boundary conditions. Let us observe that L^∞ estimates were established in the presence of numerical quadrature by Wahlbin in ref. 13. (See Remark 5.3 for a brief description of Wahlbin's results.)

Let A denote the elliptic operator given by (2.1) with the boundary condition (2.1)(b). Suppose that A is defined on a bounded domain $\Omega \subset \mathbb{R}^N$ with $N=2$ or 3 . We assume that the coefficients of A and the boundary of Ω are sufficiently smooth that conditions (A.1), (A.2), and (A.3) hold. The precise smoothness requirements on the coefficients needed to obtain estimates for the numerical quadrature will be given in Theorem 4.1 below. As in Sections 2 and 3, we assume only for the sake of simplicity that A is symmetric.

We now define the space, S^h , as follows. Our partition, T^h , will consist of simplices, t^h , such that boundary "simplices" may have one curved edge. Suppose that the partition is quasi-uniform. Furthermore, suppose that S^h is generated by a nodal basis and consists

of continuous functions whose restriction to each t^h is a polynomial of degree $\leq K-1$ for some fixed $K \geq 2$.

We now consider conditions (A.4) to (A.8) of Section 2. Employing the argument of Schatz in ref. 18, we may see that there is an $h_0 > 0$ such that (A.4) holds $\forall h \in (0, h_0]$. Condition (A.5) holds using the Bramble-Hilbert Lemma as in ref. 12 (Section 3.2). (A.6) follows from (A.5) and the fact that $D^\alpha v(x) = 0 \forall x \in t^h, v \in S^h$, and $|\alpha| = K$. Condition (A.7) is immediate from the definition of S^h . Finally, (A.8) holds with $K_0 = 0$ for $K = 2$ and $K_0 = 1$ for $K \geq 3$ using the standard duality argument (see, e.g., ref. 11, Theorem 3.2.4).

The bilinear form, $a(\cdot, \cdot)$, associated with problem (2.1)(b) is given by

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{j=1}^N b_j \frac{\partial u}{\partial x_j} v + cuv \right) \forall u, v \in H^1(\Omega). \quad (5.1)$$

Before defining our perturbed bilinear form, $a^h(\cdot, \cdot)$, we shall define our numerical quadrature scheme over each t^h in the same way as described in ref. 11 (Section 4.1), where there is a reference element, $\hat{t}$, such that $\hat{t}$ may be mapped onto each t^h by means of an invertible affine mapping, F_{t^h} , given by

$$F_{t^h}(\hat{x}) = B_{t^h} \hat{x} + b_{t^h}, \quad (5.2)$$

where B_{t^h} is an $N \times N$ matrix and b_{t^h} is an N -vector. The determinant

of B_{t^h} is denoted by $\det(B_{t^h})$. It is clear that for any integrable function $\phi(x)$ defined on t^h , we have

$$\int_{t^h} \phi(x) dx = \det(B_{t^h}) \int_{\hat{t}} \hat{\phi}(\hat{x}) d\hat{x}, \quad (5.3)$$

where ϕ and $\hat{\phi}$ are in the usual correspondence:

$$\phi(x) = \hat{\phi}(\hat{x}) \forall x = F_{t^h}(\hat{x}), \hat{x} \in \hat{t}. \quad (5.4)$$

We define our quadrature scheme over the reference element, $\hat{t}$, by means of a formula of the following form:

$$I_{\hat{t}}(\hat{\phi}) = \sum_{j=1}^J \hat{w}_j \hat{\phi}(\hat{b}_j). \quad (5.5)$$

We assume that the nodes, $\hat{b}_j$, belong to $\hat{t}$. In accordance with (5.3) and (5.4), we define our quadrature scheme over each t^h by means of the formula:

$$I_{t^h}(\phi) = \sum_{j=1}^J w_{j,t^h} \phi(b_{j,t^h}), \quad (5.6)$$

where $w_{j,t^h} = \det(B_{t^h}) \hat{w}_j$ and ϕ is defined at each node $b_{j,t^h} = F_{t^h}(\hat{b}_j)$, $1 \leq j \leq J$. Set

$$E_{t^h}(\phi) = \int_{t^h} \phi(x) dx - I_{t^h}(\phi)$$

and define $E_{\hat{t}}(\hat{\phi})$ similarly. Note that $E_{t^h}(\phi) = \det(B_{t^h}) E_{\hat{t}}(\hat{\phi})$.

We now define our perturbed bilinear form, $a^h(u, v)$, by replacing each integral in (5.1) by numerical quadrature. Hence

$$a^h(u, v) = \sum_{t^h \in T^h} I_{t^h} \left(\sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{j=1}^N b_j \frac{\partial u}{\partial x_j} v + cuv \right), \quad (5.7)$$

provided $u, v \in C^1(t^h)$ and a_{ij}, b_j , and $c \in C^0(t^h) \forall t^h \in T^h, i, j=1, \dots, N$.

It will follow from Theorem 4.1 that condition (A.4) holds with $a(,)$ replaced $a^h(,)$ under suitable assumptions. Assuming this to be true for the time being, we define our approximate solution, u^h , to (2.1)(b) as follows:

$$a^h(u^h, v) = (f, v) \forall v \in S^h. \quad (5.8)$$

Since we observed above that (A.4) holds with respect to $a(,)$, there exists a unique function $\tilde{u}^h \in S^h$ such that

$$a^h(\tilde{u}^h, v) = (f, v) \forall v \in S^h. \quad (5.9)$$

Note: In the following theorem we assume that (f, v) is computed exactly $\forall v \in S^h$. In Theorem 5.2 below, we remove this restriction.

Theorem 5.1: Suppose that conditions (A.1) to (A.3) of Section 2 hold and u satisfies (2.1)(b).

(a) Suppose that $K \geq 3$, $N=2$ or 3 , $a_{ij} \in W_\infty^{2K-5}(\Omega)$, $b_j \in W_\infty^{2K-4}(\Omega)$, $c \in W_\infty^{2K-3}(\Omega)$, $i, j=1, \dots, N$, $u \in W_\infty^K(\Omega)$, and

$$E_t(\hat{\phi}) = 0 \forall \hat{\phi} \in P_{2K-4}(\hat{t}). \quad (5.10)$$

Then for $h_0 > 0$ sufficiently small, there exists a unique $u^h \in S^h$ such that u^h satisfies (5.8) $\forall h \in (0, h_0]$.

Furthermore, we have

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K |\log h| \|u\|_{W_\infty^K(\Omega)}, \text{ where } \bar{K} = \begin{cases} 1 & \text{for } K=3 \\ 0 & \text{for } K>3. \end{cases}$$

(b) Suppose that $K=N=2$, $a_{ij} \in C^0(\Omega)$, $b_j \in W_\infty^1(\Omega)$, $c \in W_\infty^2(\Omega)$, $u \in W_\infty^K(\Omega)$, and

$$E_{\hat{t}}(\hat{\phi}) = 0 \epsilon \hat{\phi} \in P_1(\hat{t}). \quad (5.11)$$

Then for $h_0 > 0$ sufficiently small, there exists a unique $u^h \in S^h$ such that u^h satisfies (5.8) $\forall h \in (0, h_0]$. Furthermore, we have

$$\|u - u^h\| \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)}.$$

Proof: (a) Suppose for the time being that (5.8) holds for some $u^h \in S^h$. We begin with the following observation. Set $\Omega_1 = \{x \in \Omega \mid \text{dist}((x, \partial\Omega) \geq c_0 h\}$ for some fixed, sufficiently small constant c_0 . It now follows readily as in

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)} + C \|u - u^h\|_{L^\infty(\Omega_1)}. \quad (5.12)$$

There exists a point $x_0 \in \bar{\Omega}_1 \subset \Omega$, such that $\|u - u^h\|_{L^\infty(\bar{\Omega}_1)} = |u(x_0) - u^h(x_0)|$.

In view of (5.12), we thus see that

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)} + |u(x_0) - u^h(x_0)|. \quad (5.13)$$

Since it was observed above that conditions (A.4) to (A.8) hold with respect to $a(\cdot, \cdot)$, we may employ Lemma 2.1 to obtain a function $G^h = G_{x_0}^h \in S^h$. We now employ (3.7) to see that

$$|u(x_0) - u^h(x_0)| \leq |a(u - u^A, G - G^h)| + |(a^h - a)(u^h, G^h)|. \quad (5.14)$$

We may take $u^A = u^I$ (the interpolate of u with respect to S^h) in (5.14).

The first term on the right is estimated exactly as in the proof of Theorem 2.1. Hence it follows from (5.13) and (5.14) that

$$(5.15) \quad \|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)} + |(a^h - a)(u^h, G^h)|. \quad (5.15)$$

We next estimate the last term in (5.15). Suppose that $t^h \in T^h$ and consider

$$E_{t^h} \left(\sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \right) + E_{t^h} \left(\sum_{j=1}^N b_j \frac{\partial v}{\partial x_j} w \right) + E_{t^h} (cvw) \quad \forall v, w \in S^h.$$

We estimate the first term as follows. Employing the affine mapping, F_{t^h} , given by (5.2) as a map of $\hat{t}$ onto t^h , it may be readily seen using the arguments of ref. 11 that $|\det(B_{t^h})| \leq Ch^N$ and

$$|E_{t^h} \left(\sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \right)| \leq Ch^{N-2} |E_{\hat{t}} \left(\sum_{i,j=1}^N \hat{a}_{ij} \frac{\partial \hat{v}}{\partial \hat{x}_i} \frac{\partial \hat{w}}{\partial \hat{x}_j} \right)|. \quad (5.16)$$

We may apply the Bramble-Hilbert Lemma and the hypothesis, (5.10), to obtain

$$|E_{\hat{t}}(\hat{\phi})| \leq c |\hat{\phi}|_{W_1^{2K-3}(\hat{t})} \forall \hat{\phi} \in W_1^{2K-3}(\hat{t}). \quad (5.17)$$

Set $\hat{\phi} = \sum_{i,j}^N \hat{a}_{ij} \frac{\partial \hat{v}}{\partial \hat{x}_i} \frac{\partial \hat{w}}{\partial \hat{x}_j}$ and substitute in (5.17) to deduce

$$|E_{\hat{t}}(\hat{\phi})| \leq c \sum_{i,j=1}^N \sum_{\substack{\alpha+\beta+\gamma=2K-3, \\ \beta+1 \leq K-1, \\ \gamma+1 \leq K-1}} |\hat{a}_{ij}|_{W_{\infty}^{\alpha}(\hat{t})} |\hat{v}|_{W_{\infty}^{\beta+1}(\hat{t})} |\hat{w}|_{W_1^{\gamma+1}(\hat{t})}. \quad (5.18)$$

We now map $\hat{t}$ back onto t^h and combine (5.16) and (5.18) with the arguments of ref. 11 to conclude that

$$\begin{aligned} |E_{t^h}(\sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j})| &\leq ch^{N-2} \sum_{i,j=1}^N \sum_{\substack{\alpha+\beta+\gamma=2K-3 \\ \beta+1 \leq K-1, \\ \gamma+1 \leq K-1}} \\ &h^{-N+\alpha+\beta+\gamma+2} \{ |a_{ij}|_{W_{\infty}^{\alpha}(t^h)} |v|_{W_{\infty}^{\beta+1}(t^h)} |w|_{W_1^{\gamma+1}(t^h)} \} \\ &\leq ch^{2K-3} \max_{i,j=1,\dots,N} \|a_{ij}\|_{W_{\infty}^{2K-5}(t^h)} \|v\|_{W_{\infty}^{K-1}(t^h)} \|w\|_{W_1^{K-1}(t^h)}. \quad (5.19) \end{aligned}$$

Employing the same arguments as in the derivation of (5.19), we obtain

$$\begin{aligned}
 & |E_t^h(\sum_{j=1}^N b_j \frac{\partial v}{\partial x_j} w)| + |E_t^h(cvw)| \\
 & \leq Ch^{2K-3} (\max_{j=1, \dots, N} \|b_j\|_{W_\infty^{2K-4}(t^h)} + \|c\|_{W_\varphi^{2K-3}(t^h)}) \|v\|_{W_\infty^{K-1}(t^h)} \|w\|_{W_1^{K-1}(t^h)}
 \end{aligned} \tag{5.20}$$

We now combine (5.19) with (5.20) and sum over all $t^h \in T^h$ to conclude that

$$|(a^h - a)(v, w)| \leq Ch^{2K-3} \|v\|_{W_\infty^{K-1}}^h \|w\|_{W_1^{K-1}}^h. \tag{5.21}$$

Note that $\|\cdot\|_{W_\infty^{K-1}}^h$ and $\|\cdot\|_{W_1^{K-1}}^h$ are defined as in Section 4 and we are using the fact that

$$\sum_{i,j=1}^N \|a_{ij}\|_{W_\infty^{2K-5}(\Omega)} + \sum_{j=1}^N \|b_j\|_{W_\infty^{2K-4}(\Omega)} + \|c\|_{W_\infty^{2K-3}(\Omega)} < \infty.$$

We next combine (5.15) with (5.21) to see that

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)} + Ch^{2K-3} \|u\|_{W_\infty^{K-1}}^h \|G^h\|_{W_1^{K-1}}^h. \tag{5.22}$$

Employing Corollary 2.1, we obtain

$$\|G^h\|_{W_1^{K-1}}^h \leq Ch^{-(K-3)} |\log h|^{\bar{K}}. \tag{5.23}$$

We next consider

$$\|u^h\|_{W_\infty^{K-1}}^h \leq \|u^h - \tilde{u}^h\|_{W_\infty^{K-1}}^h + \|\tilde{u}^h\|_{W_\infty^{K-1}}^h, \quad (5.24)$$

where $\tilde{u}^h$ satisfies (5.9). It follows from (2.3) that

$$\begin{aligned} \|u^h - \tilde{u}^h\|_{W_\infty^{K-1}(\Omega)} &\leq Ch^{-(K-1)} \|u^h - \tilde{u}^h\|_{L^\infty(\Omega)} \\ &\leq Ch^{-(K-1)} (\|u - u^h\|_{L^\infty(\Omega)} + \|u - \tilde{u}^h\|_{L^\infty(\Omega)}). \end{aligned} \quad (5.25)$$

We observe that $\tilde{u}^h$ constitutes the "unperturbed" finite element approximation described in Section 2. Hence we may apply Theorem 2.1 and (2.3) to deduce

$$\begin{aligned} \|\tilde{u}^h\|_{W_\infty^{K-1}}^h &\leq \|\tilde{u}^h - u^A\|_{W_\infty^{K-1}}^h + \|u^A - u\|_{W_\infty^{K-1}}^h + \|u\|_{W_\infty^{K-1}(\Omega)} \\ &\leq Ch^{-(K-1)} (\|u - u^A\|_{L^\infty(\Omega)} + \|u - \tilde{u}^h\|_{L^\infty(\Omega)}) \\ &\quad + \|u - u^A\|_{W_\infty^{K-1}}^h + \|u\|_{W_\infty^{K-1}(\Omega)} \leq C(h\|u\|_{W_\infty^K(\Omega)} + \|u\|_{W_\infty^{K-1}(\Omega)}) \end{aligned} \quad (5.26)$$

Finally, we substitute (5.23) -

2.1. This yields

$$\begin{aligned} \|u - u^h\|_{L^\infty(\Omega)} &\leq Ch^K \|u\|_{W_\infty^K(\Omega)} + Ch^K |\log h|^{\bar{K}} (h\|u\|_{W_\infty^K(\Omega)} + \|u\|_{W_\infty^{K-1}(\Omega)}) \\ &\quad + Ch |\log h|^{\bar{K}} \|u - u^h\|_{L^\infty(\Omega)}. \end{aligned} \quad (5.27)$$

We thus see from (5.27) that when $h > 0$ is sufficiently small, we have

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^K \|u\|_{W_\infty^K(\Omega)}.$$

The last estimate implies the uniqueness of u^h as a consequence of condition (A.1). Since S^h is finite dimensional, this also implies its existence. This proves (a).

(b) The proof is essentially the same as in (a), except that the last term in (5.22) is replaced by $Ch^2 \|u^h\|_{W_\infty^1(\Omega)} \|G^h\|_{W_1^1(\Omega)}$. It follows from Theorem 2.2 and (A.3) that

$$\|G^h\|_{W_1^1(\Omega)} \leq \|G - G^h\|_{W_1^1(\Omega)} + \|G\|_{W_1^1(\Omega)} \leq Ch + C. \quad (5.28)$$

The remainder of the proof follows as in (a). Q.E.D.

Remark 5.1: If we replace the Neumann boundary conditions by the Dirichlet boundary condition in Theorem 5.1, our arguments are still applicable. In this case, however, the trial functions must satisfy the essential boundary condition. For example, if Ω is a polygon, we can impose the boundary condition on the trial functions exactly. In this case the arguments proceed as in Theorem 5.1 provided the regularity and Green's function estimates hold. If $\partial\Omega$ is curved, however, we could employ one of the nonstandard finite element methods for treating essential boundary conditions. For example, we might combine the argument of Theorem 5.1 with the methods of ref 1 to obtain the desired estimates.

Remark 5.2: The presence of $|\log h|$ in Theorem 5.1(a) when $K=3$ appears to be due to a technical difficulty in the proof. It is easily seen that this is removed if we improve the numerical quadrature for $K=3$, so that (5.10) is replaced by the following condition: $E_{\hat{t}}(\hat{\phi}) = 0 \forall \hat{\phi} \in P_3(\hat{t})$.

Remark 5.3: In Theorem 5.1 we assumed that the right side of (5.8) was computed exactly. Wahlbin in ref. 13, established L^∞ estimates for the finite element method applied to the Dirichlet problem, (2.1)(a), in R^2 using quadratic triangular isoparametric elements. He employed a numerical quadrature scheme sufficiently accurate that $E_{\hat{t}}(\hat{\phi}) = 0 \forall \hat{\phi} \in P_2(\hat{t})$. He assumed that $f \in W_1^3(\Omega)$ and obtained the error estimate:

$$\|u - u^h\|_{L^\infty(\Omega)} \leq C_\epsilon h^{2-\epsilon} \|f\|_{W_1^3(\Omega)} \quad \text{for arbitrary } \epsilon > 0.$$

The arguments of ref. 13 may also be shown to yield the estimate

$$\|u - u^h\|_{L^\infty(\Omega)} \leq C_\epsilon h^{2-\epsilon} \|f\|_{W_1^2(\Omega)}$$

in R^2 using piecewise linear functions in S^h and a quadrature scheme with the same accuracy as in Theorem 2.1(b). Furthermore, analogous estimates in R^2 follow from his arguments using higher degree finite elements and more accurate quadrature schemes. The estimates will, in general, be of the form

$$\|u - u^h\|_{L^\infty(\Omega)} \leq C_\epsilon h^{K-\epsilon} \|f\|_{W_1^K(\Omega)}.$$

The next result shows that optimal L^∞ estimates hold in our case with the right side of (5.8) also approximated by numerical quadrature. In order to eliminate the factor $h^{-\epsilon}$ present in the results of ref. 13, we assume that the data, f , is slightly smoother. The argument is analogous to that in ref. 13. For technical simplicity, we restrict ourselves to $N=2$ and $K=2$.

To begin with, let us denote $\int_{\Omega} f v$ by $f(v)$ and $\sum_{t \in T^h} I_{t^h}(f v)$ by $f^h(v) \forall v \in S^h$, $f \in C^0(\Omega)$. Recall that $I_{t^h}(f v)$ is defined using (5.5). We now define our approximation, u^h , to the solution of (2.1)(b) as follows:

$$a^h(u^h, v) = f^h(v) \forall v \in S^h. \quad (5.29)$$

Theorem 2.2: Suppose that conditions (A.1) to (A.3) of Section 2 hold and u satisfies (2.1)(b). Furthermore, suppose that $K=N=2$, $a_{ij} \in C^0(\Omega)$, $b_j \in W_{\infty}^1(\Omega)$, $c \in W_{\infty}^2(\Omega) \forall i, j=1, 2$, and Eq. (5.11) holds. Finally, suppose that $f \in W_{p_0}^2(\Omega)$ for some $p_0 > 1$. Then for $h > 0$ sufficiently small, there exists a unique $u^h \in S^h$ such that u^h satisfies (5.29). Furthermore, we have

$$\|u - u^h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h| \|u\|_{W_{\infty}^2(\Omega)} + Ch^2 \|f\|_{W_{p_0}^2(\Omega)}.$$

Proof: Suppose for the time being that (5.29) holds for some $u^h \in S^h$. Estimate (5.13) follows exactly as before for some point $x_0 \in \bar{\Omega}_1 \subset \Omega$. Equation (3.7) combined with (5.13) yields

$$\begin{aligned} \|u-u^h\|_{L^\infty(\Omega)} &\leq Ch^2 \|u\|_{W_\infty^2(\Omega)} + |a(G-G^h, u-u^A)| \\ &+ |(a^h-a)(G^h, u^h)| + |(f-f^h)(G^h)|. \end{aligned} \quad (5.30)$$

We may estimate the second and third terms on the right as in Theorem 5.1 to obtain

$$\|u-u^h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h| \|u\|_{W_\infty^2(\Omega)} + |(f-f^h)(G^h)|. \quad (5.31)$$

We are thus left with estimating the last term in (5.31).

Suppose that $t^h \in T^h$ and employ the mapping, F_{t^h} , given in (5.2) as well as the arguments of Theorem 5.1. We thus deduce

$$|E_{t^h}(fG^h)| \leq Ch^2 |E_{\hat{t}}(\hat{f}\hat{G}^h)| \leq Ch^2 |\hat{f}\hat{G}^h|_{W_1^2(\hat{t})}, \quad (5.32)$$

where we used the Bramble-Hilbert Lemma and (5.11) in the last step.

Suppose that $p=2+\epsilon$ with $\epsilon>0$ sufficiently small and define q_0 and q by the equations $1/q_0 + 1/p_0 = 1$ and $1/q + 1/p = 1$. Since $\hat{G}^h$ is linear in t^h , we may employ the properties of F_{t^h} as in the proof of Theorem 5.1 to obtain

$$\begin{aligned} |\hat{f}\hat{G}^h|_{W_1^2(\hat{t})} &\leq |\hat{f}|_{W_{p_0}^2(\hat{t})} |\hat{G}^h|_{L^{q_0}(\hat{t})} + |\hat{f}|_{W_p^1(\hat{t})} |\hat{G}^h|_{W_q^1(\hat{t})} \\ &\leq C(|f|_{W_{p_0}^2(t^h)} |G^h|_{L^{q_0}(t^h)} + |f|_{W_p^1(t^h)} |G^h|_{W_q^1(t^h)}). \end{aligned} \quad (5.33)$$

We next sum over all $t^h \in T^h$ and combine (5.32) and (5.33) with Sobolev's inequality to deduce

$$|(f-f^h)(G^h)| \leq Ch^2 \|f\|_{W_{p_0}^2(\Omega)} \|G^h\|_{W_q^1(\Omega)}. \quad (5.34)$$

Applying Lemma 2.1, we see that

$$\begin{aligned} \|G^h\|_{W_q^1(\Omega)} &\leq \|\tilde{G}-G^h\|_{W_q^1(\Omega)} + \|\tilde{G}\|_{W_q^1(\Omega)} \\ &\leq Ch\|\tilde{G}\|_{W_2^2(\Omega)} + \|\tilde{G}\|_{W_q^1(\Omega)} \leq C + \|\tilde{G}\|_{W_q^1(\Omega)}. \end{aligned} \quad (5.35)$$

In order to estimate the last term in (5.35) we write $\Omega = \Omega_0^h \cup \Omega^h$, where Ω_0^h is defined as the union of t^h and each of its neighboring triangles and $\Omega^h = \Omega - \Omega_0^h$. We also define $q' = 2 - \epsilon'$ with $\epsilon' > 0$ sufficiently small that Sobolev's inequality yields

$$\|\tilde{G}\|_{W_q^1(\Omega)} \leq C\|\tilde{G}\|_{W_1^{q'}(\Omega)} \leq C(\|\tilde{G}\|_{W_1^{q'}(\Omega_0^h)} + \|\tilde{G}\|_{W_1^{q'}(\Omega^h)}). \quad (5.36)$$

Employing Lemma 2.2, we obtain

$$\|\tilde{G}\|_{W_1^{q'}(\Omega_0^h)} \leq Ch\|\tilde{G}\|_{H^2(\Omega_0^h)} \leq Ch\|\tilde{\delta}\|_{L^2(\Omega)} \leq C. \quad (5.37)$$

To estimate the last term in (5.36), we note with the aid of Lemma 2.1 that $\tilde{G}(x) = \int_{\Omega} G_x(x') \tilde{\delta}(x') dx' \forall x \in \Omega$. We thus combine condition (A.3)

and Lemma 2.1 with the definition of Ω^h to deduce

$$\|\tilde{G}\|_{w_1^q(\Omega^h)} \leq C. \quad (5.38)$$

Combining (5.37) and (5.38) with (5.36), we obtain

$$\|\tilde{G}\|_{w_q^1(\Omega)} \leq C. \quad (5.39)$$

Estimates (5.34), (5.35), and (5.39) together yield

$$|(f-f^h)(G^h)| \leq Ch^2 \|f\|_{w_{p_0}^2(\Omega)}. \quad (5.40)$$

Finally, we combine (5.31) and (5.40) to see that Theorem 2.2 holds

Q.E.D.

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