TITLE: APPLICATION OF THE BOSON POLYNOMIALS OF U(n) TO PHYSICAL PROBLEMS

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APPLICATION OF THE BOSON POLYNOMIALS
OF U(n) TO PHYSICAL PROBLEMS

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I. Introduction

The use of boson operators as a method for studying the representations of the unitary groups is well known from the original work of Jordan [1] and Schwinger [2]. The effectiveness of this technique is found in four basic properties of boson operators under unitary transformations: (i) the number operator is invariant; (ii) the boson operator commutation relations are form invariant; (iii) polynomial forms are mapped into polynomial forms; and (iv) there is a natural invariant inner product defined for pairs of polynomials.

The purpose of this talk is (i) to review the properties of a general class of polynomials in the boson operators which have been found useful for obtaining the explicit unitary irreducible representations (irreps) of the unitary group itself; and (ii) to show how these same polynomials provide a unified approach for obtaining the explicit solutions to several classic problems in physics and chemistry.

The literature relating to (i) above and to the problems alluded to in (ii) is enormous, and space permits us to list only selected references where further sources may be found. In keeping with the general review nature of this talk, no detailed proofs are given and no specific credits cited. (Confer, however, the listed references.)

II. The U(n) * U(n) boson polynomials

Let A denote the n x n matrix of boson operators, \( A = (a_{ij}^\dagger), \)
\( (i,j = 1,2,...,n) \), and \( \hat{A} \) the matrix of conjugate bosons, \( \hat{A} = (\hat{a}_{ij}^\dagger) \).
The boson operators in each of the sets \( \{a_{ij}^\dagger\} \) and \( \{\hat{a}_{ij}^\dagger\} \), respectively, commute, and otherwise they satisfy the commutation relations \( [\hat{a}_{ij}^\dagger,a_{kl}^\dagger] = \delta_{ji}\delta_{lk} \).
In terms of this notation, it is the conjugate boson operators which annihilate the "vacuum" state \( |0\rangle \), that is, \( \hat{a}_{ij}^\dagger|0\rangle = 0 \) all \( i,j \).

We denote polynomials in the boson operators \( a_{ij}^\dagger \) and over the complex numbers by the notation \( P(A) \), and call such a polynomial a boson polynomial. The inner product of two such polynomials is written in two alternative forms, \( \langle P,P' \rangle \) or \( P|P'\rangle \), and is defined by
\[ (P, P') = \langle P | P' \rangle \equiv \langle 0 | P^* (\bar{A}) P' (A) | 0 \rangle, \] (1)

where \( P^* \) denotes the complex conjugate polynomial to \( P \).

In applications to physical problems, it is often the case that the superscripts \( j = 1, 2, \ldots, n \) and the subscripts \( i = 1, 2, \ldots, n \) label the boson operators \( \{ a_j^i \} \) have distinct roles, e.g., indices labelling particles and indices labelling components of operators relative to a laboratory frame. It is with this possibility in mind that we are led to consider unitary transformations of the boson matrix \( A \) of the form

\[ A + \bar{U} A V, \quad U, V \in U(n) . \] (2)

Thus, for \( V = I_n \), each column of \( A \) undergoes the same transformation \( \bar{U} \), while, for \( U = I_n \), each row of \( A \) undergoes the same transformation \( V \). The transposed matrix \( \bar{U} \) of \( U \) is used in Eq. (2) so that when the transformation (2) is followed by \( A + \bar{U}' A V' \), the composed transformation is \( A + (\bar{U} \bar{U}) A (V V') \).

We seek now to classify the set of all polynomials \( \{ P(A) \} \) according to their transformation properties under the linear substitution of boson operators given by Eq. (2). As a first step in this classification we observe that the mapping (2) carries homogeneous polynomials into homogeneous polynomials. Thus, the problem is reduced to one of classifying all polynomials which are homogeneous of degree \( N \), \( P(AA) = \lambda^N P(A) \), where \( \lambda \) is an arbitrary nonnegative integer.

The set of polynomials which are homogeneous of degree \( N \) in the \( n^2 \) bosons \( \{ a_j^i \} \) defines a vector space \( V^\{N\} \) of dimension \( \dim V^\{N\} = (n^2+N-1)!/(n^2-1)!N! \). Furthermore, \( V^\{N\} \) is the carrier space of irreducible representation \( |N\rangle = |N_0, \ldots, N_n\rangle \) of the unitary group \( U(n^2) \). This representation of \( U(n^2) \) is obtained from the linear transformation of polynomials of \( V^\{N\} \) given by \( \phi(a) = \sum_{W \in U(n^2)} \phi(\tilde{W})a \), where \( \tilde{a} \) denotes the column vector of boson operators: \( \tilde{a} = \text{col} \{ a_1^1 \ldots a_n^1, \ldots, a_1^n \ldots a_n^n \} \). Observing that the transformation (2) may be written in an alternative form: \( \bar{a} + (\bar{W} \bar{U}) \bar{a} \), where \( \otimes \) denotes the matrix direct product, we may now state in group theoretical terms the classification problem for boson polynomials under the transformation (2): Split the carrier space \( V^\{N\} \) of irreducible representation \( |N\rangle \) of \( U(n^2) \) into a direct sum of carrier spaces of irreps of the direct product group \( U(n) \times U(n) \times U(n^2) \).

In order to state the solution to this classification problem, let us first recall that the single-valued irreps of \( U(n) \) are in one-to-one correspondence with the \( n \)-tuples \( \{ m_1^1 m_2^1 \ldots m_n^1 \} \) in which the \( m_i \) are integers (positive, zero, or negative) which satisfy \( m_1^1 \geq m_2^1 > \cdots > m_n^1 \). Thus, the inequivalent unitary matrix irreps of \( U(n) \) may be denoted by \( D^{[m]} = \{ D^{[m]}(U) | U \in U(n) \} \), where \( [m] = [m_1^1 \ldots m_n^1] \) runs over all
n-tuples of ordered integers.

An abstract characterization of the splitting problem posed above is contained in the result: Each irrep of $U(n) \times U(n)$ of the form $D^{[m]} \otimes D^{[m]} = (D^{[m]}(U) \otimes D^{[m]}(V))|_{V : n}$, where each set of irrep labels satisfies $m_{1n} \geq \ldots \geq m_{nn} > 0$ and $m_{nn} = N$, occurs exactly once in the restriction of irrep $[N]$ to the subgroup $U(n) \times U(n)$.

The result stated above implies that the space $V^{[N]}$ splits into a direct sum of subspaces $H^{[m]}$, that is, $V^{[N]} = \bigoplus H^{[m]}$, where $H^{[m]}$ is the carrier space of irrep $D^{[m]} \otimes D^{[m]}$ of $U(n) \times U(n)$. Our problem of classifying boson polynomials under the transformation (2) has now been reduced to determining those polynomials which belong to the space $H^{[m]}$. We call these polynomials $U(n) \ast U(n)$ boson polynomials, the * designating that the irrep of $U(n) \times U(n)$ carried by the space $H^{[m]}$ "shares" the $U(n)$ irrep label $[m]$. The remainder of Sec. II is devoted to a description of the properties of $U(n) \ast U(n)$ boson polynomials.

We first describe two alternative notations for enumerating a set of basis vectors of the space $H^{[m]}$.

Gel'fand pattern scheme. The basis vectors of $H^{[m]}$ are in one-to-one correspondence with the set of double Gel'fand patterns:

$$\begin{align*}
\{ (m') \} = \{ (m') \mid (m') \text{ is a Gel'fand pattern} \}.
\{ (m) \} = \{ (m) \mid (m) \text{ is a Gel'fand pattern} \}.
\end{align*}$$

A Gel'fand pattern

$$
\begin{pmatrix}
m_{1n} & m_{2n} & \ldots & m_{n-1} & m_{nn} \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
m_{1n-1} & m_{2n-1} & m_{n-1} & n-1 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
m_{12} & m_{22} \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
m_{11} \\
\end{pmatrix}
$$

is a triangular array of nonnegative integers containing $n$ rows in which $i$ is the set of ordered integers denoting an irrep of $U(n)$ and the integers $m_{ij}$, $i, j = 1, 2, \ldots, n-1$ in $(m)$ may assume all sets of values consistent with the "betweenness" conditions:

$$m_{ij+1} \geq m_{ij} > m_{i+1j+1} \quad (5)$$

$(m')$ is an array of the same type as (4) which for notational
convenience is inverted over \([m_1] \) in (3), the common irrep labels \([m]\) being written only once.

**Example.** For \(n = 3\) and \([m_{13} m_{23} m_{33}] = [210]\) there are eight Gel'fand patterns

\[
\begin{pmatrix}
2 & 1 & 0 \\
2 & 1 & 1 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 \\
2 & 1 & 0 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 \\
2 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 \\
2 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 \\
2 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 \\
2 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 \\
2 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 \\
2 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

\(\text{(6)}\)

and sixty-four double Gel'fand patterns.

The significance of the betweeness conditions (5) and of the integers in the Gel'fand pattern (4) is readily understood as a geometrical realization of the Weyl branching law for the chain of unitary subgroups \(U(n) \supset U(n-1) \supset \ldots \supset U(2) \supset U(1).\)

The **weight or content** of a Gel'fand pattern is the row vector \(W = (w_1, w_2, \ldots, w_n)\) where \(w_j\) is defined to be the sum of the entries in row \(j\) minus the sum of the entries in row \(j-1\): \(w_j = \sum m_{ij} - \sum m_{i,j-1}\) (\(w_1 \equiv m_{11}\)).

**Weyl tableau scheme.** The basis vectors of \(U[m]\) are in one-to-one correspondence with the set of double standard Weyl tableaux of shape \([\lambda_1 \lambda_2 \ldots \lambda_n]\) (\(\lambda_i \equiv m_{i1} m_{nn} > 0\)):

\[
\begin{pmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\vdots \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\end{pmatrix}
\]

\(\text{(7)}\)

For completeness we recall that a **Young frame** of shape \([\lambda_1 \lambda_2 \ldots \lambda_n]\) has \(\lambda_1\) boxes (nodes) in row 1 (top row), \(\lambda_2\) boxes in row 2, \ldots, \(\lambda_n\) boxes in row \(n\). A **standard Weyl tableau** is a Young frame which has been "filled in" with integers selected from 1, 2, \ldots, \(n\) in such a way that the sequence of integers in each row is nondecreasing as read from left to right and the sequence of integers in each column is strictly increasing as read from top to bottom. The **weight or content** \(W\) of a standard Weyl tableau is the row vector \(W = (w_1, w_2, \ldots, w_n)\), where \(w_k\) equals the number of times \(k\) appears in the tableau. [A standard Young
The standard Weyl tableau corresponding to the Young frame [210] are:

\[
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 3 \\
3 & 3 & 2
\end{array}
\quad \begin{array}{ccc}
1 & 1 & 2 & 2 \\
2 & 3 & 3 & 3
\end{array}
\]

Remark. There is a one-to-one correspondence between Gel'fand patterns (4) having fixed labels \([m]\) and the standard Weyl tableaux of shape \([m]\) \((m_{nn} > 0)\). It is this result which allows one to go back and forth between the notations (3) and (7) for the basis vectors of \(H[m]\).

Row \(j\) of the standard Weyl tableau corresponding to the Gel'fand pattern (4) is

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & \ldots & j & j+1 & \ldots & n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & \ldots & j & j+1 & \ldots & n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & \ldots & j & j+1 & \ldots & n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & \ldots & j & j+1 & \ldots & n
\end{array}
\]

Observe then that the weight of a Gel'fand pattern and that of the corresponding standard Weyl tableau agree.

Let us now describe the \(U(n) \times U(n)\) polynomials which span the space \(H[m]\). We begin with the description of the simplest polynomials which are those corresponding to the Young frame having 1 row and \(p\) boxes so that \([m] = [p0...0] = [p0];

\[
B \left( \begin{array}{c}
[m'] \\
[p0]
\end{array} \right) (A) = \left[ \frac{n}{i=1} (w_i)^{1/2} \right]^{1/2} \sum_{\alpha} \frac{n}{i,j=1} (a_i^j)^{1/2} \left( \omega_i^j \right)! ,
\]

where \(W\) and \(W'\) are the weights of the lower and upper Gel'fand patterns, respectively, and \(\alpha\) denotes a square array of the nonnegative integers \((a_i^j)\) in which the entries in each row \(i\) must sum to \(w_i\) and the entries in each column \(j\) must sum to \(w_j'\). These "magic square" constraints are symbolized by the notation:

\[
\begin{array}{cccc}
a_1 & a_1 & \ldots & a_n \\
\vdots & \vdots & \ddots & \vdots \\
a_n & a_n & \ldots & a_1 \\
\end{array}
\begin{array}{c}
w_1 \\
\vdots \\
w_n
\end{array}
= \left( \begin{array}{ccc}
1 & a_1 & \ldots \\
a_1 & 2 & \ldots \\
\vdots & \vdots & \ddots \\
a_n & \ldots & n \\
\end{array} \right)
\begin{array}{c}
w_1 \\
\vdots \\
w_n
\end{array}
\begin{array}{c}
w_1 \\
\vdots \\
w_n
\end{array}

The summation in Eq. (10) is over all \((a_i^j)\) which satisfy these constraints.
The general $U(n)\times U(n)$ boson polynomial is only slightly more complicated in appearance than (10):

\[ B \left( \frac{(m')}{(m)} \right) (A) = \mu^{1/2}(m) \sum_{(\alpha)} C \left( \frac{(m')}{(m)} \right) (\alpha) \prod_{i,j=1}^{n} (a_i^{a_j})^{1/2} \left| [a_i^{1}] \right|^{-1/2}, \quad (12) \]

where the significance of $[a_i^{1}]$ is as before [Eq. (11)]. $\mu(m)$ is a normalization factor, $\mu(m) = \prod_{i} (m_{i+n-i})/\prod_{i < j} (m_{i+n-j} - m_{j+n-i})$, and the C-coefficient is a Wigner coefficient for the subduction $[N] + U(n) \times U(n)$ where $N = \sum m_{i+n}$. These coefficients are given explicitly by the matrix element

\[ C \left( \frac{(m')}{(m)} \right) (\alpha) = C \left( \frac{(m)}{(m')} \right) (\alpha) = \left< \left( \frac{m'}{(m)} \right) \left( \begin{array}{c} \gamma_n \\ \gamma_0 \end{array} \right) \right| \left( \begin{array}{c} \gamma_2 \\ \gamma_0 \end{array} \right) \cdots \left| \left( \frac{m}{(m')} \right) \left( \begin{array}{c} \gamma_2 \\ \gamma_0 \end{array} \right) \right| (0) \right>, \quad (13) \]

where the symbol

\[ \left< \left( \frac{\gamma_1}{\gamma_0} \right) \right| = \left| \begin{array}{cccc} \gamma_1 & 0 & \cdots & 0 \\
0 & \gamma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma_n \end{array} \right| \]

denotes a $U(n)$ Wigner operator. We shall say more about the coefficients (13) later. For now it suffices to observe that the coefficients (13), while complicated, are completely known.

The double Gel'fand pattern boson polynomials possess a number of important properties which are summarized below:

(i) Pairs of polynomials having distinct Gel'fand patterns are orthogonal [scalar product given by (1)].

(ii) The polynomials of given weight $(W,W')$ are a basis of the vector space spanned by all monomials in the $(a_i^{a_j})$ which contain $w_i^1$ occurrences of the subscript $i$ and $w_j^1$ occurrences of the superscript $j$.

(iii) The polynomials corresponding to all partitions $(m)$ of $N$ and all patterns $(m')$ and $(m)$ are a basis for all homogeneous polynomials of degree $N$ in the $(a_i^{a_j})$.

(iv) Replacing the boson matrix $A$ by a unitary matrix $U \in U(n)$ yields the unitary irrep $(m)$ of $U(n)$, that is,

\[ D^{(m)}(U) = B \left( \frac{(m)}{(m')} \right) (U) \quad (15) \]
(v) The polynomials corresponding to fixed \( [m] \) are a basis of the carrier space of irrep \( D^{[m]} \) of \( U(n) \times U(n) \), that is,

\[
B^{(m')}(U) = \sum_{m} D^{[m]}(U) D^{[m]}(V) B^{(m')}(V),
\]

(16)

**Example.** As a simple, nontrivial, example of the boson polynomials, we obtain the unitary irreps of \( SU(2) \) in the form:

\[
D^{J}_{jm}(U) = \sqrt{\frac{(j+m)!(j-m')!(j-m')!}{a_1!a_2!}}.
\]

(17)

Remark. In an alternative theory to that above, one associates to each pair of corresponding columns in the double standard tableau (7) a determinantal boson:

\[
\{ (i_1i_2...i_k) ; (j_1j_2...j_k) \}.
\]

Forming the product over all column pairs then yields a Weyl boson polynomial. These polynomials are, in general, nonorthogonal, but possess properties (i)-(v) above when "orthogonal" is replaced by "linear independent" and "unitary" is dropped.

III. Applications to physical problems

1. The Yamanouchi real, orthogonal representations of \( S_n \).

Let \( \{ \Gamma_p | \Gamma \in S_n \} \) denote the Cayley representation of the symmetric group \( S_n \) by \( n \times n \) permutation matrices:

\[
\Gamma_p = \begin{bmatrix} c_1 & e_2 & ... & e_n \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 & ... & n \\ 1 & 2 & ... & i_n \end{bmatrix},
\]

(18)

where \( e_i \) denotes the unit column vector, \( e_i = \text{col} [0...010...0] \) (1 in position \( i \)). We now replace the boson matrix \( A \) by \( \Gamma_p \) in Eq. (12) and restrict the Gel'fand patterns \( (m) \) and \( (m') \) to those having weight \( \omega = (1,1,...,1) \). The result simplifies to [cf. Eq. (15)]

\[
D^{[m]}(\Gamma_p) = \left[ \begin{array}{c} n ! \\ D^{[m]}(\Gamma_p) \end{array} \right] \left[ \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{array} \right] \left[ \begin{array}{c} n_0 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array} \right]_{\text{inverted}} \left[ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \right],
\]

(19)

where: (i) \( \text{dim } [m] \) denotes the dimension of irrep \( [m] \) of \( S_n \); (ii) \( \left[ \begin{array}{c} \gamma_i \\ n_0 \end{array} \right] \) denotes a fundamental \( U(n) \) Wigner operator in which \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \) and \( \left[ \begin{array}{c} \gamma_i \\ 0 \end{array} \right] \) (inverted) denote \( n \)-rowed patterns having weights \( [0...010...0] \) with 1 in position \( i \) and \( \gamma_i \), respectively; and (iii) the sequence of integers \( (\gamma_n,...,\gamma_2,\gamma_1) \) is the Yamanouchi symbol of the Gel'fand pattern \( (m')_p \), \( \omega' = (1,...,1) \). (\( \gamma_n \) is the number of the row in which integer \( s \) appears in the standard tableau.)
Let $D^{[m]}(P)$ denote the matrix of dimension, $\dim(m)$, with elements in row $(m)$ and column $(m')$ given by Eq. (19). Then the result we have obtained may be summarized as: $\{D^{[m]}(P)|P \in S_n\}$ is the Yamanouchi real, orthogonal representation of $S_n$.

Remarks. While each $\gamma_k$ appearing in Eq. (19) may assume the values $1, 2, \ldots, n$, the matrix element of the string of fundamental Wigner operators is automatically zero unless $(\gamma_n, \gamma_2, \gamma_1)$ is the Yamanouchi symbol described above. The matrix elements of the fundamental $U(n)$ Wigner operators are completely known so that Eq. (19) is a completely general and explicit result yielding all irreps of $S_n$.

2. Explicit $N$-particle states transforming irreducibly under $S_N$ and $SU(n)$. We consider that $N \geq n$ (the case $N < n$ may be treated similarly) and specialize the polynomials (12) to those having irrep labels of the form $[m_1 m_2 \ldots m_{nN} 0 \ldots 0]$, $\sum \Delta = n$, and upper Gel'fand patterns $(m')$ having the weight $W' = (1, \ldots, 1, 0, \ldots, 0)$ containing $N$ ones and $n-N$ zeroes. We obtain:

$$
\sum_{k_1 \ldots k_N} \left( \begin{array}{c}
\sum_{k_1 \ldots k_N} \left( \left( [m_1 \ldots m_{nN} \delta] \right)^{-1/2} \left( [m_1 \ldots m_{nN} \delta] \right)^{(m)} \left( [m_1 \ldots m_{nN} \delta] \right)^{(m')}
\end{array} \right)
$$

where $\{\gamma_1, \gamma_2, \gamma_1\}$ is the Yamanouchi symbol of the standard Young tableau corresponding to the $N$-rowed Gel'fand pattern $\left( [m_1 \ldots m_{nN}] \right)$, $W' = (1, \ldots, 1)$.

The transcription of the boson polynomials (20) into basis vectors of the Hilbert space $H$ of the union of $N$ physical systems, considered as a single system, is accomplished by the correspondence

$$a_1^{k_1} a_2^{k_2} \ldots a_N^{k_N} \leftrightarrow | k_1 \otimes k_2 \otimes \ldots \otimes k_N \rangle,$$

where $\{\gamma_1, \gamma_2, \gamma_1\}$ is the Yamanouchi symbol of the standard Young tableau.

This result follows by considering the unitary transformations of single-particle states given by $U$: $|i\rangle \rightarrow \Sigma_{j=1}^n |j\rangle$, each $U \in U(n)$, and the permutations of identical particles which induce the transformations $P$: $|k_1^{(i)} \otimes k_2^{(i)} \otimes \ldots \otimes k_N^{(i)} \rangle \rightarrow |k_1^{(i)} \otimes k_2^{(i)} \otimes \ldots \otimes k_N^{(i)} \rangle$ of the tensor product space, where $P$ is the rearrangement $1 \rightarrow 1_1, 2 \rightarrow 2_1, \ldots, N \rightarrow N_1$ of the subscripts $1, 2, \ldots, N$. The resulting transformation of the basis vector $a_1^{k_1} a_2^{k_2} \ldots a_N^{k_N}$ is then exactly the same as that of the boson product $a_1^{k_1} a_2^{k_2} \ldots a_N^{k_N}$ under $A \rightarrow \tilde{U} A \tilde{U}$, where $A$ now denotes the $n \times N$ boson matrix $a_{ij}^{j}$, $i=1,\ldots,n; j=1,\ldots,N$. (Since the boson operators having $j > N$ do not appear
in the right-hand side of (20), we can replace the nxn matrix boson by
the nxN matrix boson.) We thus obtain an explicit expression for the
orthonormal N-particle states which transform irreducibly under \( U(n) \)
and \( S_N \):

\[
\left( T'_{U_n} | T_{S_N} \right) = \sum_{k_1 \ldots k_N} ^n \left( \begin{array}{c}
\left( \frac{m}{(m)} \right) \\
\left( \frac{1}{k_N} \right) \\
\left( \frac{1}{k_1} \right)
\end{array} \right) \left( \begin{array}{c}
\left( \frac{0}{(0)} \right)
\end{array} \right)
\]

\[
\times |k_1 \rangle \otimes |k_2 \rangle \otimes \ldots \otimes |k_N \rangle .
\]

(21)

In stating this result, we have replaced the somewhat cumbersome (be-
cause of repetition of labels) Gel'fand pattern notation by the double
tableau notation: \( T \) denotes a standard tableau of shape \( \left[ m_{1n} \ldots m_{nn} \right] \),
\( \Sigma m_{nn} = N \); \( T_{U_n} \) denotes the standard Weyl tableau corresponding to \( \left( \frac{m}{(m)} \right) \);
and \( T_{S_N} \) denotes the standard Young tableau which has the Yamanouchi sym-
bol \( (\gamma_N \ldots \gamma_1) \). (There is no restriction between \( N \) and \( n \) in the
final result (21) other than \( \mu = m_{1n} = N \).)

In terms of the double tableau notation, the orthogonality and
transformation properties of the basis vectors (21) take the forms:

\[
\left< \left( T'_{U_n} | T_{S_N} \right) \left| T'_{U_n} | T_{S_N} \right) \right> = \delta(T'_{U_n}, T_{U_n}) \delta(T'_{S_N}, T_{S_N}) ,
\]

where \( \delta(T', T) = 0 \) for distinct tableaux and \( \delta(T', T) = 1 \) for identical
tableaux.

\[
U: \left( T'_{U_n} | T_{S_N} \right) \left( T_{U_n} | T_{S_N} \right) \left( \begin{array}{c}
\left( \frac{m}{(m')} \right) \\
\left( \frac{m'}{(m)} \right)
\end{array} \right) \left( U \right) \left( T'_{U_n} | T_{S_N} \right) ,
\]

(23)

\[
P: \left( T'_{U_n} | T_{S_N} \right) \left( T_{U_n} | T_{S_N} \right) \left( \gamma_1 \ldots \gamma_N \right) \left( \gamma_1 \ldots \gamma_N \right) \left( P \right) \left( T'_{U_n} | T_{S_N} \right) .
\]

3. Spin states for \( N \) particles of spin \(-\frac{1}{2}\) which transform irre-
ducibly under \( S_N \). These states are obtained as a special case of Eq.
(21), namely, \( n = 2 \), \( m_{12} + m_{22} = N \), \( 2S = m_{12} - m_{22} \) so that \( m_{12} = \frac{N}{2} + S \),
\( m_{22} = \frac{N}{2} - S \), where the total spin \( S \) may assume the values \( S = \frac{N}{2}, \frac{N}{2} - 1, \ldots, \frac{N}{2} \) or \( 3 \). Thus, the tableau \( T \) has the shape

\[
T = \begin{array}{cccc}
\ldots & \ldots & \ldots & \ldots \\
N_2 + S & \ldots & \ldots & \ldots \\
N_2 - S
\end{array}
\]

(24)

and the orthonormalized basis vectors are:

\[
\left( \gamma_{U_2} | T_{S_N} \right) = \left( \gamma_N \ldots \gamma_2 \gamma_1 \right) ; \left( S^S M^S \right)
\]

\[
= \sum_{\sigma_1 \ldots \sigma_N} ^2 \left< \left( \frac{N}{2} + S \right) \left( \frac{N}{2} - S \right) \left( \frac{1}{\sigma_N} \right) \ldots \left( \frac{1}{\sigma_0} \right) \right> \times \left| \sigma_1 \right\rangle \otimes \left| \sigma_2 \right\rangle \otimes \ldots \otimes \left| \sigma_N \right\rangle,
\]

(25)
where $|1\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle$, $|2\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$ denote the two single-particle spin states. [Observe that $\gamma_1 = 1$ and 2 correspond, respectively, to the patterns $\left\{ 1, 0 \right\}$ and $\left\{ 0, 1 \right\}$]

The evaluation of the matrix elements of the $U(2)$ Wigner operators in Eq. (25) is discussed below.

4. Equivalent electron configurations $i^N$. Let the single-electron states $|n\rangle$ of fixed energy $E_n$ and fixed angular momentum $l$ be denoted by $|k\rangle = |l - m + 1\rangle$ so that $k = 1, 2, \ldots, 2l + 1$, and let the two spin states be denoted by $|\sigma\rangle = \left| \frac{3}{2} - \mu \right\rangle \equiv \left| \frac{1}{2}, \mu \right\rangle$ so that $\sigma = 1, 2$. Thus, the set of single-particle states of the electron is $\{|k\sigma\rangle = |k\rangle \otimes |\sigma\rangle |k = 1, 2, \ldots, 2l + 1; \sigma = 1, 2\}$.

The Slater determinantal states for the configuration $i^N$ are given by

$$\langle k_1 \sigma_1, k_2 \sigma_2, \ldots, k_N \sigma_N \rangle = \frac{1}{\sqrt{N!}} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_N} (-1)^{P} \langle k_1 \sigma_1, \ldots, k_N \sigma_N \rangle.$$  

If we order the pairs of integers $(k, \sigma)$ by the rule $(1, 1) < (2, 1) < \ldots < (2l + 1, 1) < (1, 2) < (2, 2) < \ldots < (2l + 1, 2)$, then an orthonormal basis of the space $V_A(i^N)$ of Slater states is obtained from the states (26) by imposing $(k_1 \sigma_1, k_2 \sigma_2, \ldots, k_N \sigma_N)$. Hence, the space has dimension, $\dim V_A(i^N) = (4l + 2)^N$.

A basic problem in the LS-coupling scheme is to introduce a new basis into the space $V_A(i^N)$ such that the total orbital angular momentum $l$ and the total spin $s$ of the $N$ spin-1/2 particles have the standard irreducible action on the new basis ($l^2, l^3, s^2, s^3$ diagonal).

This problem is partially solved by the following well-known procedure: We introduce the tableau $\tilde{T}$ which is conjugate (or dual) to the tableau $T$ given by (24), and we use the fact that the antisymmetric irrep $\left[ 1 \ldots 1 \right]$ of $S_N$ occurs only in the direct product of conjugate irreps, and then exactly once. Furthermore, the coefficients for this reduction are given by the simple formula $C(T) = \sigma(T)/\dim \left[ N, S_n \right]$, where $\sigma(T)$ is the signature of the tableau $T$. (The signature of a standard tableau is the signature of the permutation $t_1 \ldots t_N$ where $t_1, t_2, \ldots, t_N$ are the entries in row 1, followed by those in row 2, \ldots.) Carrying out this procedure, using the vectors (25) and the vectors (21) for the tableau conjugate to (24), we obtain:

$$\left\langle \frac{\tilde{T}}{U_{2l+1}}, \frac{T}{U_2} \right\rangle = \sum_{T} C(T) \left\langle \frac{\tilde{T}}{U_{2l+1}}, \frac{T}{U_2} \right\rangle T \left( T, S \right) \left\langle \frac{\tilde{T}}{U_{2l+1}}, \frac{T}{U_2} \right\rangle T,$$

where the summation is over all standard Young tableaux of shape $T$ [Eq. (24)]

The vectors (27) corresponding to all $\tilde{U}_2$ Weyl tableaux of shape $\tilde{T}$ [Eq. (24)], to all $U_{2l+2}$ Weyl tableaux of shape $T$, and to all shapes
given by $S = \sum_{i=1}^{N} N - 1, \ldots, \frac{1}{2}$ or 0 are an orthonormal basis of the space $V_{\lambda}(l^N)$, and the total spin $S$ has the standard irreducible action $(S^2, S_3$ diagonal) on this basis.

The result (27) does not solve the problem of constructing states of good orbital angular momentum $(L, M_L)$. This requires reducing irrep $[\lambda] \equiv [2, \lambda, \frac{1}{2}, s]$ of $U(2l+1)$ into irreps of the rotation group.

More precisely, since the single-particle states undergo the unitary transformation $V \equiv D_{2l+1}(U) \subset U(2l+1)$, each $U \in SU(2)$, the problem is to reduce the representation $D[\lambda](V)$ of the rotation group into irreducible constituents $D^L(U)$. Alternatively, when expressed in terms of the Lie algebras $\{L_{ij} | i=1,2,3 \}$ of $SU(2)$ and $\{E_{ij} | i,j=1,2,\ldots,2l+1 \}$ of $U(2l+1)$, the problem is to determine the linear combinations of the vectors $\left(\tilde{T}_{2l+1}(V)_{\lambda} \right)$ on which $L_{+} = \sum_{m}^{\lambda}(\lambda-m)(\lambda+m+1)^{1/2} E_{\lambda-m, \lambda+m+1}$ $L_{-} = \sum_{m}^{\lambda}(\lambda+m)(\lambda-m+1)^{1/2} E_{\lambda-m-1, \lambda-m+1}$, have the standard irreducible action. Since this reduction entails summing only over the tableaux $\tilde{T}_{2l+1}$, the antisymmetry and spin properties of the states (27) are preserved, and the states (27) retain their importance in the LS-coupling problem. [Equivalent electron configurations $j^N$ may also be obtained from Eq. (20) by identifying the states $|k\rangle$ with the $ls$-coupled states $|jm\rangle$, choosing $n = 2j+1$ and $[m] = (\ldots,10\ldots,0)$ (N ones, 2j + 1 - N zeroes), $\gamma_k = k$, and reducing irrep $[1^N]$ of $U(2j+1)$ into irreps of the rotation group.]

IV. Evaluation of the coefficients. The matrix elements of the Wigner operators which appear in Eqs. (18), (20), and (24) may be evaluated using the rules of the pattern calculus (Ref. 12). These rules are simple to apply and give the following form for the nonzero matrix elements of a fundamental Wigner operator between arbitrary Gel'fand states (for specific evaluations, it is easier to apply the pattern calculus rules directly than to specialize the general result below):

$$\left\langle \left[ \begin{array}{c} |m\rangle \\ |m\rangle \\ \vdots \\ |m\rangle \end{array} \right] \middle| Y_n \cdots Y_1 \right\rangle \left\langle \left[ \begin{array}{c} |1^\delta\rangle \\ |1^\delta\rangle \\ \vdots \\ |1^\delta\rangle \end{array} \right] \middle| \left[ \begin{array}{c} |m\rangle \\ |m\rangle \\ \vdots \\ |m\rangle \end{array} \right] \right\rangle = \left( \begin{array}{c} \prod_{k=1}^{n} S(\gamma_{k-1}-\gamma_k) \\ \prod_{s=1}^{k-1} (p_{Y_k-1-k-1} p_{sk+1}) \\ \prod_{t=1}^{k-1} (p_{Y_k-k-t} p_{tk-1}) \\ \prod_{t\neq k-1}^{k-1} (p_{Y_k-1-k-1} p_{tk-1+1}) \end{array} \right)^{1/2},$$

where: (i) the final pattern is obtained from the initial one by shifting row $j$ ($j=n,n-1,\ldots,i$) to $[m_{1j}+\delta_j, \ldots, m_{ij}+\delta_j]$ ($\gamma_j = 1, \ldots, j$), $\gamma_n = \gamma$; (ii) $S(j-i) = +1$ for $j > i$ and $-1$ for $j < i$; (iii) the $k=i$ factor in the product has sign $S(i-\gamma_i)$ and all factors containing $p_{\gamma_i-1-i}$ are to be omitted; and (iv) $p_{ij} \equiv m_{ij}+j-i$.

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REFERENCES