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REDUCED EQUATIONS FOR FINITE BETA
TEARING MODES IN TOKAMAKS

By

R. Izzo, D. Monticello, J. DeLucia, W. Park, and C.M. Ryu

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ABSTRACT

The equations of resistive magnetohydrodynamics (MHD) are recast in a form that is useful for studying the evolution of those toroidal systems where the fast magnetosonic wave plays no important role. The equations are exact and have $\nabla \cdot \mathbf{B} = 0$ satisfied explicitly. From this set of equations it is a simple matter to derive the equations of reduced MHD to any order in the inverse aspect ratio ϵ of the torus, and for $\beta \sim \epsilon$ or smaller. We demonstrate this by deriving a reduced set of MHD equations that are correct to 5th order in ϵ . These equations contain the exact equilibrium relation and as such can be used to find 3-D stellarator equilibria. In addition, if a subsidiary ordering in η , the resistivity, is made, the equations of Glasser, Greene, and Johnson¹ are recovered. This set of reduced equations has been coded by extending the initial value code, HILO.² Results obtained, for both ideal and resistive linear stability from the reduced equations are compared with those obtained by solving the full set of MHD equations in a cylinder. The agreement is shown to be excellent for both zero and finite beta calculations. Comparisons are also made with analytic theory illuminating the present limitations of the latter.

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I. INTRODUCTION

The equations of magnetohydrodynamics, while providing a simple single fluid description of plasma dynamics, is complicated by the presence of multiple time scale phenomena. When studying the stability of tokamak fusion equilibria, it is beneficial to remove analytically the fastest time scale from these equations, that of the fast compressible magnetosonic wave. Since there are no unstable modes on this time scale, elimination of it simplifies numerical calculations without removing relevant physics from the MHD model. A commonly used method of filtering out the fast wave is to expand the equations of MHD in powers of the inverse aspect ratio ϵ . By ordering the plasma beta (i.e., the ratio of thermodynamic to magnetic pressure) to be of $O(\epsilon)$, we can generate a set of equations, which to leading order [$O(\epsilon^2)$] has been used to study the linear and nonlinear ideal or resistive stability of finite-beta tokamak plasmas.³ If a more realistic ordering for present day tokamaks is assumed with $\beta \sim O(\epsilon^2)$, we find that to leading order [$O(\epsilon^2)$] pressure does not enter the equations. The resulting set of equations has been used to study external kinks and tearing modes in low-beta tokamaks as straight cylinders.⁴ This lower order set of reduced equations has been compared with the full MHD equations in a cylinder.⁵ Toroidal and finite beta effects enter in next order [$O(\epsilon^3)$]. The third order equations have been used to study the behavior of resistive tearing modes in finite aspect ratio low-beta tokamaks.⁶ Including next order [$O(\epsilon^4)$] corrections introduces substantial new physics. The correct Mercier stability criterion to ideal interchange modes is recovered. Also the linear and nonlinear dynamics of ideal internal kink modes is accurately modeled.² Taking the expansion one order further, [$O(\epsilon^5)$], we claim to have sufficient terms to study the effects of finite beta on the linear and nonlinear evolution of resistive modes for

tokamak plasmas of finite aspect ratio.

While the primary motivation for deriving a fifth order set of equations and developing a code to solve them is to study finite aspect ratio toroids, a good starting point to check the formulation is to study the cylindrical tokamak. In this paper, the linear stability of the plasma to the $m = 2/n = 1$ mode is analyzed. For a cylinder, m is the usual poloidal mode number and n is related to the axial mode number, $k_z = n/L$ where the length of the cylinder is $2\pi R$. Codes exist which solve the linearized version of the full set of MHD equations in a cylinder.^{5,7} The fifth order reduced equations have been coded by extending the initial value code, HILO. We have compared results given by the full set of equations with those of HILO. A second comparison can also be made with the asymptotic theory of Glasser, Greene, and Johnson¹ for resistive modes in a cylinder. Results obtained clearly illustrate the regions of validity for the various methods.

All calculations presented in this report are for a particular profile of the safety factor, $q = q_0 (1 + r^2/r_0^2)$. We then vary the pressure profile, resistivity, and aspect ratio to study the evolution of the ideal interchange, resistive interchange, and resistive tearing modes. For all of the stability analyses excellent agreement is obtained between HILO and the full set of equations.

The asymptotic matching theory was shown to provide extremely accurate growth rates and eigenfunctions over a wide range of magnetic Reynold's number, $S \gtrsim 10^5$, for resistive interchange modes in a spheromak.⁷ We have been able to reproduce this agreement with the reduced equations for interchange modes in a tokamak. However, discrepancies arise when studying finite beta stabilization of tearing modes. While the asymptotic theory gives the correct gross features of the mode, values of the linear complex growth

rate differ appreciably from those generated by HILO and the full equations. The impact of this discrepancy is given later in this paper.

The remainder of the paper is organized as follows. In Sec. II we recast the equations of resistive magnetohydrodynamics and proceed to derive our fifth order set of equations. Linear, cylindrical stability analyses were performed for the purpose of code verification. In Sec. III we show comparisons of cylindrical results obtained using boundary layer theory, full MHD, and reduced MHD. Our conclusions are stated in Sec. IV. A discussion of the wave properties of the fifth order set is given in Appendix A. In Appendix B we perform a subsidiary ordering in η , the plasma resistivity, and describe how the fifth order set can yield the equations of Glasser, Greene, and Johnson.¹

II. FIFTH ORDER EQUATIONS

The equations of resistive MHD in rationalized emu units are

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \vec{\nabla} p \quad (1)$$

$$\vec{J} = \vec{\nabla} \times \vec{B} \quad (2)$$

$$\frac{dp}{dt} = -\Gamma_p \vec{\nabla} \cdot \vec{v} \quad (3)$$

$$\frac{d\rho}{dt} = -\rho \vec{\nabla} \cdot \vec{v} \quad (4)$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B} - \eta \vec{J}) \quad (5)$$

Here ρ is the mass density, $\vec{v}$ is the fluid velocity, $\vec{J}$ the plasma current, $\vec{B}$ is the magnetic field, p is the scalar pressure, Γ is the ratio of specific heats, η is the plasma resistivity, and d/dt is the full convective derivative. The usual cylindrical coordinate system R, Z, ζ is employed (see Fig. 1). R_0 is used to denote the major radius of the device and a the minor radius.

We start the derivation of our equations by using six scalar quantities to define general velocity and magnetic vector fields:

$$\vec{v} = R^2 \vec{v}_u \times \vec{v}_\zeta + \vec{v}_\perp \chi + R v_\zeta \vec{v}_\zeta \quad , \quad (6)$$

$$\vec{B} = \vec{v}_\psi \times \vec{v}_\zeta + \frac{1}{R} \vec{v}_\perp F + F \vec{v}_\zeta \quad . \quad (7)$$

We have chosen this form for $\vec{v}$ and $\vec{B}$ since for large aspect ratio ($R_0/a \equiv 1/\epsilon \gg 1$) devices

$$\vec{v} \cdot \left(\frac{\vec{v}_\perp}{R^2} \right) \sim O(\epsilon^2) \frac{|\vec{v}_\perp|^2}{R^2} \quad (8)$$

$$\vec{v} \cdot \vec{B}_\perp \sim O(\epsilon^3) |\vec{B}_\perp| \quad (9)$$

and hence χ and F will be small quantities. Here, the subscript $\perp$ denotes components in the poloidal R, Z plane.

Equations of motion for the three scalar velocity field quantities can be easily written down from the momentum equation. We find

$$\frac{\partial \Delta \vec{u}}{\partial t} = -\vec{v}_\zeta \cdot \vec{v} \times \left(\frac{\vec{v}_\perp}{\rho} \right) \quad , \quad (10)$$

$$\frac{\partial \Delta^*}{\partial t} = R^2 \vec{\nabla} \cdot \left(\frac{\vec{L}_1}{\rho R^2} \right) , \quad (11)$$

$$\frac{\partial v_z}{\partial t} = \frac{\vec{L}_1 \cdot \vec{e}_z}{\rho} , \quad (12)$$

where

$$\Delta^+ = \nabla_1^2 + \frac{1}{R} \frac{\partial}{\partial R} , \quad (13)$$

$$\Delta^* = \nabla_1^2 - \frac{1}{R} \frac{\partial}{\partial R} , \quad (14)$$

$$\vec{L} = \vec{J} \times \vec{B} - \vec{\nabla} p - \vec{v} \cdot \vec{\nabla} \vec{v} , \quad (15)$$

Next, the equation of motion for ψ is obtained from Faraday's law in its integrated form

$$\frac{\partial \vec{A}}{\partial t} = \vec{v} \times \vec{B} + I_0 \vec{\nabla} u + \vec{\nabla} \phi - \eta \vec{J} , \quad (16)$$

where,

$$\vec{B} = \vec{\nabla} \times \vec{A} ,$$

$$I_0 = R_0 B_0 . \quad (17)$$

Here, B_0 is the vacuum toroidal field at the wall and we have taken the scalar potential $\vec{\nabla} = I_0 u + \phi$ for convenience (ϕ is now small for a large aspect ratio device). From Eq. (16) we have

$$\frac{\partial \psi}{\partial t} = R \vec{v} \times \vec{B} \cdot \hat{\zeta} + I_0 \frac{\partial u}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} - \eta J_{\zeta} R \quad (18)$$

and

$$\nabla_{\perp}^2 \phi = \vec{\nabla}_{\perp} \cdot (\eta \vec{J}_{\perp}) - \vec{\nabla}_{\perp} \cdot (\vec{v} \times \vec{B})_{\perp} - I_0 \nabla_{\perp}^2 u \quad (19)$$

where the last equation follows from the gauge condition

$$\vec{\nabla}_{\perp} \cdot \vec{A}_{\perp} = 0 \quad (20)$$

which in turn follows from our representation of $\vec{B}$ (Eq. 7) and the identification of ψ as RA_{ζ} . The $\hat{\zeta}$ component of Faraday's law yields an equation for I :

$$\frac{\partial I}{\partial t} = -I(\Delta^* \chi) + R(\vec{B}_{\perp} \cdot \vec{\nabla}) v_{\zeta} - v_{\zeta} \vec{B} \cdot \hat{R} - \vec{\nabla} \times (\eta \vec{J}) \cdot \hat{\zeta} R \quad (21)$$

Finally, F is found from $\vec{\nabla} \cdot \vec{B} = 0$

$$\nabla_{\perp}^2 f = - \frac{(I - I_0)}{R} \quad (22)$$

where

$$F = \frac{\partial f}{\partial \zeta} \quad (23)$$

and where the constant of integration was again chosen to give the large aspect ratio results ($F = 0$ when $I = I_0$). Expressions for the current components found from Eqs. (2) and (7) are

$$J_z = -\frac{\Delta^* \psi}{R} - \frac{1}{R^2} \frac{\partial^2 f}{\partial z \partial \zeta} \quad (24)$$

$$\vec{J}_I = \frac{1}{R^2} \vec{\nabla}_I \left(\frac{\partial \psi}{\partial \zeta} \right) - \frac{1}{R} \vec{\nabla}_I \left(\frac{\partial^2 f}{\partial \zeta^2} \right) \times \vec{\nabla}_\zeta + \vec{\nabla}_I \times \vec{\nabla}_\zeta \quad (25)$$

We close these equations with the following boundary conditions at the plasma-wall interface:

$$\begin{aligned} \text{a) } \psi &= 0, \quad \frac{\partial f}{\partial r} = h(t) \\ \text{b) } u &= 0, \quad \frac{\partial \chi}{\partial r} = 0 \end{aligned} \quad (26)$$

$$\text{c) } \phi = 0$$

where $h(t)$ is chosen to satisfy Gauss's theorem. These boundary conditions follow from a) $\vec{B} \cdot \hat{n}|_D = 0$, b) $v_r|_D = 0$, and c) $\vec{B} \times \hat{n}|_D = 0$, where $\hat{n}$ is the normal to the boundary. We have assumed that the plasma is in contact with a rigid, perfectly conducting wall.

This set of Eqs. (6) through (22) is exact and equivalent to the primitive Eqs. (1) through (5) and can be solved using spectral techniques. Each Fourier component can be computed separately and rapidly. Unfortunately, the presence of the fast magnetosonic wave allows disturbances to propagate radially across flux surfaces. This puts a prohibitive limit on the allowed time step since flux surfaces must be packed closely together in order to describe accurately the physics of tearing modes in the vicinity of the rational surfaces. This restriction can be overcome by numerically integrating Eqs. (11) and (21) implicitly. This has been accomplished and will be reported in a separate publication.

We adopt here another method of solution that overcomes the time step problem by analytically eliminating the unwanted fast wave, which uniformly recovers the large aspect ratio equations that have been successfully employed in modeling tokamaks.

The approximation made is to replace Eq. (1) by

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = \vec{J} \times \vec{B} - \nabla p \quad (27)$$

where

$$\vec{V}_I = \vec{V} - \vec{V}_I \chi = R^2 \vec{V}_u \times \vec{\nabla} \zeta + v_\zeta \hat{\zeta} \quad (28)$$

This leaves all equations exact and unchanged except Eqs. (10) and (11). We now proceed as in Ref. 6 by multiplying Eq. (27) by R^2 and taking the $\hat{\zeta}$ component of the curl, resulting in

$$\rho_0 R^3 \frac{\partial \Delta^+ u}{\partial t} = -\hat{\zeta} \cdot \vec{\nabla} \times (R^2 \vec{L}) \quad (29)$$

for a constant density model or

$$\rho_0 R^3 \frac{\partial \Delta^+ u}{\partial t} = -\hat{\zeta} \cdot \vec{\nabla} \times (R^2 \vec{L}) \quad (30)$$

for the density model used in Ref. 6 [$\rho \propto (1/R^2)$]. Here $\Delta^+ = \Delta^+ + (2/R)(\partial/\partial R)$. Writing out the magnetic and pressure contribution to the right-hand side of Eq. (29) and (30), we find

$$\hat{\zeta} \cdot \vec{\nabla} \times R^2 (\vec{J} \times \vec{B} - \nabla p) = +R\vec{B} \cdot \vec{\nabla} (R J_\zeta) - R(\vec{J} \cdot \vec{\nabla})(R B_\zeta) - 2R \frac{\partial p}{\partial z} \quad (31)$$

The equations can now be closed by taking the divergence of Eq. (27) after multiplying by R^2

$$\nabla_{\perp}^2 \left(\frac{I^2}{2} \right) = \vec{\nabla} \cdot \vec{k} \quad (32)$$

where

$$\vec{k} = -R^2 \vec{\nabla}_{\perp} P - R^2 J_{\zeta} (\vec{B} \times \hat{\zeta}) + I \left(\frac{\partial B_R}{\partial \zeta} \hat{R} + \frac{\partial B_z}{\partial \zeta} \hat{z} \right) - \rho R^2 [(\vec{v} \cdot \vec{\nabla} \vec{v})_{\perp} + \frac{\partial \vec{v}}{\partial t} I] \quad (33)$$

The scheme is then the same as in Ref. 2 where Eq. (29) or Eq. (30) is used to advance u , while Eq. (32) is used to advance I , and Eq. (21) becomes the advancement equation for χ . The boundary condition for I is

$$\frac{I^2}{2} = \int \vec{k} \cdot d\vec{\ell} + C(t)$$

where $C(t)$ is chosen to keep

$$\int \frac{I}{R} dA = \text{constant} \quad .$$

This follows from the conservation of toroidal flux with $\vec{E} \times \hat{n}|_b = 0$.

The motivation for using Eq. (27) comes from the aspect ratio ordering of the scalar variables given in Ref. 2. From the procedure used there it is clear that Eqs. (10-12), (18), (19), (20), (22), and (32) are formally 5th order accurate since we have now included inertia in Eq. (32).

There are a number of other reasons for using Eq. (29). First, as stated above, these equations have only four waves instead of the usual six and none of them travel across flux surfaces in the poloidal plane (Appendix A). This

is due to the fact that poloidal plane compression has been eliminated from the dynamics but not from the magnetics or the equation of state. Second, we are able to recover the boundary layer equations of Glasser, Greene, and Johnson (Appendix B). Third, only the dynamics of the primitive equation have been modified; the equilibrium that is produced by these equations is exact and could be used to find stellarator equilibria. Lastly, since the lower order reduced equation can be recovered easily from this formulation, the source of new physics can be identified by systematically turning off higher order terms.

III. CYLINDRICAL STUDIES

While the set of equations discussed above has been derived for finite aspect ratio toroids, it is instructive to analyze the equations in the simpler geometry of a straight cylinder.

In the remainder of this section the accuracy of the 5th order equations is demonstrated by calculating ideal and resistive stability of cylindrical tokamaks. All figures are drawn in scaled units. We take $\hat{\gamma} = T_{Ap} \gamma$ and $\hat{n} = n(a^2/T_{Ap})$ where $T_{Ap} = a\sqrt{\rho}/eB_0$ is the poloidal Alfvén time, and γ is the linear growth rate. Henceforth we omit the caret on the variables.

As stated previously, we restrict our attention to a particular q-profile given by

$$q = 1.6 \left(1 + \frac{r^2}{0.64} \right) \quad 0 \leq r \leq 1$$

This is a standard peaked current profile in the sense of Ref. 8.

a) Ideal Stability

We first consider ideal interchange modes, $\eta \equiv 0$. While fixing q , we are free to vary β and ϵ . In particular, we set

$$\beta(r) = \beta_0 (1 - 3r^2 + 2r^3)$$

where β_0 is the central plasma beta. For a fixed aspect ratio of 10 we compare the linear growth rate obtained by using the reduced equations with those generated by solving the full MHD equations by quadrature. The comparison is made by plotting γ vs β_0 in Fig. 2. To within 0.5% the two results lie directly on top of one another.

In the next set of curves (Fig. 3) we have fixed β_0 and varied ϵ . We see that even for aspect ratios as low as two, the reduced equations and the full set give nearly identical results for the computed linear growth rates. The eigenfunctions from the two codes are also nearly identical and for clarity we display only the eigenfunctions produced by the reduced equations. In Fig. 4 we exhibit the perturbed quantities for the case $\epsilon = 0.1$ and $\beta = 0.2$. Note that the higher order field quantities f and ϕ are indeed one to many orders down from the leading order ψ contribution. The same is true of χ when compared to u . Also note that $v_{\perp} \sim \partial u / \partial r$ is indeed $O(v_{\perp} / \epsilon)$. So as expected, in the limit as $\epsilon \rightarrow 0$ we recover the lower order equations but for finite ϵ the higher order terms become important. Another advantage of the fifth order equations is that we can uniformly recover correct high-beta results. Note that $I_{2/1} \sim -P_{1/2}$ for this case. This is expected since $\epsilon \beta_p \sim 5$.

By a simple manipulation of the linearized momentum equation, we may write

$$\vec{B} \cdot \frac{\partial \vec{v}}{\partial t} = -\vec{v}_p \cdot \vec{B} \quad .$$

In our notation, $\vec{B} = \vec{B}_\perp + \vec{B}_z$ and $\vec{v} = \vec{v}_\perp + \vec{v}_z$, which gives

$$\gamma(\vec{B}_\perp \cdot \vec{v}_\perp + B_z v_z) = -\vec{B} \cdot \vec{v}_p$$

so that

$$v_z = -\frac{1}{B_z} \left(\frac{\vec{B} \cdot \vec{v}_p}{\gamma} + \vec{B}_\perp \cdot \vec{v}_\perp \right) \quad .$$

Therefore, linearly we need not advance an equation for v_z but can compute it directly. However, we have opted to advance the equation for v_z and construct the above diagnostic as an internal consistency check. These results are also shown in Fig. 4 and the agreement is seen to be quite good. We have done convergence tests and shown that any discrepancies are due to finite grid space. Lastly, we note that $\Psi_{2/1}$ passes through zero at $r = r_g = 0.4$ as it should for ideal modes.

b) Resistive Stability

While the ideal results show that the higher order terms are small corrections which depend upon aspect ratio, the picture is quite different for resistive calculations.

A simple resistive stability analysis is for a zero-beta plasma in a cylinder. In Fig. 5 we compare the linear growth rate as a function of plasma resistivity obtained by three different methods: the reduced equations (HILO), integrating the full MHD equations using an initial value code, and the analytic results of Ref. 8. We see that for an aspect ratio of 10 the

initial value codes agree for all values of η while the analytic theory is accurate only for $\eta \leq 10^{-7}$.

A more important test of the fifth order equations is that of finite beta resistive interchange modes. Using the same pressure profile as for the ideal interchange studies, we vary β_0 and η while keeping a fixed aspect ratio of 2. Once again the reduced equations give identical results to the full set (Fig. 6). A numerical solution of the layer equation of Coppi, Greene, and Johnson⁹ is performed using an asymptotic matching code. The layer equations give good agreement over the range of resistivity considered ($\eta \leq 10^{-5}$) and are extremely accurate for $\eta \leq 10^{-7}$ for the modes considered here. The results clearly illustrate the destabilizing effect of increasing beta. It is interesting to note that the linear growth rate does not scale as $\eta^{1/3}$ until we reach $\eta \leq 10^{-15}$ (not shown in Fig. 6 but obtained using matching code). This is due to the fact that the equilibrium used is tearing mode unstable which at zero β scales as $\eta^{3/5}$. Therefore, the resistive interchange scaling will dominate only at sufficiently low η .

In Fig. 7 we show the computed linear growth rate for a slightly larger β_0 at an aspect ratio of 10. Here we plot γ vs η on a semilogarithmic scale to show more clearly the convergence of the initial value codes with the boundary layer approach as resistivity decreases. We reserve discussion of the perturbed quantities to later in this section.

The final analysis we perform is for a stabilizing pressure profile, $p'(r_g) > 0$. In this fashion we may simulate the effects of good average curvature. There are two distinct cases to be treated, Γ (ratio of specific heats) equal 5/3 or zero. From the analytic theory we expect the latter to yield resistive modes which scale as η^1 and the former to result in overstable modes for sufficiently small η . The pressure profile used is

$$\beta = 0.5 (0.001 + 0.028 r^2 - 0.059 r^4 + 0.03 r^6)$$

and the aspect ratio is fixed at 5.

Figure 8 shows the computed linear growth rates for several different methods as a function of η . First, we direct our attention to the $\Gamma = 0$ results obtained with the fifth order equations. As predicted by Ref. 10 the mode shows no indication of going overstable. We obtain a scaling of $\gamma \sim \eta^{0.86}$ for $\eta \sim 2 \times 10^{-8}$. This resistivity is a practical lower limit where we can operate the initial value codes (both HILO and the full equations). The curve indicates that the scaling is still changing and may indeed ultimately give the predicted η^1 result. There is no indication of overstability at low resistivity.

Four separate calculations with $\Gamma = 5/3$ were performed. Results from the two initial value codes are again practically identical. For simplicity we display only the HILO results. There are substantial difficulties using the initial value method near the value of resistivity (η_c) where the mode first goes overstable. In order to calculate the complex growth rate of overstable modes we plot $\psi_{2/1}(t)$ at two arbitrary radial positions. Figure 9 is such a plot clearly showing the oscillatory behavior of the mode. To get accurate estimates of γ and ω we must run a minimum of one full period and preferably more. However, near η_c we have ω small, which requires large amounts of computer time when using this approach. Thus, in Fig. 8 we display a cross-hatched region where all modes to the right (larger η) are purely growing and all modes to the left (smaller η) are overstable. We also solve the full, incompressible MHD equations using a shooting technique.¹¹ We expect the compressible and incompressible results to converge at lower η . Indeed we see

that η_c from the shooting technique does lie within the cross-hatched region. It is interesting to note that the reduced equations and the full equations give significantly different results from the boundary layer approach. This discrepancy is greater than in any other resistive calculation and shows no sign of improvement with decreasing resistivity. The matching code result gives an appreciably different η_c (Fig. 10). There are several possible reasons for this discrepancy between the boundary layer approach and other methods of solving for the complex growth rate. First, the resistivity may not be small enough to be in the asymptotic regime. However, one would expect the agreement to get better as the resistivity decreased but this does not seem to be the case. Secondly, the numerical method used to apply the boundary layer analysis is difficult and may be at fault, although all tests for convergence indicate it is working properly. Thirdly, there may be something wrong with the layer equations themselves when applied to this pressure stabilized mode. In fact it appears that all terms of the appropriate order in η may not have been kept. Higher order terms are included when the matching to the exterior region is done but no higher order terms are kept in the layer equations themselves. If this were true, the qualitative agreement could be explained by the fact that the equations are correctly ordered for very small and very large resistivity (small and large compared to η_c). In any case, the importance of this finding is that tokamaks have average good curvature and the resistive PEST code PEST-III¹² uses the same boundary layer technique used here to solve for stabilizing values of resistivity. Therefore, it is important to have another approach, such as the initial value approach used here, to verify calculations of resistive growth rates in toroidal tokamaks.

In Fig. 11 we display the perturbed variables as produced by HILO. Once again we note that they are nearly identical to those generated by the full MHD equations. For simplicity we display only the one result. A clear distinction from the ideal results is that now $v_z \geq v_1$. This implies that the toroidal flow is an essential (leading order) ingredient to model accurately the effects of finite β on resistive modes. We now note that such was also true for the destabilizing pressure profile discussed earlier. In order to compute v_z it was necessary to include inertia in the elliptic equations used to compute I . Thus, the fifth order equations indeed include all the relevant physics needed to study finite-beta resistive instabilities in a finite aspect ratio tokamak.

IV. CONCLUSIONS

A reduced set of resistive MHD equations have been derived and coded to study any finite aspect ratio, finite- β tokamak. The equations have been verified in a cylinder by comparison with boundary layer theory and the full MHD equations. Excellent agreement has been obtained with the full equations for both ideal and resistive studies. The equations also converge to the resistive asymptotic theory for zero beta and destabilizing pressure profiles as the resistivity is decreased. Quantitative discrepancies exist for stabilizing pressure profiles which are intended to simulate the effects of average good curvature. However, the predicted qualitative behavior has been observed both for $\Gamma = 5/3$ and $\Gamma = 0$ for sufficiently small resistivity. Overstable modes have been obtained when $\Gamma = 5/3$ and purely growing modes for $\Gamma = 0$.

These comparisons of our reduced equations with the full MHD equation in the linear regime have served the purpose of providing confidence that the 5th

order reduced equations do indeed contain all of the relevant physics for MHD stability studies. The power of these equations is that they can be integrated in time in a torus almost as easily as in a cylinder, i.e., without serious time step restrictions. Equations similar to ours have been derived by other workers in the community.¹³⁻¹⁵ We will proceed from here to study the linear and nonlinear development of resistive instabilities in finite-beta, finite aspect ratio tokamaks.

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REFERENCES

- ¹A. H. Glasser, J. M. Greene, and J. L. Johnson, *Phys. Fluids* 8, 875, *Phys. Fluids* 19, 567 (1967).
- ²R. Izz., D. A. Monticello, H. R. Strauss, W. Park, M. Manickam, R. C. Grimm, and J. DeLucia, *Phys. Fluids* 26, 3066 (1983).
- ³W. Park, D. A. Monticello, R. B. White, K. McGuire, and M. Reusch, in *Proceedings of the U.S.-Japan Theory Workshop on 3-D MHD Studies for Toroidal Devices* (Oak Ridge National Laboratory, Oak Ridge, TN, May 1981).
- ⁴M. N. Rosenbluth, D. A. Monticello, H. R. Strauss, and R. B. White, *Phys. Fluids* 19, 1987 (1976); H. R. Strauss *ibid.* 19, 134 (1976).
- ⁵J. A. Holmes, B. A. Carreras, T. C. Hender, H. R. Hicks, V. E. Lynch, and B. F. Masden, *Phys. Fluids* 26, 2569 (1983).
- ⁶R. Izzo, D. A. Monticello, W. Park, J. Manickam, H. R. Strauss, R. C. Grimm, and K. McGuire, *Phys. Fluids* 26, 2240 (1983).
- ⁷J. DeLucia, S. C. Jardin, and A. H. Glasser, *Phys. Fluids* 27, 1470 (1984).
- ⁸H. P. Furth, P. H. Rutherford, and H. Selberg, *Phys. Fluids* 16, 1054 (1973).
- ⁹B. Coppi, J. M. Greene, and J. L. Johnson, *Nucl. Fusion* 6, 101 (1966).
- ¹⁰H. P. Furth, J. Killeen, and M. N. Rosenbluth, *Phys. Fluids* 6, 459 (1963).
- ¹¹C. M. Ryu and R. C. Grimm, Princeton Plasma Physics Laboratory Report PPPL-2020, July 1983.
- ¹²R. C. Grimm, R. L. Dewar, J. Manickam, S. C. Jardin, A. H. Glasser, and M. S. Chance, *IAEA International Conference on Plasma Physics and Controlled Nuclear Fusion*, 1982, IAEA-CN-41/P-2-1.
- ¹³M. S. Chu, T. H. Jensen, and J. S. Kim, *Phys. Fluids* 27, 1194 (1984).

- ¹⁴L. A. Charlton, B. A. Carreras, J. A. Holmes, H. R. Hicks, and V. E. Lynch,
in Proceedings of U.S.-Japan Workshop on 3-D MHD studies (Oak Ridge
National Laboratory, Oak Ridge, TN, March 1984).
- ¹⁵J. F. Drake and T. M. Antonsen, Phys. Fluids 27, 898 (1984).

APPENDIX A

To show that no waves propagate perpendicular to the magnetic surfaces we write out the equations for the propagation of small disturbances. We assume a uniform, static equilibrium and take a time and space dependence of $\exp[i(\vec{k} \cdot \vec{r} - \omega t)]$ for all perturbed quantities. The linearized, ideal MHD equations then give

$$-\omega^2 \vec{Q} + (s^2 + v_A^2)(\vec{k} \cdot \vec{v})\vec{k} + (\vec{v}_A \cdot \vec{k})[(\vec{k} \cdot \vec{v})\vec{v} - (\vec{v}_A \cdot \vec{v})\vec{k}] - (\vec{k} \cdot \vec{v})\vec{v}_A = 0 \quad (A-1)$$

Here $\vec{Q} \approx \vec{v}$ if the full momentum equation [Eq. (1)] is used and $\vec{Q} \approx \vec{v}_\perp$ if instead Eq. (27) is used. We then proceed to take the dot product of Eq. (A-1) with $\vec{k}$, $\vec{v}_A$, and $\vec{k} \times \vec{v}_A$. If we set $\vec{Q} = \vec{v}$, we obtain the usual phase velocities

$$v_p^2 = \frac{s^2 v_A^2 \pm [(s^2 + v_A^2) - 4s^2 v_A^2 \cos^2 \theta]^{1/2}}{2}$$

$$v_p^2 \approx v_A^2 \cos^2 \theta$$

where

$$\vec{v}_A = \frac{\vec{B}}{\sqrt{\rho}}, \quad s^2 = \frac{\Gamma p_0}{\rho}$$

and

$$\cos \theta = \frac{\vec{k} \cdot \vec{B}}{|\vec{k}| |\vec{B}|}$$

For the special case, $\vec{k} \cdot \vec{B} = 0$, we obtain

$$v_p^2 = \begin{Bmatrix} s_A^2 + v_A^2 \\ 0 \\ 0 \end{Bmatrix}.$$

This is easily recognized as the usual fast magnetosonic wave propagating perpendicular to the magnetic field.

If we now consider Eq. (A-1) with $\vec{Q} = \vec{v}_I = \vec{v} - \vec{v}_I \chi$, we find there are only zero frequency waves traveling perpendicular to the magnetic surfaces, i.e., with $\vec{k} \perp \vec{B}$ and $\vec{k} \perp \vec{v}_I$. By writing

$$\vec{v} = R^2 \vec{v}_u \times \vec{v}_I + R v_I \vec{v}_I + \vec{v}_I \chi$$

we note that the first and second terms are each perpendicular to $\vec{k}$ while the third term is along $\vec{k}$. Thus we may express the velocity as

$$\vec{v} = \alpha \vec{v}_A + \beta \vec{k} \times \vec{v}_A + i \chi \vec{k}.$$

We may then write

$$\vec{v}_I = \vec{v} - \vec{v}_I \chi = \alpha \vec{v}_A + \beta \vec{k} \times \vec{v}_A.$$

Proceeding as before, we take the dot product of Eq. (A-1) with $\vec{k}$, $\vec{v}_A$, and $\vec{k} \times \vec{v}_A$. We obtain

$$(s^2 + v_A^2) i k^4 \chi = 0$$

$$- \omega^2 \alpha v_A^2 = 0$$

$$-\omega^2 \beta k^2 v_A^2 = 0 \quad .$$

Therefore, the desired result that only

$$v_P^2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

remains.

APPENDIX B

The linearized equations of motion are given for a time dependence of $e^{\gamma t}$ by

$$\rho \gamma \vec{Q} = (\vec{v} \times \vec{b}) \times \vec{B} + \vec{J} \times \vec{B} + \vec{v} [P(\vec{v} \cdot \vec{v}) + \frac{\vec{v} \cdot \vec{v}_P}{\gamma}] \quad (B-1)$$

and

$$\vec{b} - \frac{\eta}{\gamma} \vec{v} \times (\vec{v} \times \vec{b}) = \vec{v} \times \frac{(\vec{v} \times \vec{B})}{\gamma} \quad (B-2)$$

where equilibrium quantities are capitalized and perturbed quantities are written in lower case. Here $\vec{Q} = \vec{v}$ if Eq. (1) is used and $\vec{Q} = \vec{V}_I$ if Eq. (29) is used. If $\vec{Q} = \vec{v}$, Eqs. (B-1) and (B-2) are the same as those used by Glasser, Greene, and Johnson in their derivation of the layer equations.¹ The derivation is accomplished by expanding all quantities in power of $\gamma \sim \eta^{1/3} \ll 1$. However, only two components of $\vec{Q}$ are used in the derivation. These components are Q_B and Q_L where

$$\vec{Q} = Q_\psi \frac{\vec{v} \times \vec{\psi}}{|\vec{v} \times \vec{\psi}|^2} + Q_L \frac{\vec{B} \times \vec{v} \times \vec{\psi}}{|\vec{B}|^2} + Q_B \frac{\vec{B}}{|\vec{B}|^2}$$

and $\vec{B} = \vec{v} \times \vec{\psi} + g(\psi) \vec{v} \times \vec{\zeta}$. We will show that to leading order $v_B = v_{IB}$ and $v_L = v_{IL}$. Thus, the layer equations that result from Eqs. (B-1) and (B-2) are to leading order independent of whether $\vec{Q} = \vec{v}$ or $\vec{Q} = \vec{V}_I$.

We proceed by first showing that $\chi \sim \gamma^2$ if v_L and v_B are taken of order 1 and v_ψ is taken as order γ as in Ref. 1. Working in a ψ, θ, ζ flux coordinate system, we have

$$v_\psi = R^2 \frac{\partial u}{\partial \theta} \hat{\psi}_\theta \times \hat{\psi}_\zeta + \hat{\psi}_\psi + \frac{\partial \chi}{\partial \psi} |\hat{\psi}_\psi|^2 .$$

But from Ref. 1 we also have $\partial/\partial\psi \sim O(1/\gamma)$ while $\partial/\partial\theta \sim \partial/\partial\zeta \sim O(1)$ and $u \sim O(\gamma)$. Therefore, if $v_\psi \sim O(\gamma)$, we must have $\chi \sim O(\gamma^2)$. Next we write

$$v_{I_B} = v_B - \hat{\psi}_\perp \chi \cdot \hat{B}$$

and

$$\hat{\psi}_\perp \chi \cdot \hat{B} = \frac{\partial \chi}{\partial \theta} \hat{\psi}_\theta \cdot \hat{\psi}_\zeta \times \hat{\psi}_\psi \sim O(\gamma^2) .$$

Therefore to leading order $v_{I_B} = v_B$. Finally,

$$v_{I_\perp} = v_\perp - \hat{\psi}_\perp \chi \cdot \frac{\hat{B} \times \hat{\psi}_\psi}{|\hat{\psi}_\psi|^2}$$

and

$$\hat{\psi}_\perp \chi \cdot \frac{\hat{B} \times \hat{\psi}_\psi}{|\hat{\psi}_\psi|^2} = \frac{g}{|\hat{\psi}_\psi|^2} \frac{\partial \psi}{\partial \theta} \hat{\psi}_\theta \cdot \hat{\psi}_\zeta \times \hat{\psi}_\psi \sim O(\gamma^2)$$

which gives the desired result $v_{I_\perp} = v_\perp$ to leading order.

FIGURE CAPTIONS

FIG. 1. Toroidal coordinate system (r, θ, ζ) used in the initial value code HILO.

FIG. 2. Comparison of linear growth rates for an ideal, $m/n = 2/1$, interchange mode computed using the full MHD equations (Shooting Code) and the reduced MHD equations (HILO). The equilibrium is described by $q = 1.6(1 + r^2/0.64)$; $\beta = \beta_0(1 - 3r^2 + 2r^3)$; $\epsilon = 0.1$; and β_0 varying.

FIG. 3. Comparison of linear growth rates for an ideal, $m/n = 2/1$, interchange mode computed using the full MHD equations (Shooting Code) and the reduced MHD equations (HILO). The equilibrium q and pressure profile are the same as in Fig. 2 but here $\beta_0 = 0.2$ is fixed and the aspect ratio is allowed to vary.

FIG. 4. Eigenfunctions for a typical ideal, $m/n = 2/1$, interchange mode produced by the reduced equations. Results from the full equations are identical.

FIG. 5. Comparison of linear growth rates for a zero-beta, resistive tearing mode computed using the full MHD equations (dots), the reduced equations (solid line), and the analytic results of Ref. 7 showing the $\eta^{3/5}$ scaling (dashed line). While the reduced equations and full agree for all η , agreement with the analytic theory improves with decreasing η . The q -profile is given in Fig. 2. The aspect ratio is fixed at ten.

FIG. 6. Comparison of linear growth rates for a finite-beta, resistive interchange mode computed using the reduced equations (solid line) and a boundary layer-matching analysis (dashed line). The q and pressure profile are given in Fig. 2 with two different central beta values ($\beta_0 = 0.03; 0.05$) displayed here. The aspect ratio is set equal to 2.

FIG. 7. Comparison of linear growth rates for a finite-beta resistive interchange mode computed using the reduced equations (HILO) and a boundary layer-matching analysis (Match Code). The q and pressure profiles are given in Fig. 2 with $\beta_0 = 0.06$. The aspect ratio is set equal to 10.

FIG. 8. Computed linear growth rates for a resistive tearing mode under the influence of a stabilizing pressure profile. The q -profile is as in Fig. 2 but the pressure is given by $\beta = 0.5 (0.001 + 0.28 r^2 - 0.059 r^4 + 0.03 r^6)$. The aspect ratio is set equal to five. Four results are displayed for comparison: from boundary layer equations (Match); from the full, incompressible MHD equations (Shooting); and from the reduced equations for both $\Gamma = 0$ and $\Gamma = 5/3$ (HILO). The $\square$ at the bend in the two curves denote the η at which the mode becomes overstable.

FIG. 9. The $m/n = 2/1$ harmonic of the poloidal flux function versus time showing the expected growing oscillating behavior for an overstable mode.

FIG. 10. Linear frequency for overstable modes produced by finite-beta stabilization of resistive tearing modes. The equilibrium is the same as for Fig. 8. The results from the reduced equations (HILO) are shown as two points. The $\square$ in Fig. 8 correspond to the resistivity at which the curves leave the x-axis here.

FIG. 11. Eigenfunctions for a typical overstable $m/n = 2/1$ tearing mode generated by the reduced equations. Integrating the full MHD equations produces identical results.

#84P0261

Fig. 1

Fig. 2

Fig. 3

#84P0147

Fig. 4

#84P0107

Fig. 4

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

#84P0262

Fig. 10

#84P0145

Fig. 11

Fig. 11

Fig. 11

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