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1

CONSTRUCTION OF THE FUNCTIONAL INTEGRAL REPRESENTATION
FOR FERMION GREEN'S FUNCTIONS*

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Abstract

A direct derivation is given of the functional integral representation for the Green's functions of quantum theories built from a finite number of canonical fermion operators Q_α, Q_α^+ with $\{Q_\alpha, Q_\beta^+\} = \delta_{\alpha\beta}$.

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MASTER

The Green's functions of a quantum field theory are often represented as functional integrals. For example, in a theory with a scalar field ϕ and a Dirac field $\Psi(x)$, one has

$$\langle 0 | T \{ \Psi_\alpha(x) \bar{\Psi}_\beta(y) \phi(z) \} | 0 \rangle = Z^{-1} \int d[\phi] d[\eta] d[\bar{\eta}] e^{iS[\phi, \bar{\eta}, \eta]} \eta_\alpha(x) \bar{\eta}_\beta(y) \phi(z), \quad (1)$$

where

$$S[\phi, \bar{\eta}, \eta] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{M^2}{2} \phi^2 + \bar{\eta} (i \not{\partial} + m) \eta \right] + \mathcal{L}_I(\phi, \bar{\eta}, \eta)$$

and Z is a normalization factor. The integral is over all classical field configurations $\phi(x)$ of the boson field and over anticommuting field variables $\eta(x)$ and $\bar{\eta}(x)$ at each point in space-time for the fermion field.

Integral representations such as (1) have proved to be useful for performing formal manipulations on the theory, including the derivation in a simple way of the Feynman rules for perturbation theory. In the case of boson fields, the integral representation is also useful as the starting point for various semiclassical approximations.

In most interesting field theories, $\mathcal{L}_I$ is quadratic in the fermion field variables. Thus in many applications one can perform the η and $\bar{\eta}$ integrations exactly. This leaves only the ϕ integral, which is susceptible to approximation schemes. Indeed, the paper¹ that first introduced the fermion functional integral also showed how to eliminate it in this way.

The derivation of the functional integral representation for quantum theories with a finite number of canonical boson operators ϕ_α, Π_α with $[\phi_\alpha, \Pi_\beta] = i\delta_{\alpha\beta}$ is well known and very simple². From there, one passes easily to the limit of an infinite number of field operators and thus to a boson quantum field theory. (That is, this step is easy if one assumes the existence of this limit.)

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The functional integral representation for a quantum theory with a finite number of canonical fermion operators $Q_\alpha, Q_\alpha^\dagger$ with $\{Q_\alpha, Q_\beta^\dagger\} = \delta_{\alpha\beta}$ is also well known. It has been established by showing that it leads to the Schwinger action principle¹, that it solves the field equations of motion written as functional differential equations for the generating function of Green's functions, or that it leads to the correct Feynman rules for perturbation theory. In this paper, another derivation, which is rather similar to the derivation in the boson case, is provided. The aim of the paper is to base the construction of the fermion functional integral representation directly on the underlying quantum mechanics, avoiding use of the field theoretical superstructure used in other derivations. One practical advantage of this method is that the operator ordering in the Hamiltonian corresponding to a given path integral is specified. In addition, a precise definition of the path integral as the limit of its lattice approximation as the lattice spacing Δt tends to zero is given, and the existence of the limit (for a finite number of Q_α) is proved.

One can easily combine the boson and fermion constructions so as to cover the case of a finite number of boson and fermion operators, then pass to the limit of an infinite number of operators and thus to a general quantum field theory. In the interest of simplicity, these steps are not discussed in this paper.

The objects for which we construct a functional integral representation are the finite temperature, imaginary time Green's functions $\text{Tr}\{e^{-\beta H} T[Q_\beta(-it_B) Q_\alpha^\dagger(-it_A) \dots]\} / \text{Tr}\{e^{-\beta H}\}$. These Green's functions are directly relevant to statistical mechanics. To obtain the usual Euclidean Green's functions used in particle physics, one simply takes the limit $\beta \rightarrow \infty$. This leaves only the contribution from the vacuum state, $\langle \text{vac} | T[Q_\beta(-it_B) Q_\alpha^\dagger(-it_A) \dots] | \text{vac} \rangle$. Finally, if one wants the real time

(Minkowski space) Green's functions, one analytically continues back to real times using the Wick rotation $-it \rightarrow e^{-i\theta} t \rightarrow t$ with $\pi/2 \geq \theta > 0$. The resulting Green's function can be written formally as a functional integral with $i \int dt$ in place of $\int dt$ and $-i\partial/\partial t$ in place of $\partial/\partial t$ but, in addition to making these replacements, one must remember the direction of the Wick rotation by occasionally inserting $i\epsilon$ terms in calculations.

We begin the construction by considering a quantum system described by a set of operators Q_α with $\alpha = 1, \dots, d$ and their adjoints $Q_\alpha^\dagger$. The operators are assumed to satisfy the canonical anticommutation relations

$$\begin{aligned} \{Q_\alpha, Q_\beta^\dagger\} &= \delta_{\alpha\beta} \\ \{Q_\alpha, Q_\beta\} &= \{Q_\alpha^\dagger, Q_\beta^\dagger\} = 0. \end{aligned} \quad (2)$$

The Hilbert space describing the states of the system has the usual Fock basis

$$\begin{aligned} |0\rangle \\ |\alpha\rangle &= Q_\alpha^\dagger |0\rangle \\ |\alpha, \beta\rangle &= Q_\alpha^\dagger Q_\beta^\dagger |0\rangle \\ &\vdots \\ |1, 2, \dots, d\rangle &= Q_1^\dagger Q_2^\dagger \dots Q_d^\dagger |0\rangle \end{aligned} \quad (3)$$

with $Q_\alpha |0\rangle = 0$. Of course, the ground state of the system need not be the state $|0\rangle$.

We consider the most general possible self adjoint Hamiltonian, H , for the system, subject to the condition that H be unchanged under the transformation $Q \rightarrow Q, Q^\dagger \rightarrow Q^\dagger$. We choose to write H in the form of a normal ordered polynomial

$$H = H(Q, Q) = A + B^{\alpha\beta} Q_\alpha Q_\beta + C^{\alpha\beta} Q_\alpha^\dagger Q_\beta^\dagger + D^{\alpha\beta} Q_\alpha^\dagger Q_\beta + E^{\alpha\beta\gamma\delta} Q_\alpha Q_\beta Q_\gamma Q_\delta + F^{\alpha\beta\gamma\delta} Q_\alpha^\dagger Q_\beta^\dagger Q_\gamma Q_\delta + \dots$$

(The choice of which operators of the theory are Q 's and which are $Q^\dagger$'s is arbitrary, but once we have made this choice we stick to it and write $H(Q^\dagger, Q)$ with all of the Q 's to the right of the $Q^\dagger$'s. In the case of a Dirac field, the most convenient choice is to let each component of $\Psi_\rho(\vec{x}, 0)$ at each point in space be a Q_α .)

Our object is to write a functional integral representation for the finite temperature, imaginary time Green's functions

$$G(t_L, \dots, t_A) = Z^{-1} \text{Tr} \{ e^{-\beta H} T[Q_\lambda(-it_L) \dots Q_\alpha(-it_A)] \},$$

where $Z = \text{Tr} \{ e^{-\beta H} \}$, T denotes time ordering, and it is assumed that

$-\beta/2 < t_n < \beta/2$.⁴ In order to keep the notation simple, we will consider a definite example, the two point function for $t_B > t_A$:

$$G_{\beta\alpha}(t_B, t_A) = Z^{-1} \text{Tr} \{ e^{-\frac{\beta}{2} H} Q_\beta e^{-(t_B - t_A) H} Q_\alpha^\dagger e^{-\frac{\beta}{2} H} \}.$$

(4)

Let us divide the "time" interval $(-\frac{\beta}{2}, \frac{\beta}{2})$ into a large number, N , of small intervals (t_j, t_{j-1}) with $-\frac{\beta}{2} = t_0 < t_1 < t_2 < \dots < t_N = +\frac{\beta}{2}$. We choose the lattice points t_j so that two of them fall at the values $t_{j_A} = t_A$ and $t_{j_B} = t_B$. The evolution operators can then be written as products of small interval evolution operators $\exp[-(t_j - t_{j-1})H]$.

We now make the essential approximation of the present derivation. We replace $\exp[-\Delta t_j H]$ by $:\exp[-\Delta t_j H]:$, where $::$ denotes normal ordering. This gives the "lattice Green's function"

$$\tilde{G} = Z^{-1} \text{Tr} \{ : e^{-H\Delta t_N} : \dots : Q_\beta : \dots : Q_\alpha^\dagger : \dots : e^{-H\Delta t_2} : : e^{-H\Delta t_1} : \} \quad (5)$$

where $\tilde{Z} = \text{Tr} \{ : e^{-H\Delta t_N} : \dots : e^{-H\Delta t_1} : \}$. Since $H(Q^\dagger, Q)$ is already normal ordered, the normal ordering instruction in $:\exp[-\Delta t_j H]:$ affects only the terms of second or higher order in the small quantity Δt_j . Thus the approximate Green's function $\tilde{G}$ will approach the exact Green's function G as the maximum lattice spacing Δt_j tends to zero⁵.

The rest of the analysis is purely algebraic: we write an exact representation of the lattice Green's function G as an "integral" over a set of anticommuting objects $\eta_\alpha(J), \eta_\alpha^*(J)$ with $\alpha = 1, \dots, d$ and $J = 1, \dots, N$. Each of these objects anticommutes with each other object; they are the generators of the Grassmann algebra of order $2Nd$.

Let us briefly review⁶ the definition of integration on a Grassmann algebra $\mathcal{H}_{2Nd}$. For this purpose we denote the generators by $Z_1, Z_2, \dots, Z_{2Nd}$. The algebra $\mathcal{H}_{2Nd}$ consists of all polynomials

$$P(Z) = A + B^j Z_j + C^{jk} Z_j Z_k + \dots + L Z_1 Z_2 \dots Z_{2Nd} \quad (6)$$

together with self-evident rules for addition and multiplication of polynomials (using $Z_j Z_k + Z_k Z_j = 0$). Differentiation on $\mathcal{H}_{2Nd}$ is defined by

$$\frac{\partial}{\partial Z_j} Z_k = \delta_k^j, \quad \frac{\partial}{\partial Z_j} 1 = 0$$

together with the rules

$$\frac{\partial}{\partial Z_j} [aP(Z) + bQ(Z)] = a \frac{\partial}{\partial Z_j} P(Z) + b \frac{\partial}{\partial Z_j} Q(Z)$$

$$\frac{\partial}{\partial Z_j} Z_k F(Z) = \delta_k^j F(Z) - Z_k \frac{\partial}{\partial Z_j} F(Z).$$

The standard definition of "integration" on $\mathcal{H}_{2Nd}$ can be stated very simply:

$$\int dZ_j P(Z) \equiv \frac{\partial}{\partial Z_j} P(Z)$$

(It is only for historical reasons that one uses the integration symbol for this operation.) In particular, if one integrates $P(Z)$, Eq. (6), over all of the anticommuting variables, one obtains an ordinary number, the coefficient of the highest order term in $P(Z)$:

$$\int dZ_{2Nd} \dots dZ_1 P(Z) = L.$$

We make use of the Grassmann algebra by mapping the space of states of the quantum system onto the space of polynomials $P(\eta^*)$ in N anticommuting variables η_α^* . Let the basis states $|0\rangle, |\alpha\rangle, |\alpha, \beta\rangle, \dots$ correspond to the monomials

$$\begin{aligned}
|0\rangle &\leftrightarrow 1 \\
|\alpha\rangle &\leftrightarrow \eta_\alpha^* \\
|\alpha, \beta\rangle &\leftrightarrow \eta_\alpha^* \eta_\beta^* \\
&\vdots \\
|1, 2, \dots, N\rangle &\leftrightarrow \eta_1^* \eta_2^* \dots \eta_N^*
\end{aligned} \quad (7)$$

More generally, let the polynomial $F(\eta^*)$ correspond to the state $P(\eta^*)|0\rangle$.

Now consider a quantum operator F , which we write as a normal ordered polynomial

$$F(Q^\dagger, Q) = A + B^\alpha Q_\alpha + \dots + L Q_1^\dagger \dots Q_N^\dagger Q_1 \dots Q_N.$$

The corresponding operator on the space of polynomials $P(\eta^*)$ can be written as an integral operator: if $FP(Q^\dagger)|0\rangle = \hat{P}(Q^\dagger)|0\rangle$, then

$$\hat{P}(\eta^*) = \int \left(\prod_{\alpha=1}^d d\eta_\alpha^* d\eta_\alpha \right) e^{\sum_{\alpha} (\eta_\alpha^* - \eta_\alpha) \eta_\alpha} \hat{F}(\eta^*, \eta) P(\eta^*). \quad (8)$$

The proof of Eq. (8) is straightforward and is given in Appendix A.

Evidently the relation (8) can be used repeatedly to write a product of operators,

$$F = F_N(Q^\dagger, Q) \dots F_2(Q^\dagger, Q) F_1(Q^\dagger, Q),$$

as an integral operator on polynomials $P(\eta^*)$, in the form of Eq. (8). The combined kernel $F(\eta^*, \eta)$ then corresponds to the normal ordered form $F(Q^\dagger, Q)$ of the operator F . One finds that $FP(Q^\dagger)|0\rangle = \hat{P}(Q^\dagger)|0\rangle$ where

$$\hat{P}[\eta^*(N)] = \int \left(\prod_{\alpha} d\eta_\alpha^*(0) d\eta_\alpha(1) \right) e^{[\eta^*(N) - \eta^*(0)] \eta(1)} F[\eta^*(N), \eta(1)] P[\eta^*(0)]$$

and the new kernel $F(\eta^*, \eta)$ is

$$\begin{aligned}
F[\eta^*(N), \eta(1)] &= \int \left(\prod_{J=1}^{N-1} \prod_{\alpha} d\eta_\alpha^*(J) d\eta_\alpha(J+1) \right) e^{\sum_{J=2}^N [\eta^*(J) - \eta^*(J-1)] \eta(J)} \\
&\quad e^{[\eta^*(1) - \eta^*(N)] \eta(1)} F_N[\eta^*(N), \eta(N)] \dots F_1[\eta^*(1), \eta(1)] \quad (9)
\end{aligned}$$

Here we have used the notation

$$\eta^*(J') \eta(J) \equiv \sum_{\alpha} \eta_{\alpha}^*(J') \eta_{\alpha}(J). \quad (10)$$

To complete the construction, we need to write the trace of F as an integral. The relevant relation is

$$\text{Tr}\{F(Q^\dagger, Q)\} = \int \left(\prod_{\alpha} d\eta_{\alpha}^* d\eta_{\alpha} \right) e^{2 \sum_{\alpha} \eta_{\alpha}^* \eta_{\alpha}} F(\eta^*, \eta) \quad (11)$$

for a normal ordered polynomial $F(Q^\dagger, Q)$. This relation is proved in Appendix B.

When the relations (9) and (11) are combined, one obtains

$$\begin{aligned}
\text{Tr}\{F_N(Q^\dagger, Q) \dots F_1(Q^\dagger, Q)\} \\
= \int d\eta d\eta^* e^{-\sum_{J=1}^N \eta^*(J) [\eta(J+1) - \eta(J)]} F_N[\eta^*(N), \eta(N)] \dots F_1[\eta^*(1), \eta(1)], \quad (12)
\end{aligned}$$

where

$$\int d\eta d\eta^* \equiv \int \prod_{J=1}^N \prod_{\alpha=1}^d d\eta_{\alpha}(J) d\eta_{\alpha}^*(J)$$

and where, when $\eta(N+1)$ occurs in the exponent, it means

$$\eta(N+1) = -\eta(1) \quad (13)$$

The integral representation (12) for the trace of a product of operators can now be applied to the lattice Green's function (5) with the result

$$\tilde{G}_{\beta\alpha} = \tilde{Z}^{-1} \int d\eta \, d\eta^* e^{-S[\eta^*, \eta]} \eta_{\beta}(J_B+1) \eta_{\alpha}^*(J_A)$$

$$Z = \int d\eta \, d\eta^* e^{-S[\eta^*, \eta]}, \quad (14)$$

where

$$S(\eta^*, \eta) = \sum_{J=1}^N \Delta t_J \{ \eta^*(J) \frac{\eta(J+1) - \eta(J)}{\Delta t_J} + H[\eta^*(J), \eta(J)] \}.$$

In the limit in which the maximum time interval Δt_J goes to zero, the lattice Green's function $\tilde{G}$ approaches the continuum Green's function G . It is suggestive to write G in the standard form

$$G_{\beta\alpha}(t_B, t_A) = Z^{-1} \int d\eta \, d\eta^* e^{-S[\eta^*, \eta]} \eta_{\beta}(t_B) \eta_{\alpha}^*(t_A)$$

$$Z = \int d\eta \, d\eta^* e^{-S[\eta^*, \eta]} \quad (15)$$

where

$$S(\eta^*, \eta) = \int_{-\beta/2}^{\beta/2} dt \{ \eta^*(t) \frac{d}{dt} \eta(t) + H[\eta^*(t), \eta(t)] \}.$$

A few comments are in order:

(i) The functional integrals in Eq. (15) are defined as the $\Delta t \rightarrow 0$ limits of the finite dimensional integrals in Eq. (14). We have proved⁵ that this limit exists.

(ii) The representation (15) was derived under the assumption that $t_B > t_A$, but the same representation evidently applies for $t_A < t_B$.

(iii) The integral representations for Green's functions with more operators $Q_Y(-it_C) Q_{\delta}^{\dagger}(-it_D) \dots$ have additional factors $\eta_Y(t_C) \eta_{\delta}^*(t_D) \dots$. All of these Green's functions can be obtained by differentiating the generating functional

$$Z(J, J^*) = Z^{-1} \int d\eta \, d\eta^* e^{-S[\eta^*, \eta] + \int dt [\eta^*(t) J(t) + J^*(t) \eta(t)]},$$

where J and J^* are additional anticommuting variables.

(iv) The antiperiodic boundary condition (13) implies that the operator $\frac{d}{dt}$ in the continuum action should be defined with antiperiodic boundary conditions and that the Green's functions are antiperiodic:

$$G(t_L, \dots, t_C, +\frac{\beta}{2}, t_A) = -G(t_L, \dots, t_C, -\frac{\beta}{2}, t_A).$$

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Appendix A: Proof of Eq. (8)

Let $F(Q^\dagger, Q)$ be an operator written in the form of a normal ordered polynomial and let $F(Q^\dagger, Q)P(Q^\dagger)|0\rangle = \hat{P}(Q^\dagger)|0\rangle$. Define $R(Q^\dagger)$ by

$$R(\hat{\eta}^*) = \int \left(\prod_\mu d\eta_\mu^* \right) e^{\sum_\nu (\hat{\eta}_\nu^* - \eta_\nu^*) \eta_\nu} F(\hat{\eta}^*, \eta) P(\eta^*).$$

We seek to prove that $R(\hat{\eta}) = \hat{P}(\hat{\eta})$.

It suffices to prove this in the case that F and P are monomials

$$F(Q^\dagger, Q) = Q_{C_1}^\dagger \cdots Q_{\alpha_A}^\dagger Q_{\beta_B} \cdots Q_{\beta_1}$$

$$P(Q^\dagger) = Q_{\gamma_1}^\dagger \cdots Q_{\gamma_C}^\dagger$$

Then

$$R(\hat{\eta}^*) = \hat{\eta}_{\alpha_1}^* \cdots \hat{\eta}_{\alpha_A}^* \int \left(\prod_\mu d\eta_\mu^* \right) \prod_\rho (1 + \eta_\rho^* \eta_\rho) \prod_\sigma (1 + \hat{\eta}_\sigma^* \eta_\sigma) \eta_{\beta_E} \cdots \eta_{\beta_1} \eta_{\gamma_1}^* \cdots \eta_{\gamma_C}^*.$$

Evidently $R(\hat{\eta}^*) = 0$ if the values of the indices β are not a subset of the values of the indices γ . In this case $\hat{P}(Q^\dagger) = 0$ also, so $\hat{P} = R$. In the contrary case $\{\beta_j\} \subset \{\gamma_j\}$, one can assume without loss of generality that $\beta_1 = \gamma_1, \beta_2 = \gamma_2, \dots, \beta_B = \gamma_B$. Then

$$\begin{aligned} R(\hat{\eta}^*) &= \hat{\eta}_{\alpha_1}^* \cdots \hat{\eta}_{\alpha_A}^* \int \left(\prod_\mu d\eta_\mu^* \right) \prod_\rho (1 + \eta_\rho^* \eta_\rho) \\ &\quad \eta_{\gamma_1}^* \cdots \eta_{\gamma_B}^* \eta_{\gamma_{B+1}}^* \eta_{\gamma_{B+1}} \eta_{\gamma_{B+1}}^* \cdots \eta_{\gamma_C}^* \eta_{\gamma_C} \eta_{\gamma_C}^* \\ &= \hat{\eta}_{\alpha_1}^* \cdots \hat{\eta}_{\alpha_A}^* \eta_{\gamma_{B+1}}^* \cdots \hat{\eta}_{\gamma_C}^* \int \left(\prod_\mu d\eta_\mu^* \right) \prod_\rho (1 + \eta_\rho^* \eta_\rho) \eta_{\gamma_1}^* \cdots \eta_{\gamma_C}^* \eta_{\gamma_C} \eta_{\gamma_C}^* \end{aligned}$$

The remaining integral equals 1, so

$$R(\hat{\eta}^*) = \hat{\eta}_{\alpha_1}^* \cdots \hat{\eta}_{\alpha_A}^* \hat{\eta}_{\gamma_{B+1}}^* \cdots \hat{\eta}_{\gamma_C}^*.$$

In this case,

$$\begin{aligned} F(Q^\dagger, Q)P(Q)|0\rangle &= Q_{\alpha_1}^\dagger \cdots Q_{\alpha_A}^\dagger Q_{\gamma_B} \cdots Q_{\gamma_1} Q_{\gamma_1}^\dagger \cdots Q_{\gamma_C}^\dagger |0\rangle \\ &= Q_{\alpha_1}^\dagger \cdots Q_{\alpha_A}^\dagger Q_{\gamma_{B+1}}^\dagger \cdots Q_{\gamma_C}^\dagger |0\rangle. \end{aligned}$$

Thus $\hat{P} = R$ in this case also, and the theorem is proved.

Appendix B: Proof of Eq. (11)

Let $F(Q^\dagger, Q)$ be an operator written in the form of a normal ordered polynomial and define

$$I = \int \left(\prod_{\mu=1}^d d\eta_\mu d\eta_\mu^* \right) e^{\sum_\nu 2\eta_\nu^* \eta_\nu} F(\eta^*, \eta).$$

We seek to prove that $I = \text{Tr}\{F(Q^\dagger, Q)\}$. It suffices to prove this in the case that F is a monomial $F(Q^\dagger, Q) = Q_{\alpha_1}^\dagger \cdots Q_{\alpha_A}^\dagger Q_{\alpha_B} \cdots Q_{\beta_1}$. Evidently $\text{Tr} F = I = 0$ if the set $\{\alpha_j\}$ does not equal the set $\{\beta_j\}$. In the case $\{\alpha_j\} = \{\beta_j\}$, it suffices to assume that $\alpha_1 = \beta_1, \dots, \alpha_A = \beta_A$. In this case

$$I = \int \left(\prod_{\mu=1}^d d\eta_\mu d\eta_\mu^* \right) \prod_{\nu=1}^d (1 + 2\eta_\nu^* \eta_\nu) \eta_{\alpha_1}^* \eta_{\alpha_1} \cdots \eta_{\alpha_A}^* \eta_{\alpha_A} = 2^{d-A}$$

But in this case $\text{Tr} F = 2^{d-A}$ also, so the theorem is proved.

REFERENCES

1. P.T. Matthews and A. Salam, Nuovo Cim. 2, 120 (1955).
2. E. Nelson, J. Math. Phys. 5, 332 (1964); R.P. Feynman, Rev. Mod. Phys. 20, 367 (1948).
3. A direct construction of the fermion functional integral like the one presented here does not appear to exist in the literature. A similar construction for the Hamiltonian $Q_{\alpha\beta}^+ Q_{\alpha\beta}$ is given in S. Samuel, U.C. Berkeley preprint (1977).
4. The definition is extended outside of this region using the anti-periodicity relation $G(\dots, t_n + \beta, \dots) = -G(\dots, t_n, \dots)$.
5. Let $U \equiv \exp[-H(t_F - t_I)]$. Divide the interval (t_I, t_F) into small segments (t_{j-1}, t_j) and define $\tilde{U} = \prod_j \exp[-H\Delta t_j]$. Evidently $\|\exp(-H\Delta t_j) - \exp(-H\Delta t_j)\| < C(\Delta t_j)^2$ for some constant C. From this bound one easily derives $\|U - \tilde{U}\| < \exp[-E_0(t_F - t_I)] \{\exp[C(t_F - t_I)\max \Delta t_j] - 1\}$, where E_0 is the smallest eigenvalue of H or zero, whichever is smaller. Thus $\|U - \tilde{U}\| \rightarrow 0$, and therefore $|G - \tilde{G}| \rightarrow 0$, as $\max \Delta t_j \rightarrow 0$.
6. For further details see F.A. Berezin, The Method of Second Quantization (Academic Press, New York, 1966).