CHARACTERISTICS OF COAL/
LIGHT HYDROCARBON SLURRIES
IN SPRAY COMBUSTION

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by

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ABSTRACT

This report summarizes the progress made during the first six months of the grant period. The status of each of the three major task areas is discussed: the atomization study, the analytical droplet model, and the combustion study. The time-period has been used primarily to design and begin fabrication of the experimental atomization and combustion facilities. No data is presented in either of these areas. Significant progress has been made in the development of the numerical droplet model. The results of a preliminary parametric study are presented for a single coal particle in a one-millimeter methanol droplet. The importance of gas phase velocity and coal particle diameter bringing the solid material to the droplet surface is discussed.

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I. Objectives and Scope

The increased development of our coal resources requires solution to technical problems associated with processing, transporting, and burning the fuel. Slurries made up of varying percentages of pulverized coal, water, and liquified coal or petroleum fractions can be transported efficiently with a pipeline; although some additional processing of the mixture would probably be required at the power plant to control particle size and heating value of the slurry. Before the coal/liquid fuel furnace can be properly designed, much more must be known about the fundamentals of atomization, droplet behavior, and combustion of coal/liquid suspensions. A research program has been established with the present grant to improve our understanding of the physical phenomena controlling the combustion of coal/liquid sprays. The following primary objectives have been established:

- A. Evaluate the behavior of pulverized coal slurries undergoing atomization. In particular, determine how parameters such as atomizer design, pressure, viscosity, particle size, particle loading, and fluid composition affect droplet velocity and size distribution.
- B. Model the internal circulation of fuel droplets containing small particulate matter to see how the physical characteristics of the fluid and particulates affect vaporization rate.
- C. Determine the effect of methanol in a methanol/coal mixture on the conversion of coal-nitrogen to nitric oxide within a diffusion flame. Compare the results to NO formation using other low-boiling-point liquids such as steam and hexane.

II. Task Progress

A. Slurry Atomization Studies

Principal Investigator: C.T. Crowe

Graduate Research Assistants: E. Crowe and S. El-Haggar

The objective of this task is to measure the droplet size resulting from the atomization of coal-methanol slurries. Specifically, the effect of coal particle size and loading on droplet size is to be determined.

The initial phase of this study has been the design and construction of an apparatus capable of delivering a uniform coal-methanol slurry at a steady rate to a two-fluid atomizer and providing access to the spray field for photography of droplets. A schematic diagram of the set-up is shown in Fig. 1. Pumping circuit #1 serves to fill a 45 gal. liquid propane tank with a coal-methanol mixture having the desired properties. Pumping circuit #2 is the supply line for delivery of the mixture to the atomizer. Both pumps are positive displacement gear pumps, belt-driven by 3/4 horsepower electric motors. Continuous mixing in the tank is provided by pumping circuit #1 through recirculating the fluid and discharging it tangentially into the tank, creating rotary mixture action in the tank. A bypass is incorporated in pumping circuit #2 to allow control of the flow into the atomizer. The return line from the bypass provides further mixing in the tank in that it discharges tangentially into the tank as well.

The slurry delivery rate to the two fluid nozzle is regulated by flow control valves (ball valves). The flow rate is monitored by a pressure-regulator and ball valve.

The atomizer discharges into a 2' x 2' tank fabricated from aluminum. Plate glass windows on three sides of the tank provide illumination of the spray and access for photographing. Initial testing of the systems with water indicated significant impingement of spray on the window which was unacceptable for photography. Planar air nozzles were designed, fabricated and installed over each window to remove the liquid from the windows. These nozzles are performing effectively. Their effectiveness for coal-methanol sprays is yet to be determined.

The spray is collected at the bottom of the enclosure by a funnel arrangement fitted with a series of vanes to impart a rotational velocity to the mixture and collect the liquid on the walls. It is then discharged to a barrel. This system appears to work well.

FIG. 1. SLURRY ATOMIZATION APPARATUS

The system is now undergoing final shakedown. Calibration of the rotameter for the coal-methanol slurry and measurement of the uniformity of the slurry (effectiveness of the mixing process) are underway. Preliminary photographic measurements are underway with the pure methanol to refine the optical system and data reduction techniques. These data also will serve as baseline data for the coal-methanol slurries.

B. Analytical Droplet Model

Principal Investigator: J.N. Chung

Graduate Research Assistant: L.D. Zeller

1. Nucleation processes inside an oil droplet containing coal particles.

It has been a well-known phenomenon that the internal boiling and microexplosions associated with water-in-oil emulsions greatly enhances the combustion efficiency and reduces emissions. It is even more desirable for this process to take place in a coal/oil slurry droplet because of the presence of coal particles. Microexplosions will prevent coal-particle agglomeration. There is a limited amount of literature recently published, that deals with the internal nucleation and microexplosions of emulsion droplets, both experimental and theoretical.

From the study performed on the combustion characteristics of droplets of coal/oil and coal/oil/water mixtures (C.K. Law et al., 1979), several aspects of the process were determined to influence the combustion of coal-particle/oil droplets. The internal fluid motion of the droplet is important in determining whether the coal particles will form an agglomerate or be dispersed by fragmentation of the droplet. As the number of coal particles is increased for a given droplet size, there is an increased tendency for the particles to agglomerate prior to combustion. Also, the addition of water to the droplet will increase the probability of the occurrence of internal boiling and microexplosions within the droplet.

Recent studies have been performed to understand why and how droplet fragmentation occurs. Experimental results (Baer et al., 1980) of water-in-oil emulsion droplets indicate that rapid internal circulation causes the coalescence of water within the droplet. Homogeneous nucleation then occurs within the water, and a vapor bubble is formed inside the droplet. The pressure build-up then fractures the droplet. Other experimental studies (Sheffield et al., 1980) indicate that two events can occur to reduce the droplet size. Homogeneous or heterogeneous nucleation can occur close to the droplet surface, and a disruption of the droplet will occur. During a disruption, a relatively non-violent fragmentation occurs in which a small amount of mass is expelled from the droplet, and the droplet is reduced in size, but remains intact. Violent fragmentation of the droplet occurs when homogeneous or heterogeneous nucleation occurs deep within the droplet. Experiments (Miyasaka and Law, 1980) have been performed upon freely-falling coal particle/oil droplets in electric furnaces at atmospheric pressure. During these experiments, the coal particle/oil droplet formed agglomerates of coal particles while burning inside the electric furnace. There was little tendency for the coal particles to disperse through droplet fragmentation. Upon increasing the volatility of the oil component of the droplet, the required time for the combustion of the entire droplet was increased. Also, the addition of water did little to enhance the fragmentation.

Internal boiling and microexplosion is the only feasible way to achieve the complete burning of coal particles in an oil droplet and to prevent coal particle agglomerates. Experimental findings strongly relate the internal motion with the agglomeration. Therefore, a complete nucleation model incorporating the fluid mechanics and heat transfer is required to suggest an optimal combination of system parameters for the efficient and clean-burning of coal/oil droplets. The fluid mechanics model is essentially completed, and it is shown in the next section. The heat transfer and nucleation model are under development.

2. Fluid mechanics model of a coal/oil droplet (see Figure 2 for Schematic).

Since we have a two-phase (solid-liquid) system inside the coal/oil droplet, it is convenient to discuss the fluid mechanics of the continuous phase (oil) and mechanics of the dispersed phase (coal particles) separately. The basic assumptions in which the mathematical models are developed are listed below:

- a. The coal particle loading is light such that the presence of coal particles does not affect the flow field of the continuous phase, but the particle trajectories are completely dominated by the motion of the continuous phase.
- b. The interactions among particles are neglected because of the light loading. Moreover, the purpose of the current investigation is to predict the general distribution pattern and characteristic time of the transport of the coal particle from the interior to the droplet surface. This characteristic time of transport is an important parameter for determining whether the accumulation of significant number of coal particles on the droplet surface will take place earlier than the vaporization of most of the fuel. It is plausible to assume that the accumulation of significant amount of coal particles on the surface will block the evaporation and hence, will lead the way for internal superheating and microexplosions.
- c. The fluid properties are assumed constant and the coal particles are spherical in shape.

FIG.2 Schematic of Mathematical Model and Particle Trajectory of Preliminary Study.

The equations of motion for the flow field of the continuous phase and the motion of a single coal particle for high and low particle Reynolds numbers are developed in the Appendix.

3. Results and Discussion

The equations of motion described in the Appendix were solved numerically through a fourth-order Runge-Kutta Method. A supplementary subroutine was developed and was incorporated with the main program to optimize the size of the time steps and to keep track of the single step and accumulated errors.

A parametric study was performed for a coal particle/methanol mixture. The common parameters are the following:

density of methanol = 792 kg/m^3

dynamic viscosity of methanol = $0.5 \times 10^{-3} \text{ kg/m-s}$

density of coal particle = 1386 kg/m^3

The initial particle velocity is assumed zero. It has been found that the initial particle velocity does not have a significant effect on the particle transport time. This is because the particle will adjust to the continuous phase velocity very quickly. All particles start near the center of the vortex stream lines.

Three cases were investigated to study the effects of droplet velocity and particle size, shown in the Table.

Case	$U_{\infty}(\text{m/s})$	$R(\text{m})$	$R_p(\text{m})$	Dimensionless Time Needed for Travelling					
				$\frac{1}{2}$ Loop	1 Loop	$1\frac{1}{2}$ Loops	2 Loops	$2\frac{1}{2}$ Loops	3 Loops
1	10	1×10^{-3}	1×10^{-5}	340	680	1030	1370	1715	2075
2	1	1×10^{-3}	1×10^{-5}	3375	6795	10290	13670	17100	20670
3	1	1×10^{-3}	5×10^{-5}	120	250	360	(particle reaches surface after $1\frac{1}{2}$ loops).		

1000 dimensionless time = 3.52×10^{-5} second

The trajectory of a coal particle for case 1 is plotted in Fig. 2. It is clearly indicated by the results that each loop will take the particle approximately the same time of travelling even though the total arc length for each loop is different. This is due to the nature of the vortex stream lines. In other words, the particle motion is completely guided by the stream lines. Another proof is that the particle in case 2 takes about ten times longer to travel a loop than that in case 1. This is because the stream lines are moving ten times slower for case 2. The effect of particle size is shown between case 2 and case 3. It is apparent that bigger particles experience higher gravitational force and curvature force. Therefore, it only takes $1\frac{1}{2}$ loops of motion for case 3 to transport the particle from the center of the vortex ring to the droplet surface. It takes 7 loops and 12 loops for case 1 and case 2, respectively. In real time, it is 0.17 second, 3 seconds and 0.0127 second for case 1, 2, and 3, respectively.

It can be concluded that if the particle size is small, the gravity and curvature forces are not as dominant as drag force, and the particles follow very closely with the stream lines. If the particle size is large enough for the gravity and curvature force to balance the drag force, the transport is much faster than smaller particles. The Basset force is important for all cases investigated in this preliminary study.

C. Combustion Study

Principal Investigator: W.L. Grosshandler

Graduate Research Assistant: A. Zghoul

The first six months of the granting period have been used to design and begin construction of the opposed-flow diffusion burner, and to purchase the major equipment items necessary for the analytical measurements. A one-half scale model of the burner has been operated on methane/air mixtures to assess the two-dimensionality of the rectangular burner. There was some difficulty in attaining a flat flame for this model, but it was

apparent that a larger jet-size would reduce the wall effects and improve the situation. A full-scale plexi-glass model has also been built in which the velocity profiles were measured with a hot-wire anemometer. The flows were at the lower limit of measurement (about 5 cm/sec), but the qualitative results indicate that extreme care will be necessary to maintain low turbulence levels and a flat velocity profile.

The final design chosen is shown in Fig. 3. The coal/vaporized fuel mixture will enter vertically from the top. The small divergence angle was chosen to ensure that the coal particles will follow the gas streamlines. All transitional sections are carefully fitted, and the vanes are included to reduce any possibility of flow recirculation. Since straightening screens will interfere with the coal particle flow, it has been decided to match the Reynolds numbers in both the upper and lower jets, and to use identical geometries to maintain a two-dimensional stagnation plane. The flame-front will, then, be moved upwards or downwards, as required, by adjusting the amount of fuel, oxidizer, and inert gas in each jet stream.

The fluidized-bed feeder for the coal particle flow is still in the design stage. When completed, it will deliver coal particles in the sub 100 μm range at a rate of 1000 g/hr. The feeder will be evaluated in a full-scale mock-up to determine the distribution of coal particles across the jets and to determine the penetration depth of the coal particles through the stagnation plane under a cold-flow test. A laser-doppler-anemometer will be used to monitor the particle flow as a function of diameter, assuming that particles below about 5 μm are indicative of the gas velocity.

An infra-red Industries carbon monoxide process analyzer and a Thermo-Electron nitric oxide chemiluminescent analyzer have been received. The NO analyzer has been operated and performs as expected. A gas sampling system is being designed using a quartz microprobe, and incorporating the extensive experience gained in past laboratory measurements in jet-stirred reactors (Kramlich et al., 1981; Kramlich and Malte, 1978; Singh et al., 1979).

Figure 3. Opposed-flow diffusion burner.

A two-wavelength pyrometer, designed after Ayling and Smith (1972), is being assembled to measure the coal particle temperature. Two photomultiplier tubes (RCA 4840) are used, one with a violet filter and the other with a yellow filter, which are estimated to be able to measure surface temperatures above 1000 K with an accuracy of ± 20 K. The absolute sensitivity of the instrument varies inversely with the solid-angle collected. A beam divergence of about 0.2° has been chosen as a compromise between the spatial resolution in the flame and the detection limit of the P.M. tubes.

III. Summary of Expenditures

About 42% of the grant money has been expended halfway through the first phase of the program. The remaining funds will be sufficient to cover salary commitments and operating expenses. With careful management, it may be possible to maintain a carry-over in expendable supplies and computing funds in an amount close to \$10,000. This money would be carried forward into the second phase of the research.

A breakdown of specific expenditures is listed below:

1. Salaries, wages, and benefits--\$42,300 has been spent to date for 1/8 of the salaries of the three Principal Investigators and four graduate student Research Assistants.
2. Goods and Services--\$7,200 has been spent, primarily to cover construction material and shop time for the experimental facilities.
3. Travel--\$337 was used to cover W. Grosshandler's attendance at a meeting of the Western States Section of the Combustion Institute.
4. Computing--Approximately \$1000 of the originally-budgeted \$4000 has been spent in developing the analytical droplet model.
5. Permanent Equipment--\$18,500 has been expended for four major equipment items: carbon monoxide infrared process analyzer (\$3700), data acquisition computer (\$5600), coal crushing mill (\$2300), and nitric oxide chemiluminescent analyzer (\$6,900).

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V. Appendix: Development of Equations of Motion for Numerical Models

The Flow Field of Continuous Phase (Oil and Methanol)

The circulating motion of the continuous phase inside the droplet can be expressed by a general vortex stream function (Bignell, 1975),

$$\psi = \frac{A}{2} (R^2 r^2 - r^4) \sin^2 \theta$$

where A, the strength of the vortex, is given below.

$$A = U_{\infty}/2(k + 1)R^2 \text{ (for low Re) or } A = 3U_{\infty}/2R^2 \text{ (for high Re)}$$

The Re is defined as $\frac{U_{\infty} R}{\nu_{\infty}}$, U_{∞} is the velocity of the droplet relative to hot gas. R is the radius of the droplet and ν_{∞} is the kinematic viscosity of the hot ambient gas. k is the dynamic viscosity ratio of droplet fluid to outside gas. The velocities are found through the definition of stream function:

$$U_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \quad (\text{radial velocity}) \text{ and } U_{\theta} = - \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \quad (\text{tangential velocity})$$

It is important to note that the strength, A, given above for high Reynolds flow, is based on inviscid flow theory. That is only appropriate for a bubble translating in another liquid. For the case of an oil droplet in an inert hot gas environment, the viscous effect is important. Prakash and Sirignano (1978) suggested that $\frac{U_{\infty}}{AR^2} = 10$ for oil droplets in a combustor. The problem of a solid particle moving in a liquid is complicated, and it is even more difficult when the motion is unsteady. Most of the analyses in the literature have been concerned about the free fall of objects and oscillatory motion.

Basset (1888) was the first to investigate the unsteady motion of a rigid spherical particle in creeping flow. It is interesting that the Stokes drag on a particle which is in an unsteady motion is still the same as the steady drag, $6\pi\eta RU(t)$, where $U(t)$ is the instantaneous particle velocity. There are two new force terms resulting from the unsteady motion. First, the "added mass" term arises because of the acceleration of the particle requires acceleration of the surrounding fluid. The volume of the added mass of the surrounding fluid is one-half of the particle volume. Second, the Basset force term, which is the time integral of acceleration weighted by $(t-s)^{1/2}$, where $(t-s)$ is the time elapsed since the past acceleration, is due to the formation of a boundary layer near the particle surface. For high Reynolds number flow, Odar and Hamilton (1964) have studied the force experienced by an accelerating solid sphere on a straight line. They found the drag force, like the low Reynolds number case, still can be expressed by the steady drag formula, $C_D \frac{\pi R^2 \rho}{2} U^2(t)$, C_D is the famous drag coefficient. The added mass and Basset historical force were calculated using the creeping flow formula multiplied by an empirical correction coefficient.

These coefficients are

$$\Delta_A = 2.1 - 0.132 M_A^2 / (1 + 0.12 M_A^2) \quad (\text{for added mass})$$

$$\Delta_H = 0.48 + 0.52 M_A^3 / (1 + M_A^3) \quad (\text{for historical force})$$

$$M_A = 2R \frac{dU}{dt} / u^2 \quad \text{acceleration modulus}$$

An excellent review on the accelerated particle motion has been provided by Cliff, et al. (1978).

The Equation of Motion of a Single Coal Particle in an Oil Droplet

A new Reynolds number characterizing the relative motion between particle and continuous phase is defined

$$Re_r = \frac{|U_r - U_{rp}| R_p}{\nu}$$

$U_{rp}, U_{\theta p}$: particle radial and tangential velocities

$$Re_{\theta} = \frac{|U_{\theta} - U_{\theta p}| R_p}{\nu}$$

respectively
 U_r, U_{θ} : continuous phase radial and tangential velocities respectively

R_p : particle radius

ν : continuous phase kinematic viscosity

These particle Reynolds numbers are different from the Re_f defined previously. There is no relationship between them. The particle Reynolds number is very important in determining the forces experienced by the particle and hence the particle trajectory. If the particle Reynolds number is less than unity, we can neglect the inertia effect and use the Stokes type drag. For high particle Reynolds number, both the inertia and unsteady effects are important. These effects are included through correcting coefficients. In actual spray droplet, the particle Reynolds number is generally small except short distance from the spray nozzle.

Case I. Low Re_r and Re_{θ}

Force experienced by the particle = net gravitational force + drag force (skin friction and form drag) + added mass + Basset history force + centrifugal and coriolis force.

$$\text{net gravitational force} = \vec{g}(\rho_p - \rho)V_p$$

ρ_p, ρ : densities of particle and continuum phase

$$\text{Drag force} = 6\pi\eta R_p (\vec{u} - \vec{u}_p)$$

$\vec{u}_p$: velocity of particle

Added mass = $(\rho/2) V_p d(\vec{u} - \vec{u}_p)/dt$, V_p : volume of particle

$$\text{Basset force} = 6R_p \sqrt{\pi \mu} \int_0^t \frac{(\vec{u} - \vec{u}_p)_s}{\sqrt{t-s}} ds$$

$$\begin{aligned} \text{Centrifugal force} &= \frac{(\text{particle velocity perpendicular to line of radius of curvature})^2}{\text{radius of curvature}} \\ &= \frac{W_\theta^2}{\bar{R}} \end{aligned}$$

$$\begin{aligned} \text{Coriolis force} &= \frac{(\text{particle velocity perpendicular to line of radius of curvature} \times \\ &\quad \text{particle velocity in the direction of radius of curvature})}{\text{radius of curvature}} \\ &= \frac{W_\theta \times W_r}{\bar{R}} \end{aligned}$$

$$\rho_p V_p \frac{d\vec{u}_p}{dt} = \vec{g}(\rho_p - \rho) V_p + 6\pi \mu R_p (\vec{u} - \vec{u}_p) + \frac{\rho}{2} V_p \frac{d(\vec{u} - \vec{u}_p)}{dt} + 6R_p^2 \sqrt{\pi \mu} \int_0^t \frac{(\vec{u} - \vec{u}_p)_s}{\sqrt{t-s}} ds + f\left(\frac{W_\theta^2}{\bar{R}}, \frac{W_\theta \times W_r}{\bar{R}}\right)$$

Nondimensionalize the above through the following

$$t' = \frac{g}{2} \frac{t \nu}{R_p^2}, \quad \vec{g} = \frac{\vec{g}}{g_0}, \quad \vec{u}_p' = \frac{\vec{u}_p}{U_\infty}, \quad \vec{u}' = \frac{\vec{u}}{U_\infty}, \quad W_\theta' = \frac{W_\theta}{U_\infty}, \quad W_r' = \frac{W_r}{U_\infty}, \quad \bar{R}' = \frac{\bar{R}}{R}$$

Let us drop the prime for dimensionless quantities for simplicity. Also, note that $dU_r/dt = 0$ and $dU_\theta/dt = 0$ because the continuous phase motion is assumed steady.

$$\gamma \frac{dU_{\theta p}}{dt} = Fr Re_f \left(\frac{\rho_p - \rho}{\rho} \right) \sin \theta + (U_\theta - U_{\theta p}) - \frac{1}{2} \frac{dU_{\theta p}}{dt} - \sqrt{\frac{g}{2\pi}} \int_0^t \frac{U_{\theta p} s}{\sqrt{t-s}} ds + \frac{2}{9} \frac{R_p}{R} Re_f f_1 \left(\frac{W_\theta^2}{\bar{R}}, \frac{W_\theta \times W_r}{\bar{R}} \right)$$

$$\gamma \frac{dU_{r p}}{dt} = Fr Re_f \left(\frac{\rho_p - \rho}{\rho} \right) \cos \theta + (U_r - U_{r p}) - \frac{1}{2} \frac{dU_{r p}}{dt} - \sqrt{\frac{g}{2\pi}} \int_0^t \frac{U_{r p}}{\sqrt{t-s}} ds + \frac{2}{9} \frac{R_p}{R} Re_f f_2 \left(\frac{W_\theta^2}{\bar{R}}, \frac{W_\theta \times W_r}{\bar{R}} \right)$$

$$\text{where } Fr = \frac{2}{9} \frac{g_0 R_p}{U_\infty^2}, \quad Re_f = \frac{U_\infty R_p}{\nu}, \quad \gamma = \frac{\rho_p}{\rho}$$

f_1 and f_2 are forces due to the curvature of motion. Because the motion is unsteady and both the radius of curvature and center of curvature are also changing with time, therefore, both centrifugal and coriolis forces contribute to radial and tangential forces. The complicated mathematical procedure for the calculation of f_1 , and f_2 is omitted here due to page limitations. The Basset integral needs special attention because of the singular point at the upper integration limit.

$$\begin{aligned}
 \int_0^t \frac{\dot{u}_{op}|s}{\sqrt{t-s}} ds &= \int_0^{n\Delta t} \frac{\dot{u}_{op}|s}{\sqrt{t-s}} ds \\
 &= \frac{\Delta t}{6} \sum_{i=1}^{n-1} \left(\frac{\dot{u}_{op}((i-1)\Delta t)}{\sqrt{n\Delta t - (i-1)\Delta t}} + \frac{2(\dot{u}_{op}((i-1)\Delta t) + \dot{u}_{op}(i\Delta t))}{\sqrt{n\Delta t - (i-\frac{1}{2})\Delta t}} + \frac{\dot{u}_{op}(i\Delta t)}{\sqrt{n\Delta t - i\Delta t}} \right) + \\
 &\quad \frac{0.9\Delta t}{6} \left(\frac{\dot{u}_{op}((n-1)\Delta t)}{\sqrt{\Delta t}} + \frac{2(\dot{u}_{op}((n-1)\Delta t) + \dot{u}_{op}(n\Delta t))}{\sqrt{n\Delta t - (i-\frac{1}{2})\Delta t}} + \frac{\dot{u}_{op}(n\Delta t)}{\sqrt{0.1\Delta t}} \right) + \\
 &\quad \frac{0.1\Delta t}{2} \left(\frac{8\sqrt{2}}{3} \frac{\dot{u}_{op}(n\Delta t)}{\sqrt{0.05\Delta t}} - \frac{4}{3} \frac{\dot{u}_{op}(n\Delta t)}{\sqrt{0.1\Delta t}} \right)
 \end{aligned}$$

Case II. High Re_r and Re_θ

$$\begin{aligned}
 \gamma \frac{dU_{op}}{dt} &= Fr Re_f \left(\frac{f_p - f}{f} \right) \sin \theta + \frac{Re_f}{12} C_{d\theta} (U_\theta - U_{op}) |U_\theta - U_{op}| - \lambda \frac{\Delta A}{2} \frac{dU_{op}}{dt} - \\
 &\quad \chi \Delta_H \sqrt{\frac{g}{2\pi}} \int_0^t \frac{\dot{u}_{op}|s}{\sqrt{t-s}} ds + \frac{2}{9} \frac{R_p}{R} Re_f f_1 \left(\frac{We^2}{\bar{R}}, \frac{We \times We_r}{\bar{R}} \right)
 \end{aligned}$$

$$\begin{aligned}
 \gamma \frac{dU_{rp}}{dt} &= Fr Re_f \left(\frac{f_p - f}{f} \right) \cos \theta + \frac{Re_f}{12} C_{dr} (U_r - U_{rp}) |U_r - U_{rp}| - \lambda \frac{\Delta A}{2} \frac{dU_{rp}}{dt} - \\
 &\quad \chi \Delta_H \sqrt{\frac{g}{2\pi}} \int_0^t \frac{\dot{u}_{rp}|s}{\sqrt{t-s}} ds + \frac{2}{9} \frac{R_p}{R} Re_f f_2 \left(\frac{We^2}{\bar{R}}, \frac{We \times We_r}{\bar{R}} \right)
 \end{aligned}$$

$$C_{d\theta} = 4(0.75 Re^{0.72} + 6.0) / Re_\theta$$

$$C_{dr} = 4(0.75 Re^{0.72} + 6.0) / Re_r$$

Δ_A and Δ_H are the same as defined before, λ and χ are introduced to correct for the curved motion by Odar (1968).