1811

76
7/14/81
A.E.

(1)

DA-2817

JULY 1981

PPPL-1811
UC-206

B5642

MASTER

THE SCATTERING OF LOWER-HYBRID WAVES
BY DENSITY FLUCTUATIONS

BY

P.L. ANDREWS AND F.W. PERKINS

**PLASMA PHYSICS
LABORATORY**

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

**PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY**

This work was supported by the U.S. Department of Energy
Contract No. DE-AC02-76-CHO 3073. Reproduction, transla-
tion, publication, use and disposal, in whole or in part,
by or for the United States government is permitted.

THE SCATTERING OF LOWER-HYBRID WAVES BY DENSITY FLUCTUATIONS

P. L. Andrews and F. W. Perkins

Plasma Physics Laboratory, Princeton University,

Princeton, New Jersey 08544

The investigation of the scattering of lower-hybrid waves by density fluctuations in tokamaks is distinguished by the presence in the wave equation of a large, random, derivative-coupling term. Assuming the fluctuations to be of long wavelength compared to the incident wave the similarity of the wave equation to the Schrodinger equation for a particle in a random magnetic field is used to derive a two-way diffusion equation for the wave energy density. The diffusion constant found disagrees with earlier findings and the source of the discrepancy is pointed out. When the correct boundary conditions are imposed this equation can be solved by separation of variables. However most of the important features of the solution are apparent without detailed algebra.

DISCLAIMER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

419

I. INTRODUCTION

The use of lower-hybrid waves as a heating mechanism for confined plasmas has been of great interest for some time as simple propagation theory shows them to have several properties advantageous to their use in tokamaks.¹ However, with the recent discovery of drift wave induced large amplitude density fluctuations near the edge of tokamaks^{2,3}, doubt was cast on the validity of results derived by theories neglecting the scattering effects of such fluctuations. Several papers have already been written discussing these scattering effects giving roughly equivalent predictions as to the rate of scatter of the incident waves. In one particular regime the wave energy density will obey a diffusion equation, as has already been derived using a wave-kinetic approach.⁴ It is the purpose of this paper to derive this equation in a simple manner, pointing out that some previous results are in error, and to find the intensity and angular spectrum of the lower hybrid wave energy emerging from the scattering layer.

In Sec. II we derive the wave equation and state the parameter regime in which we are interested. We then point out the equivalence of this equation to the Schroedinger equation for a charged particle in a magnetic field. This equivalence is used in Sec. III to derive a diffusion equation for the energy density of the wave. Our result disagrees with some previous work and we clarify the origin of the discrepancy in one particular paper. In Sec. IV we show how this diffusion equation can be solved when the correct boundary conditions are imposed by separation of variables. However an approximate solution for the intensity and angular spectrum of the emerging radiation can be found without recourse to complicated algebra and this is done in section V. Section VI presents our conclusions.

II. THE WAVE EQUATION

The frequencies of drift waves in tokamaks are far lower than lower hybrid frequencies. This permits us to consider propagation of a monochromatic wave in a time-independent medium. Experimental investigation² has shown the density fluctuations to be largely confined to a relatively thin region near the outside of the tokamak allowing us to consider the magnetic field uniform over the scattering region.

We thus choose a co-ordinate system in which $\hat{B} = B_0 \hat{z}$ with B_0 a constant and $\hat{z}$ the unit vector in the z direction. Then the lower hybrid wave equation can be written as

$$\nabla \cdot \mathbf{K} \cdot \nabla \phi = 0 \quad (1)$$

where the dielectric tensor $\mathbf{K}$ is given by;

$$\mathbf{K} = \begin{pmatrix} K_{\perp} & K_{xy} & 0 \\ K_{xy}^* & K_{\perp} & 0 \\ 0 & 0 & K_{\parallel} \end{pmatrix}$$

and the individual components are

$$K_{\perp} = 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\Omega_e^2}, \quad K_{\parallel} = -\frac{\omega_{pe}^2}{\omega^2}, \quad K_{xy} = -i \frac{\omega_{pe}^2}{\omega \Omega_e}.$$

The nomenclature is standard: ω_{pe} and ω_{pi} are the local electron and ion plasma frequencies respectively, which here will be functions of position, and Ω_e is the electron gyrofrequency. Near the edge of the tokamak $|\omega_{pe}^2/\Omega_e^2 - \omega_{pi}^2/\omega^2| \ll 1$ so that $K_{\perp} \approx 1$. In addition the drift waves are aligned along the magnetic field lines with little density variation along $\hat{B}$. Hence we idealize the problem to one where the waves are incident on a two-dimensional scattering layer of finite thickness. We therefore take the density to be given by

$$n = n(x, y) = n_0[1 + \mu(x, y)]$$

where n_0 is the spatially constant average density and μ arises from drift wave density fluctuations. We take $\langle \mu \rangle = 0$ the angular brackets denoting an averaging process over realizations of n . We also introduce the two-point correlation function C of the medium, assuming the fluctuations to be isotropic in the x - y plane, and define it in such a way as to make the scaling of the problem clear;

$$\langle \mu(\tilde{x}_0 + \tilde{x}) \mu(\tilde{x}_0) \rangle = \langle \mu^2 \rangle C(\xi_0^2 x^2), \quad (2)$$

With this definition ξ_0 is a measure of the inverse scale length of the medium and C decays on a unit scale length, e.g. $C(r) = \exp(-r)$. Later we will make use of the idea of spatial averaging and will assume it to be identical to the ensemble averaging process i.e.

$$\langle \mu(\tilde{x}_0 + \tilde{x}) \mu(\tilde{x}_0) \rangle = \frac{1}{V} \int d\tilde{x}_0 \mu(\tilde{x}_0 + \tilde{x}) \mu(\tilde{x}_0)$$

where V is the two-dimensional volume of integration.

We then Fourier transform in the z direction and consider a single k_z , i.e. a wave of the form $\exp(i k_z z - i \omega t)$. The wave equation, (1), thus becomes

$$\nabla_{\perp}^2 \phi - i \vec{A} \cdot \nabla \phi + k_0^2 (1 + \mu) \phi = 0 \quad (3)$$

where

$$\vec{A} = M \hat{z} \times \nabla \mu, \quad M = \frac{\langle \omega_{pe}^2 \rangle}{\omega \Omega_e}, \quad k_0^2 = \frac{\langle \omega_{pe}^2 \rangle}{\omega^2} k_z^2.$$

The quantity M is obviously of great importance; in most cases of interest we expect it to be larger than one, sometimes much larger, making the convective term in Eq. (3) the dominant scattering term. The direct solution of Eq.(3) presents considerable difficulties; in order to make progress we restrict ourselves to cases in which the correlation length of the medium is large compared to an incident wavelength but small compared to the scattering length i.e. $k_0 \gg \xi_0 \gg 1/l_c$. The first of these requirements, $k_0 \gg \xi_0$ can be written as $(m_i/m_e) \beta_i N_{\parallel}^2 \gg \xi_0^2 \rho_i^2$ in terms of the usual tokamak parameters; m_i/m_e is the mass ratio, β_i the ion beta value, $N_{\parallel}$ the parallel index of refraction, and ρ_i the ion Larmour radius. These restrictions allow us to neglect interference effects and to treat the scattering process as a series of small, random perturbations which combine incoherently to give a diffusive process. The resultant diffusion equation could be obtained by carefully going through a general procedure to find the wave kinetic equation associated with Eq. (3). This approach is certainly feasible, and was used by Ott^{4,5}, and Sen and Fisch⁶, but it is perhaps simpler to utilize the similarity of Eq. (3) to the Schrodinger equation for a particle in a magnetic field:

$$\nabla^2 \psi - 2 \frac{ie}{\hbar c} \vec{A} \cdot \nabla \psi + k_0^2 \left[1 - V - \left(\frac{e \vec{A}}{\hbar c} \right)^2 \right] \psi = 0 \quad (4)$$

where $\vec{A}$ is the magnetic vector potential (in Coulomb gauge) and V is the electrostatic potential normalised to the free energy of the particle: $\hbar^2 k_0^2 / 2m$, m being the particle's mass. In addition c is the speed of light, e the charge on the particle, and $\hbar$ Planck's constant over 2π . In the next section we will utilize the similarity of Eqs. (3) and (4).

III. THE WAVE SCATTERING EQUATION

A comparison of equations (3) and (4) shows that our simplified lower hybrid wave equation is equivalent to the Schrodinger equation for a charged particle in a magnetic field $\vec{B}_z$ given by

$$\vec{B}_z = M \nabla^2 \mu \frac{\hbar c}{2e} \hat{z} \quad (5)$$

and a normalised potential

$$V = -\frac{M^2 (\nabla \mu)^2}{4k_0^2} - \mu. \quad (6)$$

The inequalities $k_0 \gg \xi_0 \gg 1/l_s$ allow us to identify the energy density of the lower-hybrid wave with the probability density of the scattered particles and thus achieve a diffusion equation for the wave energy density.⁷ If the momentum of one particle is $\vec{p}$ with an average r.m.s momentum p_0 then the equation of motion is

$$\frac{d\vec{p}}{dt} = -\frac{p_0^2 \nabla V}{2m} + \frac{p_0 M}{m k_0} \vec{p} \times \hat{z} \nabla^2 \mu. \quad (7)$$

The restriction that there be little scattering in a correlation length thus requires $M^2 (\xi_0/k_0)^2 (\mu^2) \ll 1$ so that to first order in the scattering we can always take $V \approx -\mu$. Therefore if we follow a particle with initial momentum $p_0 \hat{p}$ we can write an expression for the rate of angular deflection

$$\frac{1}{p_0} \frac{d\vec{p}}{dx} \approx \frac{1}{2} \nabla \mu + M \hat{p}_0 \times \hat{z} \frac{\nabla^2 \mu}{2k_0} \quad (8)$$

where we have taken the deflection to be small and the z axis to be along $\hat{p}_0$.

We can thus write an expression for the mean squared angular deflection in a length Δx which is small compared to the scattering length but long compared to the correlation length,

$$\langle \Delta \theta^2 \rangle = \frac{1}{p_0^2} \int_0^{\Delta x} \int_0^{\Delta x} \left\langle \frac{d\vec{p}(x_1)}{dx_1} \cdot \frac{d\vec{p}(x_2)}{dx_2} \right\rangle dx_1 dx_2. \quad (9)$$

We proceed by using $\vec{r} = \vec{x}_1 - \vec{x}_2$ and replacing the ensemble average by a spatial one so that the previous expression becomes

$$\begin{aligned} \langle \Delta \theta^2 \rangle \approx & \frac{\Delta x}{4V} \int_{-\infty}^{\infty} d\vec{r} \int d\vec{u} [\nabla \mu(\vec{u}) \cdot \nabla \mu(\vec{u} + \vec{r}) + (M/k_0)^2 \nabla^2 \mu(\vec{u}) \nabla^2 \mu(\vec{u} + \vec{r})] \\ & + \frac{\Delta x}{4V} \int_{-\infty}^{\infty} d\vec{r} \int d\vec{u} [\nabla \mu(\vec{u}) \cdot (M/k_0) \hat{p}_0 \times \hat{z} \nabla^2 \mu(\vec{u} + \vec{r}) \\ & + \nabla \mu(\vec{u} + \vec{r}) \cdot (M/k_0) \hat{p}_0 \times \hat{z} \nabla^2 \mu(\vec{u})]. \end{aligned}$$

This equation can be greatly simplified by a little manipulation, e.g.

$$\begin{aligned}\int d\tilde{u} \nabla \mu(\tilde{u}) \cdot \nabla \mu(\tilde{u} + \tilde{r}) &= \nabla_r \cdot \int d\tilde{u} \nabla \mu(\tilde{u}) \mu(\tilde{u} + \tilde{r}) \\ &= -\nabla \cdot \int d\tilde{u} \nabla_r \mu(\tilde{u} - \tilde{r}) \mu(\tilde{u}) = -\nabla^2 \int d\tilde{u} \mu(\tilde{u}) \mu(\tilde{u} + \tilde{r}).\end{aligned}$$

Manipulating all of the terms in this manner and utilizing the definition of the two-point correlation function gives;

$$\langle \Delta \theta^2 \rangle = \frac{1}{4} \Delta x \langle \mu^2 \rangle \int_{-\infty}^{\infty} [-\nabla^2 C(\xi_0^2 r^2) + (M^2/k_0^2) \nabla^4 C(\xi_0^2 r^2)] dr \quad (10)$$

with no contribution from the cross terms.

The Fokker-Planck equation for the particle distribution function, or equivalently the lower-hybrid wave kinetic equation, is thus

$$\frac{\partial F}{\partial t} + \frac{\omega}{k_0} \cos \theta \frac{\partial F}{\partial x} = D \frac{\partial^2 F}{\partial \theta^2} \quad (11)$$

After integration by parts the diffusion coefficient D is given by;

$$D = \langle \mu^2 \rangle \frac{\omega \xi_0}{4k_0} \left[-\int_{-\infty}^{\infty} dx \frac{dC(x^2)}{d(x^2)} + 6M^2 \left(\frac{\xi_0}{k_0} \right)^2 \int_{-\infty}^{\infty} dx \left(\frac{d}{d(x^2)} \right)^2 C(x^2) \right]. \quad (12)$$

When $C(r) = \exp(-r)$ Eq.(12) becomes

$$D = \omega \frac{\sqrt{\pi}}{4} \langle \mu^2 \rangle \frac{\xi_0}{k_0} \left[1 + 6M^2 \left(\frac{\xi_0}{k_0} \right)^2 \right] \quad (13).$$

If we restrict ourselves to time independent solutions and define an inverse scattering length l_s^{-1} as $k_0 D / \omega$ then we have the two-way diffusion equation

$$\cos \theta \frac{\partial F}{\partial x} = \frac{1}{l_s} \frac{\partial^2 F}{\partial \theta^2}. \quad (14)$$

The scattering length can be written in terms of typical tokamak parameters;

$$l_s^{-1} = \frac{\sqrt{\pi}}{4} \xi_0 \langle \mu^2 \rangle \left(1 + \frac{6\rho_i^2 \xi_0^2 n_0}{N_{||}^2 n_w} \right) \quad (15)$$

where the ion-sound density n_w is given by

$$n_w = \frac{2\omega^2 T_i m_i}{4\pi e^2 c^2 m_e} = 8.9 \cdot 10^9 \text{ cm}^{-3} \left(\frac{T_i}{100 \text{ eV}} \right) \left(\frac{\nu}{1 \text{ GHz}} \right)^2. \quad (16)$$

The relatively low value of this reference density makes it clear that the derivative-coupling term will dominate the scattering process

This scattering problem could also be approached through wave formalism, in particular the treatment of Ott⁴ is perfectly good and his original result agrees exactly with the result of this paper, Eq.(13), the numerical factors being identical when the factor of 2 difference in the definition of ξ_0 is recognized. However there were two compensating algebraic errors in the original paper, one of these was removed in a following Erratum⁵ but the other remained. So, paradoxically, originally the conclusion was correct while after the Erratum it no longer is. We now point out the second algebraic error in Ott's original paper⁴. In his Appendix C Ott makes a change of variables from $\vec{k}_\perp$ to (l, β) where $l = |\vec{k}'_\perp - \vec{k}_\perp|$ and $\vec{k}'_\perp \cdot \vec{k}_\perp = k'_\perp k_\perp \cos \beta$. He then states that under this transformation

$$\delta(\omega_k - \omega_{k'}) = \delta(l - 2k_\perp \sin \beta/2) / (v \sin \beta/2). \quad (17)$$

However this is not true. That the correct transformation is not obvious stems from the change of variables not being one to one. If we write $\mathbf{x} = \mathbf{k}_\perp - \mathbf{k}'_\perp$ then $\delta(\omega_k - \omega_{k'}) = \delta(x)/v$ and we have $l^2 = k_\perp^2 + k'^2_\perp - 2k_\perp k'_\perp \cos \beta$ which implies

$$l^2 - 4k_\perp^2 \sin^2 \beta/2 = (k_\perp - k'_\perp)(k_\perp + k'_\perp - 4k_\perp \sin^2 \beta/2)$$

so that

$$\delta[x(x - 4k_\perp \sin^2 \beta/2)] = \delta(l^2 - 4k_\perp^2 \sin^2 \beta/2) = \frac{\delta(l - 2k_\perp \sin \beta/2)}{4k_\perp \sin \beta/2}.$$

We can write this as;

$$\delta(x) = [\delta(l - 2k_\perp \sin \beta/2) - \delta(x - 4k_\perp \sin^2 \beta/2) / \sin \beta/2] \sin \beta/2, \quad (18)$$

the final delta function in Eq.(18) results from the non reversibility of the transformation and must be ignored. Then finally we have

$$\delta(\omega_k - \omega_{k'}) = \delta(l - 2k_\perp \sin \beta/2) \frac{\sin \beta/2}{v} \quad (19)$$

which differs from Eq.(17) by a factor of $\sin^2 \beta/2$. Originally this factor exactly cancelled the error from the Jacobian of the transformation which was removed in the Erratum⁵. Hence the correctness of the

result in the original paper. The same result can be found by performing the integration in Appendix C without making a change of variables.

IV. SOLUTION OF THE DIFFUSION EQUATION

Even though we may know the value of the scattering length appearing in Eq.(14), in order to predict important details of the scattered wave (e.g. how much of an incident wave will penetrate a scattering regime of length l) we must know something of the solutions of this equation. We will assume that there is a scattering block of length l , within which the diffusion rate is constant, and the unscattered wave is incident from the left at $x = 0$. Arbitrary boundary conditions cannot be imposed; Eq.(14) is a two-way diffusion equation and just as the one-way diffusion equation $\partial F / \partial x = \partial^2 F / \partial^2$ would be well posed in the same domain only when boundary conditions are applied at $x = 0$, $-\pi \leq \theta \leq \pi$ so this equation is well posed only when F is given on the intervals $x = 0$, $\theta \in [-\pi/2, \pi/2]$ and $x = l$, $\theta \in [-\pi, -\pi/2]$, $\theta \in [\pi/2, \pi]$. This restriction has the satisfying result that we can only specify waves entering the scattering block.

In our examination of the correct solution to Eq.(14) we will closely follow the treatment of Fisch and Kruskal,⁹ for convenience considering only solutions even in θ . The boundary conditions are taken to be: $x = l$, $F = F_1(\theta)$, $\theta \in [-\pi/2, \pi/2]$ and $x = l$, $F = 0$ $\theta \notin [-\pi/2, \pi/2]$. For simplicity we will change the independent variable x to $z = x/l$, so that Eq.(14) becomes

$$\cos \theta \frac{\partial F}{\partial z} = \frac{\partial^2 F}{\partial \theta^2} \quad (20)$$

There is one immediate solution of Eq.(20), the so-called linear or diffusion solution; ($z - \cos \theta$) but this is neither positive definite nor capable of meeting the boundary conditions. In order to obtain the complete solution we look for separable solutions of the form $f(\theta)e^{-\lambda z}$ resulting in the equation

$$\frac{d^2 f}{d\theta^2} + \lambda \cos \theta f = 0. \quad (21)$$

In treating this equation we can borrow from the theory of Mathieu's equation¹⁰

$$\frac{d^2 f}{d\theta^2} + (\alpha + \beta \cos 2\theta)f = 0$$

so that we know Eq.(21) will have solutions with $f(2\pi + \theta) = f(\theta)$ for only a denumerable set of distinct eigen-values; λ_n , $-\infty < n < \infty$ with $n = 0$ corresponding to the degenerate linear solution in this case and $\lambda_{-n} = -\lambda_n$. The set of functions $f_n(\theta)$, $n > 0$ are orthogonal to each other and to any constant on the interval $[-\pi, \pi]$ (with weighting function $\cos \theta$) but, with the addition of the linear solution, complete¹¹ only on $[-\pi/2, \pi/2]$. Similarly the functions $f_{-n}(\theta)$ are proportional to $f_n(\theta - \pi)$ and are complete on the rest of the interval $[-\pi, \pi]$. Hence the complete solution to the diffusion problem is

$$F(z, \theta) = A - B(z - \cos \theta) + \sum_{n=1}^{\infty} \{a_n f_n(\theta) \exp(-\lambda_n z) + b_n f_{-n}(\theta) \exp[-\lambda_n(l/l_0 - z)]\} \quad (22)$$

the coefficients A, B, a_n, b_n are chosen so as to fit the boundary conditions at $z = 0, l$. As the eigen-values come in pairs and we are dealing with even eigen-functions only in calculating the eigen-functions we will consider only positive eigen-values ($\lambda_n = k_n^2$) and positive values of θ . It is known¹² that an approximate value for λ_n is $0.43(4n+1)^2$, $n > 0$, so that the k_n quickly become large as n increases. Therefore while we may have to produce the first few eigen-functions by numerical integration of Eq.(21) we expect to be able to use a W.K.B. type treatment for the rest. If we define $S(\theta) = \int_0^\theta \sqrt{\sin t} dt$ then the W.K.B. solution has the form;

$$f_n(\theta) = \frac{\exp -k_n S(\pi/2)}{(-\cos \theta)^{1/4}} \cosh[k_n S(\pi/2) - k_n S(\theta - \pi/2)],$$

$$\frac{\pi}{2} \geq \theta - \frac{\pi}{2} \gg k_n^{-2/3}$$

$$f_n(\theta) = \sqrt{\pi k_n^{-1/3}} A_1 \{(\theta - \pi/2) k_n^{-2/3}\}, \quad |\theta - \pi/2| \ll 1$$

$$f_n(\theta) = \frac{\sin[k_n S(\pi/2 - \theta) + \pi/4]}{(\cos \theta)^{1/4}}, \quad \frac{\pi}{2} \geq \frac{\pi}{2} - \theta \gg k_n^{-2/3}$$

and the eigen-values are given by $k_n S(\pi/2) = (n + \frac{1}{4})\pi$. The accuracy of these eigen-functions can be checked either by comparison with numerically produced solutions or by verifying their orthogonality to $\cos \theta$. In Fig.1 one of these eigen-functions is displayed.

Having produced all of the required eigen-functions the problem remains that while the f_n ($n > 0$) are complete on $[-\pi/2, \pi/2]$ they are not orthogonal on that interval. Hence the usual method of

integrating out the coefficients in Eq.(22) cannot be used. The method of fitting the boundary conditions used was to make a least squares fit, for accuracy sampling the error at many more points than the number of eigen-functions involved. In Fig.2 the fit at the boundaries is shown, taking the incident energy to have a Gaussian spectrum. Once the coefficients are set the solution is known in the interval $[0, l]$ and in Fig.3 we display the the solution for $l = 10 l_e$ using 20 eigen-functions. The fit can be improved by taking more eigen-functions but their addition does not change the character of the solution. The two-way diffusion equation allows a discontinuity of slope at $\cos \theta = 0$ leading to the occurrence of a Gibbs phenomenon in the fitting of the eigen-function. In comparing Fig. 3 to a similar figure in the paper by Fisch and Kruskal it should be noted that in the calculation of their results incorrect eigen-values were used so that the higher frequency eigen-functions do not decay as quickly as they should. Fig. 4 is similar to Fig. 3 but this time the incoming flux spectrum is isotropic, as it might be after reflection from the tokamak wall. In the next section we will look more closely at the physical

V. APPROXIMATE SOLUTION

Once the method of solution is recognized the gross characteristics of the solution can be predicted from very simple reasoning without recourse to numerical computations. Our procedure depends on solving a set of four simultaneous equations which we obtain by making some assumptions about the angular dependance of the transmitted and reflected fluxes. If we denote the transmitted flux by T_f we can immediately write down one equation; as all of the Mathieu eigen-functions are orthogonal to $\cos \theta$ on $[-\pi, \pi]$ the forward flux is given by

$$T_f = B \int_{-\pi}^{\pi} \cos^2 \theta d\theta = \pi B \quad (23)$$

When l is much greater than l_e the transmitted flux spectrum will be independent of the incident flux spectrum and we choose to approximate it by $l \cos \theta$ (we examine the validity of this assumption later). Equating our two values for the transmitted flux gives us

$$l = 2B. \quad (24)$$

The reflected flux spectrum, however, will be affected by the incident flux spectrum and though the exact form of our approximation will not greatly affect our result we will treat separately the cases of peaked and isotropic spectra. When the incoming spectrum is isotropic we take the reflected flux to be isotropic also and equal to r : this gives us a criterion for flux conservation of the form

$$1 = \frac{\pi t}{2} + 2r. \quad (25)$$

In the case of a relatively narrow Gaussian incoming spectrum, as in our third figure we take the reflected flux spectrum to be $r \cos \theta$. Flux conservation then gives

$$1 = \frac{\pi}{2}(t + r). \quad (26)$$

We can derive two more equations in the following manner. All of the Mathieu eigen-functions decay rapidly outside of their domain of completeness hence at $x = l$, $\theta = 0$ only the linear solution will contribute giving $t \approx A - B(l/l_s - 1)$. The same reasoning will apply at $x = 0$, $\theta = \pi$ implying $r \approx A - B$. We can solve the above equations in the two cases we have considered to give firstly for an isotropic incoming spectrum, $B = 1/(\pi + 2l/l_s)$, $A = (l/l_s + 1)/(\pi + 2l/l_s)$, $t = 2/(\pi + 2l/l_s)$ and $r = (l/l_s)/(\pi + 2l/l_s)$. The predicted value for the flux transmission coefficient is thus

$$T_f = \frac{\pi l_s}{2l + \pi l_s}. \quad (27)$$

When we have a peaked incoming spectrum our equations give: $B = 2/\pi(l/l_s + 2)$, $A = 2(l/l_s + 1)/\pi(l/l_s + 2)$, $t = 4/\pi(l/l_s + 2)$ and $r = 2(l/l_s)/\pi(l/l_s + 2)$. In this case the predicted flux transmission coefficient is

$$T_f = \frac{2l_s}{l + 2l_s}. \quad (28)$$

In Fig.6 the ratio of the predicted and numerically produced transmitted fluxes are plotted for l from 0 to $20l_s$. Considering the simplicity of the derivation the agreement is remarkably good. In figures 7 and 8 our assumptions on the form of the transmitted and reflected fluxes are examined. For both the isotropic and peaked cases we assumed that the transmitted flux could be taken to have a $\cos \theta$ dependence. As l/l_s increases, Figs. 7b and 8b show that the transmitted flux spectrum rapidly converges to form reasonably approximated by cosine function. The same is true of the reflected flux, in

the peaked case as shown by Fig. 7a. As l/l_0 becomes large the reflected flux in the isotropic case reaches good agreement with our assumed isotropic reflection spectrum as seen in Fig. 8b.

VI. CONCLUSIONS

In this paper we have shown a straightforward method to derive the diffusion equation for the scattering of lower hybrid waves, pointing out the correct value of the diffusion coefficient. We showed how this equation could be solved numerically, with the correct boundary conditions. Our method of solution suggested a simple model that was used to provide an analytic expression for the flux transmission coefficient which agreed well with our numerical results. This model showed that for large l/l_0 the flux transmission coefficient varies inversely with l . The expansion in Mathieu eigen-functions makes it clear that in the body of the scattering medium the flux density decays linearly, while the transmitted flux spectrum rapidly loses its correlation with the incoming spectrum converging to a limiting form represented modestly well by a cosine dependence. There are several possible extensions; a more realistic geometry could be used, a small amount of time dependence could be introduced, or the diffusion constant could be allowed to slowly vary. However it is difficult to see how the scaling restrictions of the regime could be circumvented in order to treat cases of heavy scattering per correlation length. In that regime interference effects cannot be neglected and numerical solutions of the original wave equation are probably all that can be obtained.

ACKNOWLEDGMENTS

One of us (P.L.A.) would like to thank Dr. N.J. Fisch for several helpful discussions. This work was supported by the U.S. Department of Energy under Contract DE-AC02-76-C1103073.

References

- ¹T. H. Stix, Phys. Rev. Lett. **15**, 878 (1965).
- ²C. M. Surko and R. E. Slusher, Phys. Rev. Lett. **37**, 1747 (1976).
- ³R. E. Slusher and C. M. Surko, Phys. Rev. Lett. **40**, 400 (1978).
- ⁴E. Ott, Phys. Fluids **22**, 1732 (1979).
- ⁵E. Ott, Phys. Fluids **23**, 421 (1980).
- ⁶A. Gen and N. J. Fisch, Massachusetts Institute of Technology Plasma Research Report No. PRR 78/16 (1978).
- ⁷L. A. Chernov, *Wave Propagation in a Random Medium* (McGraw-Hill, New York, 1960).
- ⁸N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973).
- ⁹N. J. Fisch and M. D. Kruskal, J. Math. Phys. **21**, 740 (1980).
- ¹⁰J. Mathews and R. L. Walker, *Methods of Mathematical Physics* (W. A. Benjamin, California, 1970).
- ¹¹R. Beal, Submitted to J. Math. Phys.
- ¹²N. W. McLachlan, *Theory and Application of Mathieu Functions* (Dover, New York, 1964).

81T0174

FIG. 1. The first even eigenfunction is plotted on the interval $[-\pi, \pi]$, found by integration of Eq.(21). The eigenfunctions oscillate inside $[-\pi/2, \pi/2]$ then die away in the rest of the interval.

81T0175

FIG. 2. The fit of the solution to the boundary conditions at $x = 0$ (solid line) and at $x = l$ (dashed line) is shown using 20 eigen-functions and Gaussian incoming spectrum.

81T0180

FIG. 3. The solution to the diffusion equation is shown for $x \in [0, 1]$ and $\theta \in [-\pi, \pi]$ using the same boundary conditions as in Fig. 2.

81T0181

FIG. 4. As in Fig. 3 the solution of the diffusion equation is shown, this time for an isotropic incoming spectrum.

81T0179

FIG. 5. The transmitted intensity for a Gaussian incoming spectrum is shown as a function of L/LS for θ/π from 0 to 2.

FIG. 6. The ratio of numerically calculated flux to the predicted flux is plotted for $(I/I_s) \approx 0$ to 20 for the Gaussian (solid line) and isotropic (dashed line) cases.

FIG. 7. The reflected (figure a) and transmitted (figure b) flux spectrum (both normalized to unity) from a Gaussian incoming spectrum are displayed for $l/l_0 = 0.5, 2, 5$ and 10 . The dashed line is the corresponding normalized cosine approximation.

FIG. 8. The same information as in Fig. 7 is plotted for an isotropic incoming spectrum with the corresponding approximation shown as a dashed line.