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2-D AXISYMMETRIC LINE TRANSPORT*

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ABSTRACT. The methods used in the ALTAIR code for computing the transfer of spectral line radiation in two-dimensional axially-symmetric geometry are described. ALTAIR uses a variable-Eddington-tensor approach, in which the transfer equation of non-coherent line scattering is written in moment form, and the moments are closed with an assumed tensor relating the monochromatic pressure tensor and energy density; this Eddington tensor is obtained self-consistently using an accurate angle-dependent solution of the transfer equation. The finite element method for solving the moment system, and the discontinuous finite element method for solving the S_n equation of transfer are described. Two applications of the method are discussed: line formation in uniform cylinders with different length-diameter ratios, and monochromatic transfer on an irregular x - y mesh (the Mordant test problem).

1. Introduction

This paper is concerned with radiative transfer in axially-symmetric geometry, *i.e.*, cylindrical geometry, and in particular with spectral line transfer in that geometry. It should be said at the outset that 2-D transfer calculations are never really necessary if rough answers are satisfactory, since real problems are most often longer than they are wide, or *vice versa*, and a 1-D approximation is likely to be within a factor two of the correct answer. Furthermore, 2-D problems that are optically thick can be treated with the diffusion approximation, which entails merely the solution of an elliptic partial differential equation for the mean intensity, a rather easy thing to do; the solution in the diffusion approximation is again usually within a factor two of the correct answer. This paper is concerned with cases in which factor-of-two accuracy is not good enough, and which have important regions or frequency bands of both large and small optical depth. It will describe the 2-D version of the code ALTAIR, the 1-D version of which was mentioned earlier in this workshop. The radiative transfer techniques in the two codes are rather different, as befits the different dimensionalities.

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The methods for solving line transfer problems in 2-D ALTAIR will seem, at first sight, to be rather cumbersome, and a consideration of their operation counts will show them to be fairly expensive, so a perfectly reasonable question would be: Why not use one of the simpler algorithms in the literature, reduce the numbers of angles and spatial zones to a modest level, and get moderately accurate calculations at a small fraction of the cost? The answer is that of all possible 2-D transfer problems, the ones that can be solved with moderate accuracy with just a few angles and a modest number of zones are quite exceptional. These are problems with especially smooth spatial structure, or a high degree of symmetry. Consider a "generic" 2-D transfer problem. The spatial structure may be described with a 100×100 mesh, such as may have been used for a numerical hydrodynamics calculation; on that mesh there may be some number of emitting or absorbing features that are a few zones wide. If these features have themselves an appreciable opacity, but the medium between them is translucent, then, as viewed from a point on the far side of the problem, the radiation field will have a feature in its angular structure that is a few degrees wide. If there is no special symmetry, the resolution in angle space need to describe this feature will demand several hundred angles in order to cover the required full hemisphere. That is, a few million values of the intensity are needed to specify the radiation field for a single frequency. Clearly, problems that are well described with a few angles are lucky ones indeed!

This difficulty is illustrated by the example of an opaque, bright ball shining into the transparent medium around it. If the overall region is 40 units in radius, and the ball is two units wide, then at some points on the outside boundary of the problem the intensity as computed by a reasonable algorithm with a total of 48 angles and a 40×40 spatial mesh may be up to 10^3 times too low! With 420 angles, which give a resolution of 6° , the error on the outer boundary is reduced to 40%; this calculation is certainly not over-resolved, yet it uses 706,000 intensity values.

The lesson to be drawn from this discussion is the following: The alternatives in 2-D transfer calculations are (1) lucky geometry, (2) inaccurate results, or (3) large angle sets. The method to be described shortly is designed to make the use of fairly large spatial meshes and large angle sets practical.

The method uses approximate-operator iteration (AOI, also referred to as ALI, for accelerated lambda iteration) in a particular formulation called the double splitting iteration. The lambda operation that is part of this procedure, that is, the formal solution of the equation of transfer with a known source function, is performed with the first two angle moments of the transfer equation. The moment system is closed by taking specified values of the Eddington tensor, the ratio of the monochromatic pressure tensor to the energy density. An outer iteration updates the tensor values by performing a full angle-dependent solution of the transfer equation over a fixed large set of angles. A few (≈ 6 in a representative case) tensor updates are sufficient, and for each of these some number of AOI iterations is done, that is initially about 30, but decreases as the tensor values converge, so that about 80 AOI iterations are needed in all. Thus 80 formal solutions of the moment equations are done, but only 6 solutions of the angle-dependent transfer equation.

The moment equations represent an elliptic PDE for a single variable at each zone and frequency, in place of the hundreds of intensity values. The matrix representation of the elliptic system needs to be computed and *LU*-factored only when the tensors are updated; each AOI iteration then involves just a backsolve of the elliptic system and a few other rapid operations. The timing results show that a fraction of order 90% of the total time is used for

the angle-dependent transfer, with the bulk of the remainder going for the LU -factorization of the elliptic system. As an alternative to the use of the variable-tensor method, one could apply AOI directly to the angle-dependent transfer. In this case the full angle solution would have to be done about 30 times, instead of only 6 times, assuming the AOI iteration count of 30 still obtained. The total time would increase by 4–5-fold, owing largely to the increased number of angle-dependent solutions. In time-dependent problems the Eddington tensor from the previous time step may be a sufficiently accurate approximation of the self-consistent one that only one tensor evaluation is needed each time step; in this case the variable tensor method would allow 15–30 $\times$ fewer angle-dependent solutions.

The use of the moment equations closed by a self-consistent Eddington factor/tensor was introduced to neutron transport theory by Gol'din (1964), who referred to this method as "quasi-diffusion," and later applied to astrophysical problems by Auer and Mihalas (1970). The variable Eddington factor method is closely related to the *diffusion synthetic acceleration* (DSA) method applied in neutron transport theory. DSA has been reviewed by Larsen (1984), who also shows that the asymptotic convergence ratio is ≤ 0.2247 , a property that seems to apply to the variable Eddington factor method as well.

The algorithms to be described below are based on a spatial mesh that consists of general quadrilaterals, arranged in a logically rectangular fashion, so that each quad carries indices k and ℓ , and interior quads have four neighbors with indices $k \pm 1$ and $\ell \pm 1$. The choice of a quadrilateral mesh rather than a rectangular one brings penalties in the form of a modest increase of computation time, and an appreciable increase in algorithmic complexity. What is gained is the ease of accomodating curvilinear boundaries and curvilinear interfaces between different materials.

The following three sections of this paper describe (1) the formulation of AOI that is used to solve line transfer problems, (2) the continuous finite-element technique for solving the moment equations for the radiation field, and (3) the discontinuous finite-element method for solving the angle-dependent equation of transfer with a fixed (S_n) set of ray directions. The final section gives some illustrations of the code results for axially-symmetric and 2-D x - y problems. A full discussion of the methods in 2-D ALTAIR will be found in Dykema, *et al.* (1990).

2. Double-splitting Method for Line Scattering

The standard problem of line transfer is the so-called two-level-atom problem, which reduces to the solution of a transfer equation

$$\mathbf{M}_\nu J_\nu = S, \quad (1)$$

together with the definition of S , the complete-redistribution (CRD) source function:

$$S = (1 - \epsilon) \int d\nu \phi_\nu J_\nu + \epsilon B. \quad (2)$$

Equation (1) represents the transfer equation in differential form; the operator $\mathbf{M}_\nu$ might be the second-order partial differential operator obtained from the moment equations to be discussed in §3, below. If the assumption of complete redistribution is replaced by the one of partial redistribution (PRD), the source function, S , in equations (1) and (2) becomes frequency dependent, but the following discussion proceeds with minor modifications.

The double splitting iteration method was introduced by Klein, *et al.* (1989). The iteration proceeds as follows, in outline. A full iteration cycle consists of split 1, followed by split 2. This cycle is repeated, as needed, to convergence. Split 1 is exactly lambda iteration, so the current S is used in equation (1) to obtain J_ν . The new J_ν could be used in equation (2) to update S , but this is not needed. For the second split, M_ν in equation (1) is replaced by its spatial diagonal on the left side, and the off-diagonal part is transferred to the right side, where it operates on the J_ν resulting from split 1; this equation is solved simultaneously with equation (2). This is simple to do, since all the operations are local. This is similar to a pre-conditioning or acceleration step with a diagonal operator, but in this case the diagonal-offdiagonal split refers to the differential operator M_ν itself, not to the lambda operator that would be found by eliminating J_ν between equations (1) and (2).

The double-splitting iteration is represented as follows. Split 1:

$$M_\nu J_\nu^{n+1/2} = S^n, \quad (3)$$

and split 2:

$$\text{diag}(M_\nu) J_\nu^{n+1} = S^{n+1} - [M_\nu - \text{diag}(M_\nu)] J_\nu^{n+1/2} \quad (4)$$

$$S^{n+1} = (1 - \epsilon) \int d\nu \phi_\nu J_\nu^{n+1} + \epsilon B. \quad (5)$$

The formal solution of split 1 may be vectorized over frequency, or done in parallel over frequency, or both. The local operations of split 2 may be done in parallel over spatial zones. In the CRD case, as shown, the values of J_ν^{n+1} at different frequencies are coupled through the scalar, S^{n+1} , and the system of equations (4,5) can be solved in a number of operations, per zone, that is of the order of the number of frequencies, and therefore negligible compared with the cost of split 1. In the PRD case the frequency coupling is non-trivial, and the number of operations per zone rises to the order of the cube of the number of frequencies, which is none-the-less still usually smaller than the cost of split 1.

The iteration step as just described is followed by an application of the Orthomin acceleration technique to complete the cycle. (See Vinsome [1976], Klein *et al.* [1989] or Auer's lecture in this Workshop for details.) The convergence rate of the double-splitting method is comparable with that of the best of the other AOI methods, such as that of Olson, Auer and Buchler (1986). The number of iterations required to reach a certain relative accuracy, say 10^{-4} , depends weakly on the values of ϵ and the optical depth, while it increases with the number of zones across the problem. For a Doppler line profile the variation of iterations with zones is linear; for a Voigt profile the iteration count is less, and the variation is as the square root. The actual number of iterations ranges from 10 to 30 in typical cases. These results are also discussed by Klein, *et al.*

While the double-splitting method is much like other AOI methods in its general properties, it has some special advantages: (1) the absence of parameters that must be adjusted to balance stability against optimum convergence, (2) freedom from the necessity of estimating the diagonal elements of the matrix lambda operator, and (3) the ease of adaptation to PRD.

The next two sections of the paper discuss how equation (3) is solved on a quadrilateral r - z mesh.

3. Variable Eddington Tensor Moment Equations

The discussion of the radiation moment equations begins with the transfer equation, which, in a coordinate-free form, is

$$\Omega \cdot \nabla I = \chi S - \chi I. \quad (6)$$

All the complicating factors such as frequency dependence, the time derivative, and anisotropy of the opacity χ that might be due to a velocity field, will be omitted in order to simplify the discussion. The first two angle moments of equation (6) are

$$\nabla \cdot \mathbf{H} = \chi S - \chi J, \quad (7)$$

and

$$\nabla \cdot \mathbf{K} = -\chi \mathbf{H}. \quad (8)$$

The key assumption that is now made is that an Eddington tensor $\mathbf{T}$ has been supplied, so that the moment system (7,8) is closed by

$$\mathbf{K} = \mathbf{T}J. \quad (9)$$

The numerical differencing is easier to understand in relation to the separate equations (7) and (8), but the object is to solve the combined equation

$$\nabla \cdot \left[\frac{1}{\chi} \nabla \cdot (\mathbf{T}J) \right] = \chi J - \chi S \quad (10)$$

for the mean intensity, J . The Eddington tensor $\mathbf{T}$ is positive-definite in any real case, and therefore the partial differential equation (10) is elliptic. However, since $\mathbf{T}$ stands inside the inner divergence operator instead of between the two, the equation is not self-adjoint; this rules out the use of some of the best numerical methods for elliptic equations. Another difficulty is caused by the non-self-adjointness: In the limit $\chi \rightarrow 0$ there is a solution of equations (7)-(9) with a bounded $\mathbf{H}$ only when $\mathbf{T}$ obeys a certain condition; this makes the solution of optically-thin problems ill-conditioned with respect to perturbations of $\mathbf{T}$. This problem cannot be taken up here, but it is discussed by Dykema, *et al.*

The boundary condition required for the solution of equation (10) is a relation involving the normal component of $\mathbf{H}$ at every point on the boundary of the solution domain. The two natural forms of this relation are mirror symmetry, $H_n = 0$, and a vacuum or incident-radiation boundary, where H_n is a linear function of J with known coefficients. (One of the coefficients is related to the ratio of the first to the zeroth angle moments of the intensity emerging at the boundary; this, like $\mathbf{T}$, is an assumed quantity that must be iterated.)

The spatial mesh on which the moment system (7)-(9) is to be solved consists, as mentioned above, of quadrilaterals. The set of quadrilateral corners will be called the "A" nodes; this is to distinguish them from the "C" nodes, which are the centers of the quadrilaterals. For the solution of the moment system, the mesh is triangulated by connecting each "C" node to the four surrounding "A" nodes, thus dividing each quadrilateral into four triangles. It is required that the original quadrilaterals all be

convex (no "boomerangs" or "bowties"), so the "C" nodes indeed lie within the correct quadrilaterals.

The essence of the method used to solve the moment system is that it is a finite-element method, so J is supposed vary in some simple way within each spatial zone; here J will be assumed to be linear in r and z in each triangle. This is equivalent to a representation

$$J(r, z) = \sum_i J_i W_i(r, z), \quad (11)$$

where i runs over all the nodes and the basis function W_i is 1 at the i th node, sloping linearly to 0 at the neighboring nodes. Thus W_i is a tent-shaped function that is non-vanishing in the triangles that touch node i , in each of which it varies linearly, and which is continuous along the edges. The representation of H is simpler: it is assumed to be piecewise constant in each triangle; H is then necessarily discontinuous across the triangle edges. The absorption coefficient is assumed to be constant in each quadrilateral, and therefore also within each triangle.

The technique for finding equations for the $\{J_i\}$ is the Galerkin one (*cf.*, for example, Mitchell and Wait [1977]): The representations for the dependent variables are substituted into the differential equations, giving residual functions that, ideally, should vanish everywhere. The residual functions are projected onto each of the basis functions, and these projections are required to vanish, leading to as many equations as there are basis functions and unknown coefficients; these linear equations are the desired difference equations. In the present case the residual in equation (7) is projected on each of the $\{W_i\}$; the residual in equation (8) is projected on each of the H basis functions, which simply involves averaging equation (8) over each triangle. The latter operation leads easily to an expression for the value of H in a given triangle in terms of the J values at the surrounding nodes. Thus the fluxes can be eliminated from the projections of equation (7), which produces the desired differencing of equation (10).

An important step in forming a projection of equation (7) is an integration by parts when projecting the divergence term:

$$\iiint_R W_i \nabla \cdot H dV = \iint_{\partial R} W_i H_n dA - \iiint_R H \cdot \nabla W_i dV. \quad (12)$$

Here R is comprised of the volumes of revolution swept out by the triangles that touch node i . The surface integral vanishes, since $W_i = 0$, unless node i lies on the outer boundary of the problem and the portion of ∂R in question also lies on the outer boundary. In this case H_n either vanishes or may be expressed in terms of the J values at the nearby boundary nodes. The volume integral on the right side of equation (12) breaks up into integrals over the component swept-triangle parts of R , in each of which H is constant, so the integral is easily done.

Another important aspect of the procedure is that when the right side of equation (7), $\chi(S - J)$, is projected on W_i , the integral is modified by taking $S - J$ outside and giving it the value at node i . This procedure, called "mass-lumping", represents some sacrifice of formal accuracy in exchange for improved stability—freedom from "ringing"—in the optically-thick limit.

The description of solving the moment system for J is now complete, except for the way in which the necessary values of T and the coefficient(s) relating H_n to J are computed.

4. S_n Transport Equation

The Eddington tensor T and the boundary Eddington factor H_n^+/J^+ are computed from a solution of equation (6), which is performed after the AOI iterations of equations (1,2) have converged to an acceptable degree. The approach adopted in 2-D ALTAIR to solve equation (6) is the S_n one, in which a fixed set of ray directions is chosen for Ω , at each of which the intensity is given. It is the extension to 2-D of the method of discrete ordinates that is used a great deal in 1-D, slab geometry radiative transfer. In r - z geometry the r and z components of Ω have constant values, whereas the value of Ω_r must vary along a characteristic. Therefore the transfer equation includes an angle derivative term when expressed in this coordinate system. The vector $\Omega = (\Omega_r, \Omega_\phi, \Omega_z)$ can be separated into its poloidal part $\Omega_p = (\Omega_r, 0, \Omega_z) = (\sin \theta \cos \psi, 0, \cos \theta)$ and its toroidal part $\Omega_t = (0, \Omega_\phi, 0) = (0, \sin \theta \sin \psi, 0)$. The angle θ is the polar angle of the ray with respect to the z , or symmetry, axis, and ψ is the azimuthal angle of the ray with respect to the local radial direction. In the S_n method a fixed set of (θ, ψ) values is used. When Ω is decomposed in this way, the transfer equation becomes

$$\nabla \cdot (\Omega_p I) - \frac{\partial}{\partial \psi} \left(\frac{\Omega_t I}{r} \right) = \chi S - \chi I. \quad (13)$$

This conservative form of the equation lends itself well to the finite-element formulation. In contrast to the continuous finite element technique that works well for the moment system, a discontinuous finite element (DFE) method is needed for the first-order transfer equation; the correct diffusion limit is otherwise difficult to obtain. The DFE method for transport problems was developed by Mordant (1975, 1976, 1981), who in the last reference compares continuous and discontinuous FE and finite-difference methods.

The quadrilateral mesh described above is used as-is for the S_n transfer, rather than subdivided into triangles as it is for the moment solution. While it is possible to use the DFE method with linear elements on triangles, the number of unknowns would be triple the number for bilinear elements on quadrilaterals. Since storage for the large number of intensities is a major limiting factor in 2-D problems, ALTAIR chooses the second possibility. The bilinear element is not actually bilinear in r and z . Instead, a mapping is established between each quad and the unit square:

$$\mathbf{r} = (1 - \alpha)(1 - \beta)\mathbf{r}_1 + \alpha(1 - \beta)\mathbf{r}_2 + \alpha\beta\mathbf{r}_3 + (1 - \alpha)\beta\mathbf{r}_4, \quad (14)$$

where the $\{\mathbf{r}_i\}$ are the vertex coordinate vectors numbered in counter-clockwise order around the quad. This associates each point $\mathbf{r}$ in the quad with a point (α, β) in the unit square. The bilinear element for I is a similar expression:

$$I = (1 - \alpha)(1 - \beta)I_1 + \alpha(1 - \beta)I_2 + \alpha\beta I_3 + (1 - \alpha)\beta I_4, \quad (15)$$

where the subscripted intensities are the vertex values. Each vertex adjoins four quads, in general, and it is specifically *not* required that the vertex intensities be unique; instead, each vertex carries an intensity for each of the quads that touch it. Thus the intensity will, in general, be discontinuous across the edges of the quads. However, a rule is provided for resolving the discontinuity: At each edge an upwind and a downwind quad are defined so that Ω_p points from upwind to downwind across that edge; then the discontinuity is required to occur infinitesimally within the *downwind* quad. Another way to say this is

that the domain of a given bilinear element includes the downwind edges of its quad, but excludes the upwind edges.

The application of the finite-element method machinery now proceeds much as in the case of the moment equations: The intensity representation defined piecewise by equation (15) is substituted into equation (13), and the residual is projected onto the various basis functions, $(1 - \alpha)(1 - \beta)$, etc. If R is one of the quads, and w is one of the basis functions, then the rule for the discontinuities allows the following transformation of the poloidal divergence term:

$$\begin{aligned} \iiint_R w \nabla \cdot (\Omega_p I) dV = & \iint_{\Omega \cdot \mathbf{n} > 0} w I \Omega_p \cdot \mathbf{n} dA + \iint_{\Omega \cdot \mathbf{n} < 0} w I^* \Omega_p \cdot \mathbf{n} dA \\ & - \iiint_R I \Omega_p \cdot \nabla w dV. \end{aligned} \quad (16)$$

The symbol I^* in the surface integral over the inflow edge(s) of the quad is used to indicate that this intensity is evaluated from the bilinear element in the neighboring upwind quad, or from a supplied boundary intensity, if that edge should lie on the outside boundary of the problem. The other references to I on the right side of equation (16) are to be supplied from the bilinear element in the quad R itself.

The procedure for solving the difference equations is a sweep of the mesh in the upwind-to-downwind direction, owing to the upwind reference in equation (16). First of all, the quads of the mesh are ordered so that any quad that is an upwind neighbor of another must precede it in the list. This is possible for any mesh consisting of convex quadrilaterals. Then the four equations for the vertex intensities of the most upwind quad are set up and solved simultaneously; the I^* that appears is supplied by the boundary condition. Then the procedure is repeated for the next quad in the list, and so forth. Always I^* is available from a previously-computed quad, or from the boundary condition.

So far the angle-derivative and source-sink terms in equation (13) have not been discussed. The source-sink terms are treated with mass lumping, exactly as for the moment equations. The representation of the angle-derivative term is closely tied to the angle set that is used. In ALTAIR the hemisphere $0 \leq \theta \leq \pi$, $0 \leq \psi \leq \pi$ of angle space that must be computed (the hemisphere with $\psi < 0$ is the same because of axial symmetry) is divided into angle bins. These bins are lined up in rows bounded by chosen values of θ . The bins in one row are divided at certain values of ψ , but the number of bins and the dividing points may vary from row to row. Usually the number of bins per row is small at the poles and large at the equator, in order to preserve a roughly constant angular resolution over the hemisphere. The designation " S_n " may sometimes have the strict meaning of a triangular arrangement of angles within one octant: 1 bin in the row next to the pole, 2 in the next row, and so forth, with a total of $n/2$ rows in the octant. This means $n(n+2)/8$ angle bins in the octant, or $n(n+2)/2$ in the full hemisphere. Each angle bin has a center that is used to define the poloidal vector, Ω_p .

The ψ -derivative term in equation (13) is treated in ALTAIR at present by simple donor-cell differencing: the projections of the equation include as well as the spatial integration also an integration over the angle bin. The integral of the ψ -derivative term becomes the jump in a flux proportional to $-\Omega_t I$. This flux is always in the $-\psi$ direction, and therefore it is evaluated from the intensity in the neighboring angle bin at larger ψ . This differencing is only of first-order accuracy. It could be replaced by using a piecewise-linear instead of

piecewise-constant representation of the angle dependence, but this has two drawbacks. One is the doubling of the number of intensities per quad and per angle bin; the second and fatal drawback is that the angle and space coupling of quads become mixed in such a way that the sorting of quads in upwind-to-downwind order is impossible, unless the spatial mesh is actually rectangular.

5. Examples

Two examples will illustrate the 2-D radiative transfer capability of ALTAIR: formation of a spectral line with complete redistribution by a uniform cylinder of conservatively scattering material, and a single-frequency formal transfer calculation of a long square prism of black absorbing material wrapped in a larger square prism of translucent emitting material. The first problem illustrates the line formation capability and the axial-symmetry; the second problem, which is solved with an irregular quadrilateral mesh, shows the ability of the S_n transfer to cope with mesh distortion and retain reasonable accuracy.

5.1. LINE FORMATION IN A CYLINDER

The specifications of these problems are as follows: complete redistribution with a Doppler line profile; uniform cylinder of material; whichever is less of the optical depth across the diameter and along the length (integrated over the frequency in Doppler units) fixed at 2×10^4 ; values of diameter/length ranging from 0.1 to 10; conservative scattering ($\epsilon = 0$); uniform source of line photons in the cylinder. One of the main quantities of interest that can be derived from the solution to these problems is $\langle N \rangle$, the mean number of scatterings experienced by the line photons escaping the cylinder. If s is the contribution to the source function from the primary source of photons, then the expression for $\langle N \rangle$ is $\int \chi S dV / \int \chi s dV$. It is known from the analytic work summarized by Ivanov (1973) that $\langle N \rangle$ increases in proportion to the optical depth for a Doppler line profile; the present calculations show the manner in which $\langle N \rangle$ varies with the aspect ratio of the cylinder. The ALTAIR calculations of the cylinder problems have used a 40×40 spatial mesh and the S_{20} angle set. Several of the computed results are given in the following table.

Uniform Cylinder Results						
χD	χL	$F(\text{side})/F$	$\langle N \rangle$	$\bar{J}_c/s$	$\bar{J}_{\text{top}}/s$	$\bar{J}_{\text{cor}}/s$
2×10^5	2×10^4	0.125	4.66×10^4	6.66×10^4	3.27×10^2	8.04
10^5	2×10^4	0.236	4.18	6.54	3.24	7.85
4×10^4	2×10^4	0.479	3.06	5.46	2.82	7.31
2×10^4	2×10^4	0.686	2.00	3.70	2.08	6.53
2×10^4	4×10^4	0.834	2.36	4.16	2.16	6.61
2×10^4	10^5	0.933	2.58	4.24	2.16	6.62
2×10^4	2×10^5	0.966	2.66	4.24	2.16	6.62
1-D slab			5.3×10^4			
1-D sphere			1.9×10^4

Figure 1: Perspective plot of $\bar{J}$, on a logarithmic scale, vs. location in the equal-dimension cylinder ($D = L$). The actual mesh lines are indicated on the surface. The arrow shows the extent of one decade on the vertical axis.

Apparently the mean number of scatterings varies relatively little among cylinders of different aspect ratio that have the same smallest dimension. Analytic results from Ivanov (1973) are shown in the table for the mean number of scatterings in the 1-D slab, which should agree with the limiting cylinder $D \gg L$, and in the 1-D sphere, which is approximated by $D = L$. Also shown are the fraction of photons that escape through the curved sides of the cylinder, and the values of $\bar{J}$ at the center, the middle of a circular end, and the corner of the cylinder. The great contrast between the corner and top side values of $\bar{J}$ is interesting; it suggests that $\bar{J}_{\text{top}}/s$ scales as $O(\tau^{1/2})$, while $\bar{J}_{\text{cor}}/s$ scales as $O(1)$. The very small value of $\bar{J}$ near the corner is confined to a region that is only about one mean free path in extent; this is illustrated in Figure 1.

5.2. MORDANT 2-D X-Y TEST PROBLEM

The problem is made up of the superposition of two square rods. First a long rod is defined that is one unit square and has an opacity $\chi = 1$ and a unit Planck function. Then the center section of that rod is replaced by a 0.6 unit square rod in which the opacity is $\chi = 100$ and there is no emission, $B = 0$. This problem was first posed by Reed and Hill (1973) and discussed thoroughly by Mordant (1981); it poses two difficulties: the intensity drops very rapidly into the cold absorber, and many numerical methods produce ringing and negative intensities; also, the spatial structure of the mean intensity in the translucent material is complex, with cusps along the shadow lines. A truly accurate result for this problem would require more angles than the 312 (S_{24}) used for the calculations shown here.

One of the two calculations was made with a fine mesh—60×60 square zones; the other

Figure 2: Contours of J for the Mordant test problem. One quadrant of the problem is shown. The contour interval is $0.05J_{\max}$. Left: 20×20 distorted mesh; right: 60×60 mesh.

was a 20×20 mesh obtained from a uniform square mesh by applying random perturbations to the vertices. The r - z geometry of ALTAIR was effectively converted into x - y geometry by applying an offset to the radii. The result of the calculation is that in neither case did the intensity go negative due to the strong absorption in the center; the agreement of the coarse and fine results was of the order of two percent except just inside the absorber, which is satisfactory given the coarseness of the 20×20 mesh. The two calculations are compared in Figure 2.

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