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WHAT ARE SQUEEZED STATES REALLY LIKE?

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INTRODUCTION

This is work, the concept of which I have been mulling by myself and with Vern Sandberg over the last few years. Vern is now in the Medium Energy Physics Division at Los Alamos. He used to be in Kip Thorn's group at Caltech and before that he was with John Wheeler's group at Texas. Also, the stuff I'll be talking about tomorrow partially involves work which we jointly did with Bob Fisher, who is an experimental laser man.

In this lecture* I'm going to use what I call the "Bethe philosophy." I was a graduate student at Cornell during Hans Bethe's school of physics there. If there is one thing that the great man taught me, it is that if you really understand it, then you've got to be able to explain it simple. I remember Bethe could bring a seminar to a standstill when he would ask, with that slow teutonic voice of his, "But does this description give you the free particle?" Sometimes it didn't.

If you're doing mathematics, you can have fun with the superscripts to the subscripts. I enjoy that just as any other theoretician. Tomorrow, in the second lecture, we'll be playing that game.

However, today, after first giving the answer, I'm going to begin with the simple harmonic oscillator and some quantum mechanics. Then I'll review the special case of the squeezed states that you all know and love, the coherent states. I will start by discussing the coherent states from my simple picture, the Schrödinger point of view. This, I think, allows you to really understand things simply. Next, I'll describe the coherent states from the operator formalism. The operator formalism is more useful in the sense that once you understand what you're doing, you can calculate easier with it. Lastly today, I will describe the squeezed states from the Schrödinger point of view, and discuss their properties.

*This written version is based upon a recording of the two lectures actually delivered. The text has been edited to include topics raised by the questions and comments of the participants, both at the lectures and during breaks.

In my second lecture I'll begin by discussing harmonic motion, and coherent and "squeezed" states for general potentials. Then I'll move on to discuss the (harmonic oscillator) squeezed states from the operator point of view and some of their mathematical properties. I'll end by discussing the physics that they are useful for. In other lectures at this Institute, you'll get more detailed and perceptive discussions of the physics involved. My aim is to try to show you the path from this simple description to where you're really down to the nitty-gritty; detecting photon anti-bunching or gravitational waves. So with that and the memory that we're operating from a simple point of view, let me begin by giving the answer.

Here we go, folks. A coherent state, very simply, is a state in a harmonic oscillator potential which, at $t=0$, i) is a Gaussian that is displaced from the origin by x_0 , ii) has a phase which is proportional to the position x , and iii) the width of the Gaussian is that of the ground state, σ_0 . In particular, it is

$$\psi_{cs} = [2\pi\sigma_0^2]^{-1/4} \exp \left\{ - \left[\frac{x - x_0}{2\sigma_0} \right]^2 + \frac{i}{\hbar} p_0 x \right\} , \quad (1.1)$$

$$\sigma_0^2 = \frac{\hbar}{2m\omega} = \frac{1}{2\alpha_0} . \quad (1.2)$$

A squeezed state is the same Gaussian but the width is different than that of the ground state,

$$\sigma_0 \rightarrow \sigma = s\sigma_0 . \quad (1.3)$$

That's all it is, honest to God. You can go out to eat lunch in Santa Fe.

So why all the fuss? Well, part of the fuss, is that people like to make things complicated. (We can too, and we will a little later. As I indicated, it's fun to play with operators.) Furthermore, people are continuously rediscovering these states in different guises.

From the operator formalism, the operator transformation which creates a squeezed state,

$$a \rightarrow b = \mu a + \nu a^\dagger , \quad |\mu|^2 - |\nu|^2 = 1 , \quad (1.4)$$

is basically an updated version of the Holstein-Primakoff transformation¹ which was used to discuss magnetic solids back in 1940, or the Bogolubov transformation² which was used to discuss superfluidity and superconductivity back in 1947. [A little cultural side-point. Note the journal Bogolubov published in, "Journal of Physics, U.S.S.R." I'm now dated. At the end of World War II, the Soviet Union still had journals in foreign languages--a tradition that went back to Czarist times. But apparently someone (Stalin?) decided that shouldn't be, and it stopped.]

In 1962, before he became the President of the University of California, David Saxon published a book on quantum mechanics.³ On p. 162, in very concise Schrödinger wave function form, he describes the time evolution of squeezed states as we're going to be discussing it. The ideas involved trace back to Kennard's article.⁴ Its simply that fans of squeezed states often do not realize this early work was done.

The first (widely quoted) "modern" (re-)discovery was by Takahasi⁵ in 1965. He was interested in communication theory, discussing parametric amplifiers using wave-function theory. Next was David Stoler,⁶ who was interested in general minimum-uncertainty packets from an operator point of view. Horace Yuan⁷ was interested in two-photon coherent states. Hollenhorst⁸ invented the term "squeezed states." (That's the term which I prefer and which I use. You'll see why later.)

But since then, there have been pulsating states,⁹ correlated coherent states,¹⁰ new coherent states,¹¹ twisted states,¹² etc., etc., etc. These articles actually illuminate and emphasize things from new viewpoints, which is why new names are sometimes given for the states. However, in those cases where the connection to squeezed states was not realized, papers then came out saying the states were already known. This is confusing, and so, as I say, that's part of the reason for all the fuss.

SOME QUANTUM MECHANICS

Before we describe Schrödinger's states in our terminology, let's review some quantum mechanics, starting with the simple harmonic oscillator. This is something which you all got in your first course in quantum mechanics. You have the Hamiltonian eigenvalue equation

$$H\psi_n = E_n \psi_n, \quad (2.1)$$

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2. \quad (2.2)$$

You all know that the eigenstates are the Hermite polynomials times a Gaussian, and that the energy levels are equally spaced:

$$\psi_n = \left(\frac{a_0}{\pi^{1/2} 2^n n!} \right)^{1/2} \exp[-\frac{1}{2} a_0^2 x^2] H_n(a_0 x), \quad (2.3)$$

$$E_n = \hbar\omega(n + \frac{1}{2}). \quad (2.4)$$

Now is a good time to quickly introduce the operator formalism that we will use extensively, later. This formalism is "natural" in the harmonic oscillator system. It is natural because, for this equally-spaced energy-level system, the raising and lowering operators a^+ and a , are independent of n . That is, the same operator a^+ that takes you from the n th to the n th-plus-first state also takes you from the n th-plus-first to the n th-plus-second state. The operators are related to x and p as

$$x = \left(\frac{\hbar}{2m\omega} \right)^{1/2} (a + a^+), \quad (2.5)$$

$$p = \left(\frac{\hbar}{2m\omega} \right)^{1/2} \frac{1}{i} (a - a^+). \quad (2.6)$$

Then, since

$$[x, p] = i\hbar, \quad (2.7)$$

one has that

$$[a, a^\dagger] = 1 \quad . \quad (2.8)$$

Note, in particular, that what will be the standard notation for coherent states in the operator formalism is given by

$$\begin{aligned} \langle a \rangle_{cs} &= \alpha = \alpha_1 + i\alpha_2 \\ &= \langle x \rangle_{cs} a_0/2^{1/2} + i \langle p \rangle_{cs} / (\hbar 2^{1/2}) \quad . \end{aligned} \quad (2.9)$$

Commutators bespeak of uncertainty relations, so let me remind you about them. If we have two Hermitian operators X and P (not necessarily little x and little p), whose commutator is

$$[X, P] = iG \quad , \quad (2.10)$$

then we have that

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{1}{4} G^2 \quad . \quad (2.11)$$

Equality is obtained by those states which satisfy the eigenvalue equation

$$\left[X + \frac{i\langle G \rangle}{2(\Delta P)^2} P \right] \psi_{\text{mus}} = \left[\langle X \rangle + \frac{i\langle G \rangle}{2(\Delta P)^2} \langle P \rangle \right] \psi_{\text{mus}} \quad . \quad (2.12)$$

Because $\langle X \rangle$, $\langle X^2 \rangle$, $\langle P \rangle$, and $\langle P^2 \rangle$ are four parameters, equality in Eq. (2.11) reduces ψ to a function of three parameters.

That's going to be very important for us later. It will turn out that squeezed states satisfy (2.12) with x and p the operators, and coherent states set $(\Delta X)/(\Delta P)$ to a restricted value.

COHERENT STATES

Now let's go back to the beginning of coherent states. The beginning is Schrödinger's fourth paper in quantum mechanics.¹³ Schrödinger discovered what we now call the coherent states for the simple harmonic oscillator in 1926 in an article in Naturwissenschaften. They became known as the "coherent states" especially due to the work of Glauber¹⁴ and Klauder and Sudarshan¹⁵ in quantum optics.

What Schrödinger was interested in was to find wave functions which follow the classical motion of a particle. He wanted the shapes to remain constant in time. But note; this was 1926. So this was before Hiesenberg discovered the uncertainty principle and this was before Bohr had explained it. This was also before Born had realized that the density is $\psi^* \psi$. Schrödinger found a solution which did what he wanted, and which amounts to satisfying the uncertainty relation with a packet of ground state width. But, how did he get it?

Schrödinger thought in analogy to electromagnetic waves. What he did was to take the real part of ψ . So, instead of getting the Gaussian wave-packet, he got oscillations inside a Gaussian envelope. But, he had it. He didn't have the coherent-state wave packets, but he had the coherent states. What the hell, that's pretty good for 1926.

Schrödinger's point of view has a wave function which follows the classical motion and keeps its shape with time. That is closest to the uncertainty point of view. The commutator (2.7), $[x, p] = i\hbar$, implies that

$$(\Delta x)(\Delta p) \geq \hbar/2 . \quad (3.1)$$

The solution to the eigenvalue equation (2.12) is the Gaussian (1.3). If you further have that

$$(\Delta x)/(\Delta p) = 1/m\omega , \quad (3.2)$$

this amounts to specifying that the width of the Gaussian is that of the ground state for the harmonic oscillator; i.e., Eq. (1.1). This is the Gaussian which Schrödinger discovered, which follows the classical motion, and which does not change its shape with time.

You also know that the coherent states can be defined as eigenstates of the annihilation operator. If you just try to solve this equation,

$$a|\alpha\rangle = \alpha|\alpha\rangle , \quad (3.3)$$

you get that these coherent states $|\alpha\rangle$ can be written in terms of the number states as the particular combination

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \quad (3.4)$$

By using the Hermite polynomial generating function, one can verify that (3.4) is Eq. (1.1), up to a phase.

Notice in particular that α , which is defined by (2.9), is a two-parameter complex number, not the four parameters from $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, and $\langle p^2 \rangle$. Satisfying the equality in Eq. (3.1) reduces the four by one, and giving the particular value of the width in Eq. (3.2) reduces the parameters to two. Eqs. (1.1) and (3.4) are equivalent definitions.

Finally, pay attention to a third method for defining coherent states. This method is going to be generalized to squeezed states. Consider the displacement operator,

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] . \quad (3.5)$$

It takes the ground state $|0\rangle$ and it displaces it to $|\alpha\rangle$:

$$D(\alpha)|0\rangle = |\alpha\rangle . \quad (3.6)$$

From the Schrödinger viewpoint, the ground-state Gaussian is displaced in phase space to a beginning value x_0 and a beginning value p_0 . The Gaussian then oscillates back and forth without changing its shape, partially because the harmonic oscillator has equally-spaced levels. It turns out that you indeed get the classical motion, and the coherent states $|\alpha\rangle$ which you get from $D(\alpha)$ are the same ones we had before in (3.4).

You can sit down and have fun calculating that $\langle x(t) \rangle$ is equal to that of the classical motion, with the ground-state energy subtracted:

$$\langle \alpha | x(t) | \alpha \rangle = \left(\frac{2\hbar|\alpha|^2}{m\omega} \right)^{1/2} \sin(\omega t + \phi) . \quad (3.7)$$

Furthermore, if you calculate $\Delta x(t)$ and $\Delta p(t)$, they are constant:

$$[\Delta x(t)]^2 [\Delta p(t)]^2 = [\hbar/(2m\omega)] [m\hbar\omega/2] = \hbar^2/4 . \quad (3.8)$$

The coherent-state wave packet keeps its shape and goes back and forth forever.

SQUEEZED STATES IN THE SCHRÖDINGER PICTURE.

Now we go on to squeezed states. The squeezed states at $t=0$ are minimum-uncertainty states which contain, as a special class, the minimum-uncertainty coherent states. That is, Eqs. (1.1)-(1.3). Recall that when we got the coherent states we started with a Gaussian with 3 parameters. (Satisfying the equality in the uncertainty relation had reduced the possible 4 parameters to 3.) Having the width be that of the ground-state reduced the Gaussian to two parameters.

Well, now we're not going to reduce out that last parameter. The state no longer has the width of the ground state. We have the three parameter system defined by (1.3), which I repeat:

$$\psi_{ss} = [2\pi\sigma^2]^{-1/2} \exp \left\{ - \left[\frac{x - x_0}{2\sigma} \right]^2 + \frac{i}{\hbar} p_0 x \right\} , \quad (4.1)$$

$$\sigma = s\sigma_0 , \quad (4.2)$$

$$\sigma_0 = \frac{\hbar}{2m\omega} = \frac{1}{2a_0^2} . \quad (4.3)$$

σ_0 is the width of the ground state or the coherent state. "s" is what I call the "squeeze factor."

This shows you why I like the name "squeeze." What you're doing by putting that "s" in there is that you're either squeezing in the Gaussian ($s < 1$) or you're squeezing it out ($s > 1$), the momentum-space wave function being squeezed oppositely. You're taking the Gaussians and you're squeezing them. When "s" is equal to 1 you have a coherent state. When "s" is not equal to 1 you still have a squeezed state.

At $t=0$ both coherent states and squeezed states are minimum-uncertainty states. The difference is that for $t \neq 0$ a squeezed state will not remain a minimum-uncertainty state. What it does when it evolves away from $t=0$ is the basis of all the interest for the people in quantum optics and in gravitational wave detection. The fact that it is going to do something different is where the fun comes in.

You can sit down and, for $t=0$, you can calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, and $\langle p^2 \rangle$. (An exercise for the graduate student; he can crank it out.) You get that $[(\Delta x)^2 (\Delta p)^2]$ is the same as for the coherent states.

But what happens in time? Well, there are two ways to do it. We'll do it both ways. The first way is to calculate things with the operators being time dependent. That means I want to calculate the expectation values of

$$\delta(t) = \exp[i\hbar t/\hbar] \delta \exp[-i\hbar t/\hbar] , \quad (4.4)$$

where θ is x , x^2 , p , and p^2 . These are

$$\langle x(t) \rangle = x_0 \cos \omega t + \frac{p_0}{m\omega} \sin \omega t, \quad (4.5)$$

$$\langle p(t) \rangle = p_0 \cos \omega t - m\omega x_0 \sin \omega t, \quad (4.6)$$

$$\begin{aligned} \langle x^2(t) \rangle = (x_0^2 + \sigma^2) \cos^2 \omega t + \frac{\left(p_0^2 + \frac{\hbar^2}{4\sigma^2}\right)}{m^2 \omega^2} \sin^2 \omega t \\ + \frac{2x_0 p_0}{m\omega} \cos \omega t \sin \omega t, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \langle p^2(t) \rangle = \left(p_0^2 + \frac{\hbar^2}{4\sigma^2}\right) \cos^2 \omega t + m^2 \omega^2 (x_0^2 + \sigma^2) \sin^2 \omega t \\ - 2m\omega x_0 p_0 \cos \omega t \sin \omega t. \end{aligned} \quad (4.8)$$

Because of the $s = \sigma/\sigma_0$ in $\langle x^2 \rangle$ and $\langle p^2 \rangle$, this is already telling you that if I squeeze in x -space I squeeze in the opposite direction in p -space. When you calculate the uncertainty, you get

$$[\Delta x(t)]^2 = \frac{1}{2a_0^2} \left[s^2 \cos^2 \omega t + \frac{1}{s^2} \sin^2 \omega t \right], \quad (4.9)$$

$$[\Delta p(t)]^2 = \frac{\hbar^2 a_0^2}{2} \left[\frac{1}{s^2} \cos^2 \omega t + s^2 \sin^2 \omega t \right]. \quad (4.10)$$

Now, because cosine rotates into sin after $\pi/2$, you can already see what's going to happen. The two uncertainties are going to oscillate in width, out of phase.

When you take the uncertainty product you get

$$[\Delta x(t)]^2 [\Delta p(t)]^2 = \frac{\hbar^2}{4} \left[1 + \frac{1}{4} \left(s^2 - \frac{1}{s^2} \right)^2 \sin^2 [2\omega t] \right]. \quad (4.11)$$

Notice two things. When $s=1$, we have a coherent state and the uncertainty product is a constant in time, as advertised. The second thing, which also is as advertised, is that this uncertainty product oscillates in and out with an angular velocity which is twice the angular velocity of the particle going back and forth in the well.

To reemphasize, a squeezed state does not remain a minimum-uncertainty state as a function of time. At various times, whenever $\omega t = \pi/2$, it will be a minimum-uncertainty state. But it oscillates from being a minimum-uncertainty state. In a little bit, I'll give you a figure which describes how this comes about.

This squeezed-state Gaussian property of regaining its original shape turns out to be more general. It is the property of any wave packet in a simple harmonic oscillator well, because the energy levels

are equally spaced. Take any wave function, decompose it into eigenstates and consider it as a function of time:

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x) \exp[-i\omega t(n + \frac{1}{2})] \quad (4.12)$$

Because the decomposition is commensurate, every full period $[t = 2\pi j]$, $|\psi(x,t)|$ will return to its original shape and position, just because the eigenstates are evenly spaced. Every half period $[t = (2j + 1)\pi/\omega]$, ψ will be mirrored in the form $|\psi(-x,0)|$, because it's the harmonic oscillator. What happens is that a packet will start on the left, be mirrored on the right one half period later, and return to the left as it started. Even a delta function (considered as a limit of a Gaussian) will do this.

Now! What is $\psi_{ss}(x,t)$? To find out, first do what I just said, decompose it into eigenvectors. Combining Eqs. (2.3), (2.9), (4.1), and (4.12),

$$\begin{aligned} c_n &= \langle n | \psi_{ss} \rangle \\ &= \left[\frac{2s}{(1+s^2)^2 n!} \right]^{\frac{1}{2}} \exp \left[-\frac{\alpha_1^2 + \alpha_2^2 s^2}{1+s^2} \right] \\ &\quad \times \left[\frac{1-s^2}{1+s^2} \right]^{n/2} H_n \left(\frac{2^{1/2}(\alpha_1 + i\alpha_2 s^2)}{[(1-s^2)(1+s^2)]^{\frac{1}{2}}} \right) \end{aligned} \quad (4.13)$$

Here I have changed to the notation $\alpha = \alpha_1 + i\alpha_2$ instead of x_0 and p_0 because that allows me to get rid of some of the constants. When $s=1$, c_n just becomes the normal Gaussian coherent state factor.

$$\lim_{s \rightarrow 1} c_n = \exp[-\frac{1}{2} |\alpha|^2] \frac{\alpha^n}{(n!)^{\frac{1}{2}}} \quad (4.14)$$

Now we can write down $\psi_{ss}(x,t)$ explicitly.

$$\psi_{ss}(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x) \exp[-i\omega t(n + \frac{1}{2})] \quad (4.15)$$

What does it look like? Well, again folks, this is an exercise for the graduate student (or in this case me, when I made God knows how many mistakes--you know how that goes). This is it:

$$\begin{aligned} \psi_{ss}(t) &= \left[\frac{1}{2^{\frac{1}{2}} \pi^{\frac{1}{2}} \Delta x(t)} \right]^{\frac{1}{2}} \exp \left[-\frac{[x - x_0(t)]^2}{4[\Delta x(t)]^2} \right] \\ &\quad \times \exp \left\{ ix \left[\frac{p_0 \cos \omega t}{h} - \frac{a_0^2 x_0 \sin \omega t}{s^4} \right] - \frac{\sigma^2}{[\Delta x(t)]^2} \right\} \\ &\quad \times \exp \left\{ -i \frac{x^2 (s^4 - 1) \cos \omega t \sin \omega t}{4s^2 [\Delta x(t)]^2} \right\} \exp \{ i\phi(t) \} \end{aligned} \quad (4.16)$$

$$\phi(t) = \left[\left(\frac{x_0^2 a_0^2}{s^2} - \frac{p_0^2 s^2}{\hbar^2} \right) (\sin \omega t \cos \omega t) - 2 \frac{x_0 p_0 a_0^2}{\hbar(1+s^2)} (s^2 \cos^2 \omega t - \sin^2 \omega t) \right] [\Delta x(t)]^{-2} \quad (4.17)$$

Very interesting and ungodly. That's the reason why you eventually want to change to an operator formalism.

Let me note a few things. We've got the Gaussian shape and the width oscillates in time, just like we said it would. The momentum phase, in the second exponential of (4.16), follows the classical motion. The third exponential is technically why the packet does not remain a minimum-uncertainty state in time. The last piece is, for our purposes, an irrelevant phase factor, but to show all the people I actually did it, there it is.

Using $\psi_{ss}(x,t)$ we can now calculate $\langle x(t) \rangle$ in the opposite way. We can calculate $\langle \psi_{ss}(x,t) | x | \psi_{ss}(x,t) \rangle$. Believe it or not, it is the same. $\langle x(t) \rangle$ comes out to be what we had before. Also, $\langle x^2(t) \rangle$, $\langle p(t) \rangle$, $\langle p^2(t) \rangle$, $\Delta x(t)$, and $\Delta p(t)$ are all as before. So, indeed everything works.

SOME PROPERTIES OF SQUEEZED STATES

What we have now is a physical understanding of squeezed states. We start at $t=0$ with a displaced Gaussian with a squeezed shape. It will follow the classical motion with time, oscillating back and forth. But, what does its shape explicitly do?

Let's start it at $x=0$. In Fig. 1, the little ellipses represent uncertainties in x and p space. They are called uncertainty ellipses. What happens is, the wave packet follows the large circle with angular frequency ω , but the shape oscillates with angular frequency 2ω .

People can get the impression from some of the literature that the ellipse just rotates along like a frankfurter. It doesn't do that. It's a quadrupole motion. At 45 degrees it is a circle. The area, $A(t)$, of this circle at $\omega t = \pi/4$ is bigger than the area of the ellipse at $\omega t = 0$, since

$$A(t) = \pi \Delta x(t) \Delta p(t) \quad (5.1)$$

Then at $\omega t = \pi/2$ it becomes the same shaped ellipse, only it's rotated. Note that when it's gone one-half period, it's got the same shape and orientation as it had before. That means that the uncertainty ellipse is oscillating with twice the angular frequency. In particular, a wave packet at $t=0$ will have the same uncertainties one-half period later.

Since $t=0$ is really a phase, any Gaussian with the types of phases proportional to x and x^2 in Eq. (4.16) is a squeezed state away from minimum uncertainty; but it will rotate into minimum uncertainty in time. [Those phases prevent the Fourier transform of $\psi(x,t)$ yielding a minimum-uncertainty $\psi(p,t)$ for all t .]

Figure 1. The motion and shape with time of a squeezed-state uncertainty ellipse.

So, the general squeezed state in time is not a minimum-uncertainty state, but a Gaussian state which will evolve into a minimum-uncertainty state. People often like to say that a coherent state is a Gaussian. Stop them! A coherent state is a Gaussian in a harmonic oscillator which at $t=0$ has the width of the ground state and a phase proportional to x . But a coherent state is not a Gaussian, per se!

A couple of final points, since I am reaching the end of my allotted time.

If I define the Hamiltonian $\mathcal{H}$ as being the normal Hamiltonian H divided by $\hbar\omega$, then for squeezed states

$$\langle \mathcal{H} \rangle_{ss} = |\alpha|^2 + \frac{1 + s^4}{4s^2}, \quad (5.2)$$

$$\langle \mathcal{H}^2 \rangle_{ss} = \langle N \rangle_{ss}^2 + \alpha_1^2 s^2 + \frac{\alpha_2^2}{s^2} + \frac{(1 - s^4)^2}{8s^4}, \quad (5.3)$$

$$[\Delta \mathcal{H}]^2 = \alpha_1^2 s^2 + \frac{\alpha_2^2}{s^2} + \frac{(1 - s^4)^2}{8s^4}. \quad (5.4)$$

This reduces to the harmonic oscillator coherent state results when $s=1$.

One last thing. Last night as I was putting my head on my pillow, it dawned on me that the free Gaussian should fit into this. It occurred to me that a free Gaussian must be a squeezed state, in the limit of $\omega \rightarrow 0$ harmonic oscillator coupling. So, I quickly got out my copy of Schiff.¹⁶ If you look at Schiff, one has for $\langle x \rangle = \langle p \rangle = 0$ that the time evolution of the free Gaussian has the form

$$\psi^*(t)\psi(t) = [2\pi Q^2(t)]^{-1/2} \exp \left[-\frac{x^2}{2Q^2(t)} \right], \quad (5.5)$$

$$Q^2(t) = Q^2(0) + \frac{\hbar^2 t^2}{4m^2 Q^2(0)}. \quad (5.6)$$

But this is the functional form of $\psi_{ss}^*(t)\psi_{ss}(t)$, with $\Delta x(t)$ replacing $Q(t)$. Then, since from (4.9)

$$\lim_{\substack{\omega, s \rightarrow 0 \\ (\omega/s^2) \rightarrow \Omega}} [\Delta x(t)]^2 = \frac{\hbar}{2m\Omega} [1 + \Omega^2 t^2 + 0 \dots], \quad (5.7)$$

taking

$$\Omega = \hbar/[2mQ^2(0)], \quad (5.8)$$

gives us that you can think of free Gaussian as being a special case of a squeezed state. It spreads out, but it never gets anywhere close to the limit where it would turn around and start contracting again. It's always falling apart.

So, with that being said, let me summarize. A coherent state at $t=0$ is a Gaussian in a harmonic oscillator, with the width of the ground state. It has a phase proportional to x and the expectation values of x and p correspond to its starting position and momentum in a harmonic oscillator. It oscillates back and forth following the classical motion in time and it keeps its proper shape forever.

A squeezed state is a slight relaxation. The relaxation is simply that you allow the width of the Gaussian to be different from that of the ground state. It, too, can be displaced and the wave packet will follow the classical motion in time.

The important difference is that the squeezed states' uncertainty will oscillate back and forth. The uncertainty will oscillate between an ellipse with the small width in the x direction, to one with the small width in the p direction. That aspect of this Schrödinger formalism is, when we translate it into the operator formalism, a very useful tool. It is what makes squeezed states important in the quantum theory of measurement for gravitational wave detection and for quantum optics.

OK. That's all I'm going to talk about in the Schrödinger representation. I will soon translate this to the operator formalism and show the connection. When we discuss the beautiful, if obtuse, operators, think back to the wave function if you are getting lost. The key will always be the parameter s , which tells you how squeezed or how unsqueezed the Gaussian is.

HARMONIC MOTION, COHERENT STATES AND SQUEEZED STATES FOR GENERAL POTENTIALS

What I am going to start with today will be modified from what I had originally intended. That's the result of the questions which I received yesterday, both here and at lunch. As we have observed, a lot of the interest in coherent states and squeezed states comes from the fact that these states follow the classical motion in a harmonic oscillator. Now it's known by everybody that the harmonic oscillator has very special properties. One is that the classical period of oscillation is independent of energy. Another is that the quantum eigenvalues are equally spaced. So, the question arises if all the special properties characteristic of coherent and squeezed states are due to these properties of the harmonic oscillator. The answer is, "It helps, but it's not entirely due to this." This answer can be understood from a discussion of what I call "classical and quantum harmonic potentials "

The classical period of oscillation in a potential $V(x)$ is given by the integral

$$\tau_c(E) = 2\pi/\omega_c(E) = (2m)^{1/2} \int_{x_L}^{x_R} \frac{dx}{[E - V(x)]^{1/2}} \quad (6.1)$$

If $\omega_c(E)$ is a constant independent of energy, I call V "a classical harmonic potential." For a given ω_c , there are an uncountable number of potentials which satisfy that condition; as a matter of principle. The simple harmonic oscillator with $\omega = \omega_c$ is one of them, so the simple harmonic oscillator is a classical harmonic potential.

Supposing, I now look at the quantum eigenvalue problem for a potential, V . If the energy eigenvalues are equally spaced, proportional to a given constant, $\omega_q = (E_{n+1} - E_n)/\hbar$, I will call V a "quantum harmonic potential." Again the simple harmonic oscillator is one. The reason I call such a potential "harmonic" is because after every period $\tau = 2\pi/\omega_q$ the wave packet will always reconstruct itself as we discussed around Eq. (4.12). Again there are an uncountable number of potentials which satisfy the condition for a given ω_q .

For a given $\omega_c = \omega_q = \omega_0$ are all of the classical harmonic potentials also quantum harmonic potentials, and the reverse? The answer is,¹⁷ "No." For a given ω_0 there are three sets. There are classical harmonic potentials that are not quantum harmonic potentials, quantum harmonic potentials which are not classical harmonic potentials, and potentials which are both; and all three sets are uncountable.¹⁷

Let me give you an example of a quantum harmonic potential which is not a classical harmonic potential. I came across this particular potential by luck at the right time, when I first became interested in the question. (You've got to be good or you've got to be lucky, one of the two.) We had just finished the work on coherent states for general potentials where we had concentrated on classical motion, the harmonic oscillator being the most classical of all.

The potential was sent to me in a paper by Abraham and Moses;¹⁸ and as I always say, "How can you not believe a paper by Abraham and Moses?" They were looking at the inverse method of Gel'fand and Levitan. This method allows you to do the following: If you have a potential which, as

a matter of principle, is solvable with a complete set of eigenvectors and associated eigenvalues, you can generate a new potential which has the same spectrum, but with any particular eigenvalue and eigenvector removed. In particular, if you give me the harmonic oscillator, I can generate a potential with the ground state removed. The eigenvalues are still equally spaced. So, it's still a quantum harmonic potential, but you can show that the new potential does not have a period of oscillation which is independent of energy.

How you actually construct this is another talk. I just give you the (dimensionless) results that the potential

$$v = \frac{1}{2} y^2 + \phi [\phi - 2y] \quad , \quad (6.2)$$

$$\phi = \frac{\exp[-y^2]}{\int_y^\infty dt \exp[-t^2]} \quad , \quad (6.3)$$

has the orthonormal eigenvectors

$$\chi_n(y) = \psi_n(y) - \left(\frac{2}{n}\right)^{1/2} \phi(y) \psi_{n-1}(y) \quad , \quad (6.4)$$

with eigenvalues $\varepsilon_n = (n + \frac{1}{2})$, $n = 1, 2, \dots$, where $\psi_n(y)$ are the harmonic oscillator eigenfunctions of Eq. (2.3).

If you now put Eq. (6.2) into our integral Eq. (6.1), you find that the classical period of oscillation is a slightly varying function of E .¹⁷ [$\omega_c(E)$ never differs from ω_q by more than 7%.] Thus, V is not a classical harmonic potential but it still is a quantum harmonic potential. And you can think of an uncountable number of these things.

The reversed case is even more easily seen,¹⁹ as I mentioned at lunch yesterday. Take two half-harmonic oscillators of different width placed together. Since the harmonic oscillator classical period of oscillation is independent of energy, a particle in one side of the half-harmonic oscillator has a period independent of energy, and also when it is on the other side. Thus, it's independent of energy over both sides. However, if you think of the quantum problem, you have a parabolic cylinder function for an eigenvector on one side, which is trying to match with a different parabolic cylinder function on the other side. Matching their slopes and values at the origin, gives a solution that is not a parabolic cylinder function equal to a Hermite polynomial. This means that the eigenvalues are not equally spaced.

To summarize, the special features of the simple harmonic oscillator are partially due to its "quantum" and "classical" harmonic nature, but not entirely. It is not only the simple harmonic oscillator which can have ω_c and $(E_{n+1} - E_n)/h = \omega_q$ equal to a constant.

Another question was raised by Marlan, about if there are "squeezed states for general potentials." The answer is in the papers we did on the coherent states for general potentials.²⁰ Let me first tell you what that program was. Then I can say, "Yes, there are squeezed states for general potentials, and they're already written down in our papers; one step from the end." Our program was to find what we called "minimum-uncertainty coherent states" (MUCS).

Give me a potential. Consider the classical problem. I can find a transformation from x and p to what I call X_c and P_c (which are functions of x and p) such that they vary as $\sin[\omega_c(E)t]$ and $\cos[\omega_c(E)t]$. What that amounts to is changing variables so that the classical Hamiltonian looks like the harmonic oscillator Hamiltonian, $H \sim \frac{1}{2}X_c^2 + \frac{1}{2}P_c^2$. It's switching variables so that the classical phase-space path is a circle in the new variables, X_c and P_c .

Now I change these classical variables into quantum operators, X and P . $[X, P] = i\hbar$ defines the uncertainty relation $(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\hbar^2$ of Eq. (2.11). Find those states, which we call "minimum-uncertainty states," (MUS) which satisfy the equality in the uncertainty relation. This is a three parameter set, complex C and real B . Our last step was to set B such that the ground state is a member of that set. Those are our "minimum-uncertainty coherent states" for general potential.

The squeezed states are one step back. Those are what I called "minimum-uncertainty states." They are an arbitrary B-shape, such that the ground-state "B" is a special case.

That's exactly the same thing as for the harmonic oscillator. There, for squeezed states you've got a general Gaussian that is a "minimum-uncertainty state." Then you set the parameter so that it has the width of the ground state to make a coherent state. So, to answer Marlan's question again, "Yes, a procedure can be carried out to obtain squeezed states, for general potentials. We've already done it, except that we were interested in the last step. The squeezed states are one step back." You can look in our papers²⁰ and see that it can all be done analytically for solvable potentials.

SQUEEZED STATES IN THE OPERATOR FORMALISM

Recall what I have emphasized, that the coherent states are simply Gaussians displaced from the origin with the width of the ground state and a complex phase. The squeezed state is the same Gaussian, but with a width different from the ground state, labeled by $s \neq 1$, s being what I call the "squeeze parameter." That's all there was to it.

Remember that folks, because now we're going to go into the operator formalism. As I said, it's a lot easier to calculate with, but much less transparent in a physical sense than wave functions are.

In Sec. III we recalled that in the operator formalism the coherent state $|\alpha\rangle$ can be generated by displacing the ground state:

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad (7.1)$$

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] \quad (7.2)$$

$$= \exp[-\frac{1}{2}|\alpha|^2] \exp[\alpha a^\dagger] \exp[-\alpha^* a] \quad (7.3)$$

What I've done is to take the ground-state Gaussian and displace it. But how do I "squeeze" the ground-state Gaussian? In other words, what is the "squeeze operator?" It is this beastie:

$$S(z) = \exp[\frac{1}{2}z a^{\dagger 2} - \frac{1}{2}z^* a a^2] \quad (7.4)$$

with z the complex parameter

$$z = re^{i\theta}, \quad \theta = 2\omega t_0, \quad (7.5)$$

$$r \equiv \ln s. \quad (7.6)$$

θ amounts to an initial time, t_0 . If θ is not 0, it means you've started at some $t \neq 0$. r is equal to the logarithm of s , the squeeze factor which is in the Gaussian. That's the connection between the operator formalism and the Schrödinger formalism, and it makes sense. If $s=1$, then $r=0$ so $z=0$, so there's no squeeze.

A squeezed state can be produced one of two ways. The first way, which is analogous to Eq. (4.1), is to squeeze and then displace:

$$\psi_{ss} \rightarrow |\alpha, z\rangle = D(\alpha)S(z)|0\rangle. \quad (7.7)$$

That's the way I tend to do it because I tend to think more in terms of gravitational waves than in terms of quantum optics and also because this gives you the Schrödinger wave function (4.1) in terms of the parameters α and s . But you can equally as well do it the opposite way. The quantum optics people usually like to do it that way, a point I'll come to later.

The next question is, what is $S(z)$ in normal-ordered form? That's an important question. Well, there are two ways you can do it. One is to use something which is called McCoy's Theorem, which allows you to write an exponential of a general second-order polynomial in a and $a^\dagger$, as a normal-ordered object. (You put $S(z)$ in the formula and you get out the right answer.) McCoy was a very smart fella. But the method is obscure.

It is much more fun to do it by matrices,²¹ and that's what I'm going to do right now. First note the following identification with the algebra $SU(1,1)$:

$$\frac{1}{2}a^\dagger a^\dagger \equiv L_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (7.8)$$

$$-\frac{1}{2}aa \equiv L_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad (7.9)$$

$$\frac{1}{2}a^\dagger a + \frac{1}{2} \equiv L_3 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}. \quad (7.10)$$

These matrices and these operators satisfy the $SU(1,1)$ commutation relations

$$[L_+, L_-] = 2L_3, \quad [L_3, L_\pm] = \pm L_\pm. \quad (7.11)$$

Therefore, the squeeze operator can be written as

$$S(z) = \exp \begin{bmatrix} 0 & z \\ z^* & 0 \end{bmatrix} = I \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!} + \begin{bmatrix} 0 & z \\ z^* & 0 \end{bmatrix} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n+1)!}$$

$$\begin{aligned}
&= \begin{bmatrix} \cosh r & e^{i\theta} \sinh r \\ e^{-i\theta} \sinh r & \cosh r \end{bmatrix} \\
&= \begin{bmatrix} 1 & e^{-i\theta} \tanh r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\cosh r & 0 \\ 0 & \cosh r \end{bmatrix} \begin{bmatrix} 1 & 1 \\ e^{-i\theta} \tanh r & 1 \end{bmatrix}.
\end{aligned} \tag{7.12}$$

Now you work backwards and write these matrices as exponents of matrices giving

$$S(z) = \exp[e^{i\theta}(\tanh r) L_+] \exp[-2 \ln(\cosh r) L_3] \exp[e^{-i\theta}(\tanh r) L_-]. \tag{7.13}$$

Finally, by using the Baker-Campbell-Hausdorff-type formula

$$\exp(xa^\dagger a) = \sum_{n=0}^{\infty} \frac{(e^x - 1)^n}{n!} (a^\dagger)^n (a)^n \tag{7.14}$$

on the middle exponential in Eq. (7.13), we get

$$\begin{aligned}
S(z) &= \exp[\tfrac{1}{2}e^{i\theta}(\tanh r) a^\dagger a^\dagger] (\cosh r)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(\operatorname{sech} r - 1)^n}{n!} (a^\dagger)^n (a)^n \\
&\quad \times \exp[-\tfrac{1}{2}e^{i\theta}(\tanh r) aa] .
\end{aligned} \tag{7.15}$$

Note that in Eq. (7.15), even though you have cross terms, effectively you're dealing with pairs of a 's and $(a^\dagger)$'s. That's where the concept of "two-photon coherent states" comes in, especially when you take expectation values. The quantum optics people like to think of squeezed states as two-photon coherent states; two excitations.

MATHEMATICAL PROPERTIES OF SQUEEZED STATES

Another property of squeezed states comes from this well known theorem:

$$e^{\zeta A} B e^{-\zeta A} = B + \zeta[A, B] + \frac{\zeta^2}{2!} [A, [A, B]] + \dots \tag{8.1}$$

Using this theorem and summing the infinite series you can obtain

$$b^\dagger = S^\dagger a^\dagger S = (\cosh r) a^\dagger + e^{-i\theta}(\sinh r) a, \tag{8.2}$$

$$b = S^\dagger a S = (\cosh r) a + e^{i\theta}(\sinh r) a^\dagger, \tag{8.3}$$

$$[b, b^\dagger] = 1. \tag{8.4}$$

This is just the Bogolubov transformation² discussed many years ago for superconductivity and superfluidity.

But this is also one of the many new discoveries of the squeezed states, called twisted states.¹¹ They are the same thing. Why? You look for the eigenvector of

$$b = \mu a + \nu a^\dagger, \quad |\mu|^2 - |\nu|^2 = 1. \quad (8.5)$$

Well, $[\cosh^2 r - \sinh^2 r]$ is equal to one. Basically, this result was realized in Stoler's papers back around 1970.² (A lot of this stuff that I'm talking about right now, Vern Sandberg and I are going to be writing up.)

Finally, from the same theorem, we have two further results. Any particular function of a and $a^\dagger$, when you surround it by squeeze operators, is equal to the same operator, only as a function of b and $b^\dagger$:

$$S^{-1}(z) f(a, a^\dagger) S(z) = f(b, b^\dagger). \quad (8.6)$$

This means that D does not commute with S , but, rather that

$$D(\alpha) S(z) = S(z) D(\gamma), \quad (8.7)$$

$$\gamma = \alpha \cosh r - \alpha^* e^{i\theta} \sinh r. \quad (8.8)$$

So, it is not fundamentally important whether you prefer the gravitational wave point of view, where you apply first S then D , or whether you like to do it the quantum optics way where you first displace and then you squeeze. The two views are mathematically related by a complex scale transformation. You have to be conscious and careful about it, and which you apply first. But if you are, there is no fundamental problems.

Going on, we've really got three parameters α_1 , α_2 , and $|z| = r$, with θ corresponding only to an initial time. Now we've learned the game. Take an exponent of a and $a^\dagger$, and that displaces. Take an exponent of (aa) and $(a^\dagger a^\dagger)$, and that squeezes. So what about exponents of $(a)^3$ and $(a^\dagger)^3$, or $(a)^n$ and $(a^\dagger)^n$, for that matter? Well, let me define a k -photon unitary operator,

$$\begin{aligned} U_k(z_k) &\equiv \exp[z_k (a^\dagger)^k - z_k^* (a)^k] \\ &= \exp[iA_k]. \end{aligned} \quad (8.9)$$

U_1 is D , U_2 is S . What about U_3 , U_4 , U_5 , etc.? It turns out that they don't exist.²¹

You can show that very simply. Take the ground-state expectation value of U_k . Expand U_k and collect all the terms. It's very straightforward and looks something like this:

$$\begin{aligned} \langle 0|U_k|0\rangle &= 1 - |z_k|^2 \frac{k!}{2!} + |z_k|^4 \frac{1}{4!} [(k!)^2 + (2k)!] \\ &\quad - |z_k|^6 \frac{1}{6!} \left\{ (k!)^3 + 2k!(2k)! + \frac{1}{k!} [(2k)!]^2 + (3k)! \right\} \\ &\quad + \cdots + (-1)^n |z_k|^{2n} \frac{1}{(2n)!} C_n + \cdots \end{aligned} \quad (8.10)$$

C_n has many terms, all of which are greater than 0. The largest is of order $(kn)!$ as a matter of fact, all of them are of this order. That means that

$$\lim_{n \rightarrow \infty} |z_k|^{2n} \frac{1}{(2n)!} C_n \neq 0 \quad (8.11)$$

for all $k > 2$ and $z_k \neq 0$. Therefore, the terms of this series do not decrease monotonically. The terms are bounded below by $(kn)!/(2n)!$ and so, by the ratio test, there is no convergent limit as $n \rightarrow \infty$. The series is divergent and so $\langle 0 | U_k(z_k) | 0 \rangle$ is undefined by the series for $k > 2$.

If you want to put in some fancy language, and I thank John Klauder for emphasizing to me that you really should put it in fancy language, consult footnotes 4 and 20 of Ref. 21. It means the following: Even though A_k is Hermitian, so U_k is unitary, one is assuming that the ground state is an analytic vector of the generator A_k . It is not! In other words, $A_{k>2}$ is not self-adjoint in the harmonic oscillator basis.

This is interesting. This is one place where the difference between self-adjoint and Hermiticity makes a difference. Here we have a potential which is proportional to x^2 . So for large $|x|$ the wave function dies out as $\exp[-x^2]$. That means that D , which is an exponent of x , just moves the center of the wave function. S , which is an exponent of x^2 , changes the shape. But $U_{k>2}$, which is an exponent of x^k , can make the wave function blow up. This is an operator statement. It is saying that the expectation value of the operator expansion (of say $\exp[-x^4]$) does not converge. In general, if you have a potential which rises as $|x|^n$, then the wave function dies out as $\exp[-|x|^{(n/2)+1}]$, so you could go up to $U_{k \geq [n/2]+1}$.

Note that it was rather critical that when I talked about the ratio test, it gave you no answer for $k=2$. Remember the series summed as $(\tanh r) < 1$ for $r < \infty$. From this point of view, the displacement operator converges well, the squeeze operator just barely makes it, and higher orders just blow up. The odd-ball thing is that if you now go to the harmonic oscillator with para-bose statistics, even the squeeze operator diverges.²² That's a cute little result.

THE PHYSICS OF SQUEEZED STATES

Let's begin with a little bit of physics as to why there is a usefulness in the differing viewpoints of quantum optics and gravitational waves. First, let's look at $SD|0\rangle$. Applying the displacement operator can be thought of as filling an optical cavity. Then applying the squeeze operator can be considered as processing the light trying to make anti-bunched light, which is a low-noise object. Anti-bunching is basically a quantum-optics phenomena. Bunching can be "classical" in concept. [Remember the Hanbury-Brown Twiss experiment? That was a "classical" experiment. They had these two dishes looking at a faint object in the middle of the Australian desert. If they got a photon in one dish, they tended to get a photon in the other dish.] At this Institute, Leuchs²³ will describe anti-bunching experiments, and why he feels objections to these experiments are not valid.

Something that I know hasn't been done is to detect gravitational waves. Gravitational wave people want to apply S first and then D . They

want to take a Weber bar and perform a quantum non-demolition experiment, starting with a squeeze of the ground state. They want to read out the bar, measuring what amounts to a combination of position and momentum. Then a highly classical gravity wave comes along and displaces the bar, which is D. Without the squeeze, the noise would make the measurement impossible.

When the gravitational people at Caltech were thinking of this process, they wanted to convince their great man, Feynman, that this really is the correct way of looking at things. Of course Feynman always likes analogies to classical physical. So they thought of this one: Consider a pendulum. Since it is a harmonic oscillator, it will always leave and come back to the center in half a classical period, no matter what it's energy (or momentum). Let's say at $t=0$ the thing is standing still and I take a hammer, and I go whack! One half of a period later I go whack again. Now, I don't care how hard I hit it, all I care is that I went whack half a period later and the pendulum was there. But one time I go whack and the pendulum is not there. That means something's happened to it while it was flying around.

As a matter of principle, that's how they want to detect gravitational radiation. They want to monitor a time-dependent operator of position and momentum (which I'll come to) in very precise time steps. If one time the measurement does not give what it's supposed to, that means a gravitational wave has passed. They've got to do this at such a low noise level because the signal is so weak, as we discuss below.

Let me close by making some final comments on gravitational wave detection. (For a good understanding you should really see the article by Caves, Thorn, Drever, Sandberg, and Zimmerman in RMP.²⁴) Gravitational wave people consider a couple of explicit time-dependent operators:

$$X_1 \equiv x \cos \omega t - \frac{p}{m\omega} \sin \omega t \quad , \quad (9.1)$$

$$X_2 \equiv x \sin \omega t + \frac{p}{m\omega} \sin \omega t \quad , \quad (9.2)$$

$$[X_1, X_2] = \frac{i\hbar}{m\omega} \quad . \quad (9.3)$$

Practically, they propose that one should measure x , say with a transducer modulated by the bar's classical ω , and simultaneously measure p , say with another transducer modulated by the classical ω but with the phase 90 degrees ahead of the first. They want to couple, mix, and amplify the two signals. What they want to do is to measure X_1 . In principle, these measurements can be made arbitrarily quick. (People have also discussed Gedanken experiments.)

As a matter of principle, people believe they can do it, but they have not done it yet. The real nitty-gritty details are very involved, and I refer you to Ref. 24. When I read it, I say I understand it. Two weeks later I ask myself, "What-in-the-hell did I read?" But at least I think I understand the principle.

Now notice I said what they are trying to do is to measure X_1 . But remember,

$$\begin{aligned}
 x(t) &= e^{iHt/\hbar} x e^{-iHt/\hbar} \\
 &= x \cos \omega t + \frac{p}{m\omega} \sin \omega t
 \end{aligned}
 \tag{9.4}$$

That means that X_1 , which they're trying to measure, is $x(-t)$. So

$$\begin{aligned}
 X_1(t) &= e^{iHt/\hbar} X_1 e^{-iHt/\hbar} \\
 &= x
 \end{aligned}
 \tag{9.5}$$

They have rotated to a useful coordinate system.

Remember I was talking about the rotating uncertainty ellipse? (See Fig. 1.) $\langle x(t) \rangle$ is rotating clockwise. Of course, you'll make the "starting point" very narrow. In $\langle X_1 \rangle = \langle x(-t) \rangle$ the time is going counterclockwise. But once you put the implicit time dependence in, $\langle X_1(t) \rangle$, you add a clockwise rotation, so you "stand still" in time. So, you're always measuring something with the narrow width on the right. But you've got to do that by coupling these transducers. That's the trick.

If you can do this in practice, then you'll have a real chance of measuring gravitational waves. They haven't done it yet, not by a long shot. A sensitivity to a relative change in length of the bar of $\Delta L/L \sim 10^{-15}$ is claimed. They've got a way to go yet, but that's the idea.

Thank you very much.

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