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**A UNIQUE SU(5) AND SO(10) UNIFICATION WITH
COMPLETE DYNAMICAL SYMMETRY BREAKING***

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ABSTRACT

A unique asymptotically free, anomaly-free, SU(5) gauge theory is proposed as a possible complete unification of the standard model in which all symmetry-breaking is dynamical. The asymptotic freedom constraint is saturated, removing renormalon divergences and leaving well-defined instanton interactions as the only non-perturbative ingredient of the theory. Consequently, it is argued, topological vacuum polarization of a very heavy, unconventional quantum number, quark sector dominates the dynamics, producing SU(5) symmetry breaking and a three generation low energy spectrum. Electroweak symmetry breaking is due to a chiral condensate of color sextet quarks. The embedding of the theory in a single SO(10) representation is used for the dynamical analysis and may also have physical significance.

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MASTER

SECTION 1: INTRODUCTION

Dynamical insight into non-abelian gauge theories which distinguishes very selectively amongst candidate physical theories has, to date, proved extremely difficult to obtain. A number of general properties, such as asymptotic freedom, confinement, anomaly cancellation, chiral symmetry breaking, the dynamical Higgs mechanism (based on technicolor, say), anomaly constraints on composite models etc., are thought to fundamentally influence the dynamics--if not determine the consistency--of such theories. Nevertheless such properties seem to be far from selecting a unique candidate theory of nature on either aesthetic or technical grounds. Consequently the present comparison of gauge theories with nature is essentially phenomenological and model builders attempting to extend the standard $SU(3) \times SU(2) \times U(1)$ model have an enormous freedom. This is, in part, why the relative uniqueness promised by attempts to embed gauge theories in a larger framework--first Supergravity and now Superstring theories--is regarded as so welcome.

It is, of course, possible that it is only the unification with gravity that severely restricts or even determines the effective "low-energy" gauge theory which unifies the gauge symmetries of the standard model. Alternatively it is also possible that a better understanding of the dynamics of gauge theories will limit or even select the appropriate unified theory and in so doing, perhaps constrain the unification with gravity. Certainly the general significance of the subtle interplay between gauge-field topology and fermion anomalies which characterizes low-dimensional solvable models¹ can, at present, be only guessed at in four dimensions. Further insight into such

phenomena may well produce major constraints on candidate physical theories.

In this paper we present a unified theory that is significantly different from conventional "GUTS" in many respects. In particular our theory is uniquely determined by general dynamical requirements that we formulate. In addition, for the theory to be physically realistic, anomalous fermion phenomena must be an essential ingredient. Indeed we are led to study such phenomena in part because they are vital for a realistic particle spectrum to emerge from our theory and in part because the theory has intrinsic properties which emphasize the role of topology and fermion anomalies. A particularly vital phenomenon that we appeal to is that of soliton (or topological) vacuum polarization of heavy fermions. This has been much discussed recently² but not applied in the manner we suggest to both produce gauge symmetry breaking and to effectively obtain light composite states from "heavy" constituents.

We are led towards the unique theory we present by two, at first sight distinct, dynamical ideas. The first is relatively straightforward and is a variant of the original technicolor proposal³ for dynamical symmetry breaking of the electroweak interaction. It has been suggested⁴ that QCD itself could play the role of the technicolor gauge interaction if the quarks forming the relevant chiral condensate belong to a higher color representation. We find that there is a unique SU(5) unified theory which is both asymptotically free and anomaly-free and contains the necessary color representations. In addition no other unitary group produces such a theory. Hence we find a remarkable selection of SU(5) as a unification group for our purpose.

The second dynamical concept which independently points towards the theory we propose (although having a general formulation, as we discuss) has grown out of studies of the high-energy behavior of gauge theories (and QCD in

particular) in the Regge limit.⁵ Asymptotic behavior which is self-consistent and unitary in this limit is extremely hard to obtain since the infra-red and ultra-violet regions of a theory are coupled dynamically in a very non-trivial manner. Indeed we might well expect to see dynamical constraints in the Regge region which are not visible directly in either the infra-red or ultra-violet regions. (Note that the presence of a reggeizing spin two graviton in superstring theories implies that such theories are unstable, in perturbation theory, in the Regge region⁶--whereas they are perturbatively stable in the ultraviolet and infra-red regions!)

The Regge limit studies suggest there is a particular dynamical significance to saturating the asymptotic freedom constraint on the number of fermions in a gauge theory. In QCD we have argued⁵ that the consequent infra-red fixed-point is related to the occurrence of Critical Pomeron scaling⁷ (including KNO scaling of the multiplicity distribution) which in turn we have also argued⁸ to be related to the validity of the parton-model for hadron scattering and to chiral symmetry breaking. However, independently of these arguments it is clear that an infra-red fixed-point produced by "fermion saturation" may be vital for a dynamical solution of a theory based on the inter-relation of gauge-field topology and the anomalous properties of massless fermions. This is because instantons and not renormalons⁹ are the most important non-perturbative contribution when a theory is saturated with fermions. (In fact renormalons may very well be absent altogether from the theory.) The dynamics this produces seems necessary for our purposes and might even, we suggest, be generally necessary for a self-consistent theory! Demanding fermion saturation, together with a non-degenerate mass spectrum, would also uniquely select the theory we propose.

We shall argue that $SU(5)$ Yang-Mills theory based on the (left-handed) fermion representation

$$R_{SU(5)} = 5 + 15 + 40 + 45^* \quad (1.1)$$

provides a (potentially) complete unification of the standard model, with the whole representation playing a dynamical role in generating the observed particle spectrum and the necessary dynamical symmetry breaking. An immediately intriguing feature of $R_{SU(5)}$ is that although the criteria which select this representation make no reference to either triplet quarks or leptons, the spectrum of such fermions is remarkably close to that of the standard model. Conventional approaches to unified model building begin, of course, with the known quark-lepton spectrum and the spectrum of $R_{SU(5)}$ has sufficient discrepancies with the physical spectrum that it would be rejected as a conventional Grand Unified Theory. However, the "close resemblance" of $R_{SU(5)}$ to the physical spectrum is reflected, at the $SU(3) \times SU(2) \times U(1)$ level, in a set of "flavor" anomalies which would be reproduced by three standard generations of quarks and leptons. This raises the possibility that a dynamical solution to the theory could produce a "composite" realistic spectrum with the fermions in $R_{SU(5)}$ playing a "preonic-role".

A major part of this paper is devoted to arguing that there may indeed be such a composite solution of the $R_{SU(5)}$ theory which does not, however, completely lose the conventional quark and lepton character of the "preonic" fermions. Rather we suggest that the dynamics may produce fermionic solitons and consequent heavy-fermion vacuum polarization involving both higher color and non-weakly interacting quarks in a manner which effectively adjusts the

quantum numbers of the conventional quarks and leptons in $R_{SU(5)}$ to those of three generations. In this case the lepton and quark generations can still effectively be identified within $R_{SU(5)}$.

While it is possible to speculate how the three-generation solution of the $R_{SU(5)}$ theory originates dynamically, the complexity of the theory makes it difficult to develop any systematic procedure. Fortunately $R_{SU(5)}$ can be entirely embedded in a single complex, anomaly-free, asymptotically-free, irreducible representation of $SO(10)$ --the 144. We argue that there exists a dynamical solution of $SO(10)$ Yang-Mills theory with the (left-handed) fermion representation

$$R_{SO(10)} = 144 + 16^*, \quad (1.2)$$

in which $R_{SU(5)}$ appears at "low-energy" and in which the three-generation quark and lepton spectrum can be identified relatively easily. The dynamical solution we describe exploits the existence of gauge-invariant four-fermion condensates, instanton interactions and the phenomenon referred to above of topological heavy-fermion vacuum polarization.

An important feature of the $R_{SO(10)}$ theory is that the instanton interactions are relatively simple compared to those of $R_{SU(5)}$, although because $R_{SO(10)}$ is not saturated with fermions, such interactions have infrared divergences due to renormalons.⁹ Consequently we believe that the ultimate significance of the $SO(10)$ dynamics we elaborate may be as a high-energy description of the instanton dynamics of composite states in $R_{SU(5)}$. We do not attempt to directly derive the dynamical solution of $R_{SU(5)}$ from that of $R_{SO(10)}$, although we suspect this can be done. Instead we outline how

the dynamics of the complex $R_{SU(5)}$ instanton interaction can parallel that of $R_{SO(10)}$. This enables us to show that the $SU(5)$ symmetry breaking mechanism can be viewed as the topological vacuum polarization of a composite fermion by a topological condensate--both composed of (very) massive color octet and non-weakly interacting triplet quarks. The electroweak symmetry breaking is then directly identified with the breaking of the chiral symmetry of color sextet quarks.

Utilizing the $SO(10)$ dynamics it is possible to locate the three generations of composite states in $R_{SU(5)}$. As stated above, they have the characteristics of elementary quarks and leptons but with various combinations of octet and non weakly interacting quarks locked inside them by the mechanism (we suggest) of sextet pion Skyrmin absorption of heavy quarks.

In summary, the essential dynamical feature of our theory is that there is an additional very massive sector composed of higher color and non-weakly interacting quarks buried inside the familiar fermions. This sector lies behind the symmetry breaking (and therefore mass-generation) properties of the theory. It saturates the theory to the point where topological properties can dominate the dynamics, leading to the proposal that topological fermion polarization produces the "breaking" of the gauge symmetry at low energy. The gauge-invariant physical states have a very complex structure at high-energy with the hidden (very massive) sector dominating the degrees of freedom. What the true scales are at which these degrees of freedom would appear is just one of the multitude of questions we would hope to address as the theory is gradually understood at a more detailed level than we are able to approach in this paper. We need hardly emphasize that our theory has the great attraction that not only is it uniquely defined but it may also be a no parameter theory

with the complete dynamical generation of the mass spectrum. Certainly the complexity of the mass generation process gives it the potential to produce the full range of the physical spectrum! The price we pay, of course, is that the dynamics of the theory is so complex that extracting detailed numerical predictions, for the mass spectrum, for example, will surely require considerable theoretical effort and ingenuity.

In Section 2, the $SU(5)$ theory is first discovered by looking for an $SU(5)$ unification containing color sextet quarks. The uniqueness properties of the theory are then elaborated on. Section 3 contains a description of the $SU(3) \times SU(2) \times U(1)$ flavor symmetries and anomalies of the theory. The embedding in $SO(10)$ is described in Section 4, as well as a further embedding in E_6 which could be relevant if the theory is contained in any Superstring theory. The $SO(10)$ and $SU(5)$ dynamics alluded to above are described respectively in Sections 5 and 6.

SECTION 2: THE SU(5) THEORY

We shall first locate the SU(5) theory which is the focus of this paper by looking for an SU(5) unification which contains color sextet quarks with the correct gauge couplings to produce dynamical electroweak symmetry breaking. We shall then describe a set of properties which uniquely distinguish this theory from the complete range of possible unified theories--with an arbitrary unification group!

The $SU(3) \times SU(2) \times U(1)$ quantum numbers we require for (left-handed) sextets are first an "anti-quark" doublet

$$(U_c, D_c) \equiv (6^*, 2, \frac{1}{6}) , \quad (2.1)$$

and secondly two singlets

$$\bar{U} \equiv (6, 1, -\frac{2}{3}) \quad \text{and} \quad \bar{D} \equiv (6, 1, \frac{1}{3}) . \quad (2.2)$$

We wish to preserve asymptotic freedom and so we consider low-dimensional SU(5) representations only. Those containing appropriate color sextets, with their full $SU(3) \times SU(2) \times U(1)$ content, are¹⁰

$$15 = (1, 3, 1) + (3, 2, \frac{1}{6}) + \underline{(6, 1, -\frac{2}{3})} \quad (2.3)$$

$$35 = (1, 2, -\frac{3}{2}) + (3^*, 3, -\frac{2}{3}) + \underline{(6^*, 2, \frac{1}{6})} + (1, 1, 1) \quad (2.4)$$

$$40 = (1, 2, -\frac{3}{2}) + (3, 2, \frac{1}{6}) + (3^*, 1, -\frac{2}{3}) + (3^*, 3, -\frac{2}{3})$$

(2.5)

$$+\underline{(6^*, 2, \frac{1}{6})} + (8, 1, 1)$$

$$45^* = (1, 2, -\frac{1}{2}) + (3^*, 1, \frac{1}{3}) + (3^*, 3, \frac{1}{3}) + (3, 1, -\frac{4}{3})$$

(2.6)

$$+(3, 2, \frac{7}{6}) + \underline{(6, 1, \frac{1}{3})} + (8, 2, -\frac{1}{2}) .$$

In addition to asymptotic freedom we also, of course, require our theory to be anomaly-free. The relevant Casimirs giving the anomaly A and β -function contribution B (normalized so that $B < 55$ gives asymptotic freedom) are listed in Table 2.1.

<u>representation</u>	<u>A</u>	<u>B</u>
15	9	7
35	36	28
40	-15	22
45*	6	24

Table 2.1

To include the complete sextet set (U_c, D_c) , $(\bar{U}, \bar{D})$ we can add the 15 and 45* representations to either the 40 or the 35 representations. However, from Table 2.1

$$B(15 + 40 + 45^*) = 7 + 22 + 24 = 53 \quad (2.7)$$

$$B(15 + 35 + 45^*) = 7 + 28 + 24 = 59 \quad (2.8)$$

$$> 55 \quad (2.9)$$

and so asymptotic freedom immediately excludes the combination involving the 35 representation. From Table 2.1 we also have

$$A(15 + 40 + 45^*) = 9 - 16 + 6 = -1, \quad (2.10)$$

and so to obtain an anomaly-free theory we must add a further representation(s). Table 2.2 gives a list of low-dimensional representations of SU(5) with the corresponding values of A and B.

<u>representation</u>	<u>A</u>	<u>B</u>
5	1	1
10	1	3
24	0	10
50	15	35

Table 2.2

Clearly, since asymptotic freedom is so close to being violated in (2.7), the only possibility to obtain an anomaly-free, asymptotically free, theory is

add a 5 representation to that of (2.7) giving

$$R_{SU(5)} = 5 + 15 + 40 + 45^* . \quad (2.11)$$

the following we shall argue that a Yang-Mills theory with left-handed fermions belonging to the representation $R_{SU(5)}$ provides a complete unification of the standard model. For the moment we wish to emphasize the remarkable set of properties satisfied by $R_{SU(5)}$. They are

- i) it is asymptotically-free and anomaly-free
- ii) it is a complex representation of $SU(3) \times SU(2) \times U(1)$ but a real representation of $SU(3) \times U(1)$.
- iii) it contains a chiral doublet of color sextet quarks with the left-handed chiral symmetry coupled to an $SU(2)$ gauge field.
- iv) it is multiplicity free, that is it contains each irreducible representation only once
- v) the asymptotic freedom constraint is saturated--no more fermions can be added to the theory without violating asymptotic freedom.

Properties i)-iii) actually uniquely distinguish the theory based on (5) from any other unified gauge theory based on a simple gauge group. (i) and (ii) are commonly imposed in conventional GUT models.¹¹ Note that, for a few representations, asymptotic freedom is a much stronger constraint than anomaly cancellation. For example

$$15 + 45^* + 50 \quad (2.12)$$

is anomaly-free but it is not asymptotically free.

If we consider an arbitrary $SU(N)$ gauge group, it is shown in Ref. 12 that the only $SU(N)$ representation which satisfies i) and ii) and contains higher (than triplet) color representations of $SU(3)$ is in fact $R_{SU(5)}$. Hence $R_{SU(5)}$ is uniquely selected amongst representations of the unitary groups solely by our desire to use a higher color representation to produce electroweak symmetry breaking.

There is only a very limited number of complex, anomaly-free and asymptotically-free representations of the remaining simple groups.¹³ Amongst these only the 126 and 144 representations of $SO(10)$ contain higher $SU(3)$ color representations. The 144 actually contains $R_{SU(5)}$ entirely, as we discussed in the introduction and enlarge upon in later sections. It should therefore not be considered a distinct theory. The 126 has some similarity to the 144 but contains (2,12) and hence does not generate an asymptotically-free "low-energy" $SU(5)$ theory. Another problem is that it contains an $SU(2)$ triplet of color sextets (rather than the doublet contained in the 144) which is right-handed rather than left-handed. Also the quark and lepton spectrum is too small, requiring the addition of further low-dimensional (16's most easily) $SO(10)$ representations.

Properties iv) and v) are, separately, very restrictive. iv) is important if we wish to see a dynamical mass spectrum generated which does not have the usual generation symmetries associated with the simplest conventional unifications of the standard model. It has also been imposed as a rule for grand-unification by Georgi.¹¹ Property v), we believe, has major significance for the stability of a dynamical solution of the theory based on gauge-field topological properties. It is expected¹⁴ to be associated with an

infra-red fixed-point in any non-perturbatively defined β -function. This implies the gauge-coupling does not grow indefinitely in the infra-red region, allowing the possibility that the topological classification of gauge fields remains significant dynamically and is not destroyed by the dominance of large quantum mechanical fluctuations.¹⁵ A further closely related phenomenon produced by the addition of many fermions to a theory, is that renormalon singularities⁹ in the Borel plane (whose location depends on the number of fermions) occur much further to the right (and so are of less importance) than multi-instanton singularities. "Infra-red" renormalons are generally thought to represent the main non-perturbative ambiguity of asymptotically free gauge theories and underlie most arguments¹⁵ that gauge-field topology, and instantons in particular, are not important dynamically. Indeed, as we enlarge upon in Section 6, when there is fermion saturation (of the asymptotic freedom constraint) there may be no renormalon singularities at all. In this case instantons and the associated massless fermion phenomena will be promoted to the main non-perturbative ambiguity of the theory. In the introduction and in references 5 and 8, we have also emphasized the importance of an infra-red fixed-point in the Regge limit.

Taken in isolation properties i), iv) and v) are, in themselves, sufficient to select $R_{SU(5)}$ from any other gauge theory representation. In particular the $SO(10)$ representations are eliminated since they do not saturate the asymptotic freedom constraint. We shall return to this point later. Taken together properties i)-v) are uniquely satisfied by $R_{SU(5)}$ amongst the complete (countably infinite) list of potential unified theories.

As we emphasized in the introduction the criteria that uniquely select $R_{SU(5)}$ do not include the requirement that it contain the correct quark and lepton

spectra. The purpose of the rest of the paper will be to argue that this spectrum can emerge dynamically. As we have implied above the dynamics we appeal to will be heavily based on the inter-relation of gauge-field topology with massless fermion anomalies and the regularization of the fermion Dirac sea.

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SECTION 3: FLAVOR SYMMETRIES AND ANOMALIES AT THE $SU(3) \times SU(2) \times U(1)$ LEVEL

We can list the content of $R_{SU(5)}$ at the $SU(3) \times SU(2) \times U(1)$ level by adding to (2.3), (2.5) and (2.6) the familiar decomposition

$$5 = (1, 3, -\frac{1}{3}) + (2, 1, \frac{1}{2}) . \quad (3.1)$$

We then obtain the following spectrum.

Leptons

We have nine leptons, three $SU(2)$ doublets

$$(1, 2, \frac{1}{2}) , \quad (1, 2, -\frac{1}{2}) , \quad (1, 2, -\frac{3}{2}) \quad (3.2)$$

and one triplet

$$(1, 3, 1) . \quad (3.3)$$

Triplet quarks

We have eight quarks, two conventional $SU(2)$ doublets

$$(3, 2, \frac{1}{6}) , \quad (3, 2, \frac{1}{6}) \quad (3.4)$$

one unconventional doublet

$$(3, 2, \frac{7}{6}) \quad (3.5)$$

and two SU(2) singlets

$$(3, 1, -\frac{1}{3}) , \quad (3, 1, -\frac{4}{3}) . \quad (3.6)$$

Triplet anti-quarks

We have eight antiquarks, two conventional singlets

$$(3, 1, \frac{1}{3}) , \quad (3^*, 1, -\frac{2}{3}) \quad (3.7)$$

and two unconventional triplets

$$(3, 3, \frac{1}{3}) , \quad (3^*, 3, -\frac{2}{3}) . \quad (3.8)$$

Sextet quarks

By construction we have two singlets

$$(6, 1, \frac{1}{3}) , \quad (6, 1, -\frac{2}{3}) . \quad (3.9)$$

Sextet antiquarks

By construction we have the doublet

$$(6^*, 2, \frac{1}{6}) . \quad (3.10)$$

Octet quarks

We have a doublet

$$(8, 2, -\frac{1}{2}) \quad (3.11)$$

and also a singlet

$$(8, 1, 1) . \quad (3.12)$$

The lepton and triplet quark spectrum is tantalizingly close to that of the real world, although the $SU(2) \times U(1)$ quantum numbers are not right according to those conventionally assigned in the standard model. (3.2) gives three generations of lepton doublets but with hypercharges ranging from $-\frac{3}{2}$ to $+\frac{1}{2}$, instead of each generation having hypercharge $-\frac{1}{2}$. (3.3) gives an $SU(2)$ triplet where we would like three singlets with the same hypercharge. There are three conventional quarks with charges $-\frac{2}{3}$ and three with charge $\frac{1}{3}$. The sextet quarks are, of course as we prescribed. Both the exotically charged color triplet quarks and antiquarks and the octet quarks seem superfluous, although we have not yet envisaged any mechanism for $SU(5)$ symmetry breaking (in contrast to the $SU(2)$ breaking by a sextet chiral condensate).

The major difficulty with the above spectrum is clearly that the $SU(2)$ gauge symmetry extends to a "horizontal" symmetry linking what we would like to identify as distinct generations. The resemblance of the spectrum to the physical spectrum is, however, made explicit by considering the "flavor" symmetries exhibited (at the $SU(3) \times SU(2) \times U(1)$ level) and recording the anomalies of the associated currents. Following 't Hooft's arguments¹⁶ we expect such anomalies to constrain any composite spectrum built from $R_{SU(5)}$. That is if the (massless) fermions of $R_{SU(5)}$ are replaced by massless fermion

composites, such composites must reproduce the flavor current anomalies.

The simplest flavor symmetry of the above spectrum is that of the doublet of color triplet quarks (3.4). We could regard these as an $SU(2)$ flavor doublet. However, since there is no (perturbative) $SU(2)$ anomaly this will not lead to any constraint on the composite spectrum. To obtain such an anomaly we require at least an $SU(2) \times U(1)$ flavor symmetry. There are in fact several such symmetries, which we can identify as follows

S(1) The three lepton doublets of (3.2) can be identified as a triplet under a new $SU(2) \times U(1)$ symmetry--which we denote as $SU(2)_1 \times U(1)_1$ --if we identify the $U(1)_1$ "hypercharge" as $-\frac{1}{2}$. That is the three doublets form the $SU(3) \times SU(2) \times SU(2)_1 \times U(1)_1$ representation

$$(1, 2, 3, -\frac{1}{2}) . \quad (3.13)$$

The original $U(1)$ symmetry is now identified as a combination of $U(1)_1$ and a $U(1)$ subgroup of $SU(2)_1$.

The anomaly associated with the $SU(2)_1 \times U(1)_1$ symmetry is

$$A_1 \equiv \text{"sum over charges"} = 2 \times \left(-\frac{3}{2} - \frac{1}{2} + \frac{1}{2} \right) \quad (3.14)$$

$$= -3 . \quad (3.15)$$

Note that the same anomaly would be reproduced by three doublets under $SU(2)_1$ which had $U(1)_1$ charge $-\frac{1}{2}$ and were singlets under the gauged $SU(2)$. In this sense the anomaly A_1 can be identified with that of three generations. Before

the identification is complete, however, we will have to identify $SU(2)_1 \times U(1)_1$ with the physical electroweak symmetry.

S(2) The next flavor symmetry we identify is that of the triplet anti-quarks (3.8). We denote their flavor symmetry as $SU(2)_2 \times U(1)_2$, they form the $SU(3) \times SU(2) \times SU(2)_2 \times U(1)_2$ representation

$$(3^*, 3, 2, -\frac{1}{6}) . \quad (3.16)$$

The $SU(2)_2 \times U(1)_2$ anomaly is the "sum over charges"

$$A_2 = 3 \times 3 \times 2 \times -\frac{1}{6} = -3 . \quad (3.17)$$

This anomaly would be reproduced by three generations of the form

$$3 \times (3^*, 1, 2, -\frac{1}{6}) \quad (3.18)$$

implying that we will not want to identify $SU(2)_2 \times U(1)_2$ with the electroweak symmetry since we will want the antiquarks to be singlets under that symmetry. Clearly we will want to identify $SU(2)_2 \times U(1)_2$ with $SU(2)_R \times U(1)_{e.m.}$ where $SU_R(2)$ is the usual right-handed chiral symmetry of antiquarks and $U(1)_{e.m.}$ is electromagnetism. In this case the anomaly A_2 will also be reproduced by (three generations of) massless anti-baryons formed from three generations of antiquarks (if the chiral symmetry is not spontaneously broken--if it is the corresponding Goldstone bosons reproduce the anomaly).

The flavor symmetries $S(1)$ and $S(2)$ will be what we ultimately use to

locate the three generation solution we are seeking. Note that there are also additional flavor symmetries which can be listed as

S(3) The singlet antiquarks (3.7) have an $SU(2)_3 \times U(1)_3$ flavor symmetry, and form the $SU(3) \times SU(2) \times SU(2)_3 \times U(1)_3$ representation

$$(3^*, 1, 2, -\frac{1}{6}) . \quad (3.19)$$

S(4) The singlet quarks (3.6) form an $SU(3) \times SU(2) \times SU(2)_4 \times U(1)_4$ representation

$$(3, 1, 2, -\frac{5}{6}) . \quad (3.20)$$

S(5) Finally we note that the sextet singlets of (3.9) form an $SU(3) \times SU(2) \times SU(2)_5 \times U(1)_5$ representation

$$(6, 1, 2, -\frac{1}{6}) . \quad (3.21)$$

To exploit all of the above flavor symmetries and associated anomalies we would have to demonstrate that all of the fermions in $R_{SU(5)}$ remain massless until the $SU(2)$ gauge symmetry is broken dynamically. We would then be able to argue that the spectrum of fermions obtaining their mass entirely from the $SU(2)$ symmetry breaking process must reproduce the above anomalies. However, since the gauge symmetry is assumed to break first to $SU(3) \times SU(2) \times U(1)$ from $SU(5)$ we would in general expect some of $R_{SU(5)}$ to acquire $SU(3) \times SU(2) \times U(1)$ invariant masses in this symmetry breaking process. The only $SU(3) \times$

$SU(2) \times U(1)$ singlet appearing in the product $R_{SU(5)} \times R_{SU(5)}$ is given by

$$5 \times 45^* = 24 + \dots \quad (3.22)$$

which would allow mass terms of the form

$$(3, 1, -\frac{1}{3}) \times (3^*, 1, \frac{1}{3}) \text{ and } (1, 2, \frac{1}{2}) \times (1, 2, -\frac{1}{2}) \quad (3.23)$$

involving the corresponding quarks and leptons.

In our discussion of $SO(10)$ dynamics in Section 5 we shall argue that the mass-terms (3.23) are not generated. The reason will be that the additional $U(1)$ quantum number of the 5 and 45^* acquired from the $SO(10)$ embedding will, for the mass-term (3.22), have the wrong sign to be generated. However, since it is only the flavor symmetries $S(1)$ and $S(2)$ that we wish to exploit, we could allow the first term in (3.23) to occur--indeed we might welcome it--since it violates only $S(3)$ and $S(4)$. It will in fact occur in the $SU(5)$ dynamics we discuss in Section 6.

It is instructive to consider how the spectrum (3.2)-(3.12) might be modified dynamically to give the conventional quark and lepton spectrum. Of course, since quarks will be confined we can only ask that physical states be interpretable as formed from conventional quarks--with the flavor anomalies used to show that the counting of states is equivalent to the counting associated with conventional quarks.

We need to briefly review the salient features⁴ of dynamical symmetry breaking by a sextet chiral condensate. We introduce a sextet pion operator

$$U_6 = \exp[-i\tau \cdot \pi_6 / \pi_6] \quad (3.24)$$

which is a 2×2 matrix transforming under the $SU_L(2) \times SU_R(2)$ chiral symmetry of the sextet quarks. In the present context $SU_L(2)$ is to be identified with the $SU(2)$ gauge symmetry and $SU_R(2)$ should be identified with the $SU(2)_5$ flavor symmetry identified in (3.21). It is the expectation value

$$\langle 0 | U_6 | 0 \rangle = 1 \quad (3.25)$$

which produces the (sextet) chiral symmetry breaking and results from the sextet quark condensates, that is

$$\langle 0 | U_6 | 0 \rangle_{11} \sim \langle (6^*, 2, \frac{1}{6}) (6, 1, \frac{2}{3}) \rangle \quad (3.26)$$

$$\langle 0 | U_6 | 0 \rangle_{22} \sim \langle (6^*, 2, \frac{1}{6}) (6, 1, -\frac{1}{3}) \rangle . \quad (3.27)$$

Physical (in particular lepton) states are created by ($SU(2)$) gauge-invariant fermion operators of the form

$$\tilde{\Psi} = U_6 \Psi = \langle 0 | U_6 | 0 \rangle \Psi + \dots \quad (3.28)$$

$$= \Psi + \dots \quad (3.29)$$

where Ψ is an elementary fermion. Consequently physical lepton states can be regarded as containing an elementary fermion in a background condensate of sextet quarks having the form of (3.26) and (3.27).

Note also that the "gauge fields" which acquire masses are the $SU(2)_L$ invariant fields

$$\tilde{A}_\mu = U_6^{-1}(\partial_\mu + A_\mu)U_6 . \quad (3.30)$$

Consequently both $\tilde{\Psi}$ and $\tilde{A}_\mu$ transform under $SU(2)_R$ as the gauge-dependent fields Ψ and A_μ transform under $SU(2)_L$.

Exploiting the analogy of the sextet pion operator U_6 with the familiar pion operator we note that we expect U_6 to have a topological Skyrmon solution producing sextet baryons. The topology is associated with the non-trivial homotopy $\pi_4(SU(2)) = Z_2$. The Skyrmon lies in the non-trivial homotopy class while the condensate (3.25) lies in the trivial class. A two-Skyrmion (or Skyrmon/anti-Skyrmion) solution would also lie in the trivial class.

Next we note that octet quarks have a comparable color Casimir to that of the sextets ($C_6 = 3 \frac{1}{3}$, $C_8 = 3$) and so acquire their mass (from the sextet condensate) at a scale comparable with (or as we shall argue later, greater than) that of the condensate. Consequently the octet quarks have the right properties to be vacuum polarized by a sextet pion Skyrmon.² Because the octets will be absorbed in negative energy states, the energy of the Skyrmon will be significantly reduced by this process.

If each Skyrmon configuration absorbs a pair of octet quarks we obtain a configuration with the quantum numbers

$$(8, 2, -\frac{1}{2}) \times (8, 1, 1) \times (8, 2, -\frac{1}{2}) \times (8, 1, 1) = (1, 3, 1) , \quad (3.31)$$

we could combine such a "semi-classical" U_6 -operator with an "exotic" lepton doublet in analogy with (3.28), to obtain a conventional lepton doublet

$$(1, 2, -\frac{3}{2}) \times (1, 3, 1) = (1, 2, -\frac{1}{2}) . \quad (3.32)$$

Similarly an exotic quark/antiquark pair could combine with the other exotic lepton doublet to give

$$(1, 2, \frac{1}{2}) \times [(3, 1, -\frac{4}{3})(3^*, 1, \frac{1}{3})] = (1, 2, -\frac{1}{2}) . \quad (3.33)$$

This discussion suggests that topological properties of the sextet pion field may inter-relate with the color octet and exotic triplet fermion seas in such a manner as to compensate for the "wrong" $SU(2) \times U(1)$ quantum numbers of the above quarks and leptons. If this is the case, clearly the octet and exotic quarks will play a fundamental role in the dynamical solution we are seeking. While it is easy to speculate about such phenomena, we will be able to give a more systematic motivation for their occurrence after we have discussed the embedding of $R_{SU(5)}$ in an $SO(10)$ theory, as we now describe.

SECTION 4: THE $SO(10)$ EMBEDDING

The uniqueness properties of $R_{SU(5)}$ discussed in Section 2 imply that it cannot be embedded in any higher anomaly-free, asymptotically-free $SU(N)$ theory. It can, however, be embedded in a single, complex, anomaly-free, asymptotically-free $SO(10)$ representation. The 144 dimensional representation $SO(10)$ has¹⁰ the $SU(5) \times U(1)$ decomposition

$$144 = (5,7) + (15,-1) + (40,-1) + (45^*,3) + (5^*,+3) + (10,-1) + (24,-5) \quad (4.1)$$

$$= R_{SU(5)} + 5^* + 10 + 24. \quad (4.2)$$

Since $5^* + 10$ is well-known to be an anomaly-free representation of $SU(5)$ and the 24 has no $SU(5)$ anomaly, it is possible to regard the embedding of $SU(5)$ in this anomaly-free $SO(10)$ representation as an "explanation" of its anomaly-freeness.

Note that the 144 representation is some way from saturating the $SO(10)$ asymptotic freedom constraint. That is

$$B(144) = 63 < 88 \quad (4.3)$$

where the normalization is chosen to compare directly with that used in Section 2. (As a result $B = 88$ is the critical value for asymptotic freedom.) A particularly significant additional representation that can be added is the familiar

$$16 = (1, -5) + (5^*, 3) + (10, -1) . \quad (4.4)$$

Since

$$B(16) = 4 , \quad (4.5)$$

up to four 16's (or 16^* 's, or combinations of both) can be added to the 144 without losing asymptotic freedom. Previous attempts^{12,17} to construct Grand Unified Theories incorporating the 144 have added sufficient 16's to produce the known particle spectrum directly--with the 144 exploited for dynamical symmetry breaking in at least one case.¹⁷ Our intention is quite opposite , we wish to show that $R_{SU(5)}$ can be the low-energy remnant of the 144 and in particular produces the known particle spectrum.

For several reasons, that will be elaborated on in the following, it is most interesting for our purposes to add a single 16^* representation to the 144 giving

$$R_{SO(10)} = 144 + 16^* \quad (4.6)$$

$$\equiv R_{SU(5)} + [5 + 5^* + 10 + 10^* + 24 + 1] \quad (4.7)$$

$$\equiv R_{SU(5)} + r_{SU(5)} . \quad (4.8)$$

Since $r_{SU(5)}$ is a real $SU(5)$ representation it is straightforward, in principle at least, for all the fermions in $r_{SU(5)}$ to acquire large $SU(5)$

invariant masses. It is the purpose of the next Section to describe a dynamical framework which will indeed leave $R_{SU(5)}$ as the low-energy remnant of $R_{SO(10)}$.

It will be important in the next Section to have the decomposition of both the 144 and the 16 of $SO(10)$ under the branchings

$$SO(10) \rightarrow SU(4) \times SU(2) \times SU(2) \quad (4.9)$$

$$\rightarrow SU(3) \times SU(2) \times SU(2) \times U(1) \quad (4.10)$$

$$\rightarrow SU(3) \times SU(2) \times U(1) . \quad (4.11)$$

Under the first branching, that is (4.9), we have¹⁰

$$144 = (4, 2, 1) + (4^*, 1, 2) + (4^*, 3, 2) + (4, 2, 3) + (20, 2, 1) + (20^*, 1, 2) \quad (4.12)$$

and

$$16 = (4, 2, 1) + (4^*, 1, 2) . \quad (4.13)$$

The 144 has clear reality properties under the branching (4.12) and indeed the reality of $R_{SU(5)}$ under $SU(3) \times U(1)$ can be understood directly in terms of (4.12). We can also break the components of (4.12) and (4.13) down under the further branchings (4.10) and (4.11). We can then identify the components of the 144 under the $SU(5) \times U(1)$ branching (4.1). The result is¹⁰

$$(4, 2, 1) \rightarrow (3, 2, 1, \frac{1}{6}) + (1, 2, 1, -\frac{1}{2}) \quad (4.14)$$

$$\rightarrow [3, 2, \frac{1}{6}]_{10} + [1, 2, -\frac{1}{2}]_5^* \quad (4.15)$$

$$(4^*, 2, 1) \rightarrow (3^*, 1, 2, -\frac{1}{6}) + (1, 1, 2, \frac{1}{2}) \quad (4.16)$$

$$\rightarrow [3^*, 1, \frac{1}{3}]_5^* + [3^*, 1, -\frac{2}{3}]_{10} + [1, 1, 1]_{10} + [1, 1, 0]_{24} \quad (4.17)$$

$$(4^*, 3, 2) \rightarrow (3^*, 3, 2, -\frac{1}{6}) + (1, 3, 2, \frac{1}{2}) \quad (4.18)$$

$$\rightarrow [3^*, 3, \frac{1}{3}]_{45}^* + [3^*, 3, -\frac{2}{3}]_{40} + [1, 3, 1]_{15} + [1, 3, 0]_{24} \quad (4.19)$$

$$(4, 2, 3) \rightarrow (3, 2, 3, \frac{1}{6}) + (1, 2, 3, -\frac{1}{2}) \quad (4.20)$$

$$\rightarrow [3, 2, \frac{7}{6}]_{45}^* + [3, 2, \frac{1}{6}]_{15} + [3, 2, -\frac{5}{6}]_{24} \quad (4.21)$$

$$+ [1, 2, -\frac{3}{2}]_{40} + [1, 2, -\frac{1}{2}]_{45}^* + [1, 2, \frac{1}{2}]_5$$

$$(20, 2, 1) \rightarrow (3, 2, 1, \frac{1}{6}) + (3^*, 2, 1, +\frac{5}{6}) + (6^*, 2, 1, \frac{1}{6}) + (8, 2, 1, -\frac{1}{2}) \quad (4.22)$$

$$\rightarrow [3, 2, \frac{1}{6}]_{40} + [3^*, 2, \frac{5}{6}]_{24} + [6^*, 2, \frac{1}{6}]_{40} + [8, 2, -\frac{1}{2}]_{45}^* \quad (4.23)$$

$$(20^*, 1, 2) \rightarrow (3^*, 1, 2, -\frac{1}{6}) + (3, 1, 2, -\frac{5}{6}) + (6, 1, 2, -\frac{1}{6}) + (8, 1, 2, \frac{1}{2}) \quad (4.24)$$

$$\begin{aligned} &+ [3^*, 1, \frac{1}{3}]_{45}^* + [3^*, 1, -\frac{2}{3}]_{40} + [3, 1, -\frac{1}{3}]_5 + [3, 1, -\frac{4}{3}]_{45}^* \\ &+ [6, 1, -\frac{2}{3}]_{15} + [6, 1, \frac{1}{3}]_{45}^* + [8, 1, 1]_{40} + [8, 1, 0]_{24} . \end{aligned} \quad (4.25)$$

From the above decompositions we see that $R_{SU(5)}$ is contained entirely in the last four representations of (4.12), that is

$$\begin{aligned} R_{SU(4) \times SU(2) \times SU(2)} &= (4^*, 3, 2) + (4, 2, 3) + (20, 2, 1) + (20^*, 1, 2) \\ &\equiv R_4^* + R_4 + R_{20} + R_{20}^* \end{aligned} \quad (4.26)$$

$$= R_{SU(5)} + \tilde{r} , \quad (4.27)$$

where $\tilde{r}$ is not a complete $SU(5)$ representation but is entirely contained within the 24 of $SU(5)$ appearing in (4.1).

By comparing (3.13) with (4.20), (3.16) with (4.18), and (3.19)-(3.21) with (4.24) we see that each of the $SU(2) \times U(1)$ "flavor" symmetries of $R_{SU(5)}$ is gauged by the embedding in $SO(10)$. Indeed if we distinguish the $SU(2)$ factors in (4.9) by writing

$$SU(2) \times SU(2) \rightarrow SU_L(2) \times SU_R(2) ,$$

we can identify each of the flavor $SU(2)$ symmetries with $SU_R(2)$. Clearly the $SO(10)$ embedding automatically cancels all the flavor "anomalies". Note that

both of the anomalies A_1 and A_2 , that we wish to use to locate a three-generation spectrum, are due to fermions contained in the first two representations of (4.26), that is $R_4^* + R_4$. The remainder of (4.26), that is $R_{20} + R_{20}^*$ clearly contains all the higher-dimensional color representations and exotic triplet quarks that we either wish to form condensates or to disappear from the low-energy spectrum. This clear demarcation suggests how the three generations we would like to find might be generated.

Note first that $R_4 + R_4^*$ contains just the right number of states to generate three 16^* generations. Also

$$R_{20} \times R_{20}^* = (20 \times 20^*, 2, 2), \quad (4.28)$$

with

$$20 \times 20^* = 1 + 15_1 + 15_2 + 20' + \dots \quad (4.29)$$

Since each of 1 , 15_1 and 15_2 contains an $SU(3) \times U(1)$ singlet we have that (in terms of $SU(3) \times SU(2) \times SU(2) \times U(1)$ quantum numbers)

$$R_{20} \times R_{20}^* = 3 \times (1, 2, 2, 0) + \dots \quad (4.30)$$

(There is a fourth $SU(3) \times U(1)$ singlet generated in (4.29) but it occurs in a very high $SO(4)$ representation and will therefore not produce a condensate according to the hypotheses of the next Section.)

If we now consider three fermion states of the form

$$\bar{\Psi}_{R_{20}} \bar{\Psi}_{R_{20}^*} \bar{\Psi}_{R_4^*} + R_4 |0\rangle, \quad (4.31)$$

then (4.30) will give "three generations" of states with quantum numbers

$$\begin{aligned} (1,2,2,0) \times [R_4^* + R_4] &= (1,2,2,0) \times [(1,3,2,+\frac{1}{2}) + (3^*,3,2,-\frac{1}{6}) \\ &\quad + (1,2,3,-\frac{1}{2}) + (3,2,3,+\frac{1}{6})] \end{aligned} \quad (4.32)$$

$$\begin{aligned} &= (1,2,1,+\frac{1}{2}) + (3^*,2,1,-\frac{1}{6}) + (1,1,2,-\frac{1}{2}) \\ &\quad + (3,1,2,+\frac{1}{6}) + \dots \end{aligned} \quad (4.33)$$

This could be the three generations that we require, since (as is clear from (3.28) and (3.30), for example) the symmetry breaking effectively interchanges the role of the two SU(2) factors in (4.26), for physical states.

As we have observed, each of the SU(2) $\times$ U(1) flavor symmetries of $R_{SU(5)}$ is simultaneously gauged into the same $SU_R(2) \times U(1)$ subgroup of $SU(4) \times SU_L(2) \times SU_R(2)$ in the above. Therefore if we extend the lepton symmetry S(1) to this full symmetry the three generations of "composite leptons" in (4.33) will indeed reproduce the anomaly contribution A_1 of the "elementary" leptons. Since the "composite" antiquarks in (4.33) are SU(2) singlets under this symmetry it is useful to retain the flavor symmetry S(2) as a distinct global symmetry defined on the elementary antiquarks only. In this case the three generations of composite states formed from (4.33) will again reproduce the anomaly A_2 . Therefore the two anomalies A_1 and A_2 can be used in distinct

SECTION 5: SO(10) DYNAMICS--FOUR FERMION CONDENSATES, INSTANTON
INTERACTIONS AND HEAVY FERMION VACUUM POLARIZATION

We consider SO(10) Yang-Mills gauge theory with the left-handed fermion representation $R_{SO(10)}$ described in the previous Section. We shall be looking for a confining solution of the theory in which physical states are manifestly gauge-invariant SO(10) singlets. However, this immediately implies a difficulty for fermion states. We can define an SO(10) congruency¹⁰--for all representations--which takes values $C = -1, 0, 1, 2$ --with singlets having $C = 0$. ($C = -2$ is identified with $C = 2$ since all congruencies are defined modulo 4). Since the fermion representations in our theory satisfy

$$C(144) = +1 \quad \text{and} \quad C(16^*) = -C(16) = -1, \quad (5.1)$$

we cannot have singlet states containing an odd number of such fermions. Apparently then we cannot have bound-state, SO(10) singlet, fermions in our theory--unless the gauge field can contribute to the effective fermion number! Hence, we are led from the outset of our study of $R_{SO(10)}$ to look for the possible contribution of gauge-field topology to any (bound-state) dynamics we discuss.

We would like to develop a gauge-dependent dynamical symmetry breaking formalism based on the existence of vacuum expectation values for composite fermion operators. However, composite operators are in general difficult to define and vacuum expectation values which do not characterize the breaking of global symmetries are particularly ambiguous.¹⁸ A-priori we can at best expect only gauge-invariant composite operators to have well-defined vacuum

condensate values. In our theory we cannot form any gauge-invariant fermion bilinear operators (a common phenomenon for chiral theories) and so the composite operators we work with contain at least four-fermions. To develop a gauge-dependent symmetry breaking formalism we shall (initially) suppose that only those gauge-dependent condensates which can be identified with gauge-invariant condensates in a particular gauge (from which they can, in principle at least, be gauge-transformed to a general gauge) have a well-defined existence. The implementation of this constraint will become clear as we discuss examples below.

The existence of fermion condensates is a non-perturbative phenomenon which we assume is required by self-consistency of the theory. For the moment we also assume that instanton interactions are the main non-perturbative problem. We shall give a general discussion of the justification for this for $R_{SO(10)}$ in the next Section.

Note first that the axial anomaly mixes the fermion seas of different kinds of fermions through instanton interactions, and because of the vastly different Casimirs of the 144 and 16^* (as given in (4.3) and (4.5)) the instanton interactions conserve the "bizarre" fermion number

$$N_F = 17 N_{16^*} - N_{144} . \quad (5.2)$$

It is clearly a major non-perturbative problem of the $R_{SO(10)}$ theory to regulate the 144 and 16^* fermion seas in the presence of instanton interactions (among massless fermions) which conserve (5.2). The simplest such interaction is that illustrated in Fig. 5.1(a) (and, of course, those related to it by crossing). It seems, to us, very likely that high-order

Fig. 5.1

fermion condensates of the kind we assume to exist are actually required to reduce this interaction to a relatively simple coupling of right and left-handed fermions. Indeed as illustrated in Fig. 5.1(b) a gauge-invariant four-fermion condensate $\langle c \rangle$ would immediately reduce the interaction of Fig. 5.1(a) to that of a mass-term for a three-fermion composite state.

Fig. 5.2

If we assume that the invariant condensate is gauge-equivalent to a gauge-dependent condensate $\langle G \rangle$ then the instanton interaction can also give a gauge-dependent single-fermion mass term, as illustrated in Fig. 5.2(a), which

couples a left-handed 16^* to a right-handed 144^* . Equivalently if the invariant condensate factorizes into gauge-dependent two-fermion condensates, that is

$$\langle C \rangle = \langle g_1 \rangle \langle g_2 \rangle \quad (5.3)$$

$$= \langle G \rangle \quad \text{in a particular gauge ,} \quad (5.4)$$

then we can regard Fig. 5.2(b) as describing mass-generation for the three-fermion state

$$\bar{\psi}_{144} \bar{\psi}_{144} \bar{\psi}_{16}^* |0\rangle \sim \langle g_1 \rangle \bar{\psi}_{16}^* |0\rangle . \quad (5.5)$$

(To write (5.3) we must clearly assume that ambiguities in defining $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are absent in their product.)

The condensates $\langle g_1 \rangle$ and $\langle g_2 \rangle$ will be both gauge and scale dependent. We assume the conventional (perturbative) most-attractive channel¹⁹ hypothesis for this scale dependence. That is a fermion bilinear condensate is proportional (at large Q^2) to

$$[C_1 + C_2 - C] \alpha(Q^2) , \quad (5.6)$$

where C_1 and C_2 are the constituent fermion Casimirs and C is the Casimir of the product representation, with $\alpha(Q^2)$ being the scale-dependent (square of the) gauge coupling. As a consequence the largest high-energy condensate will be the product of the two highest dimensional representations (available) in

the lowest dimensional representation they can form. Conversely the invariant condensate $\langle C \rangle$ can be regarded as developing into an increasing sum of products of the form (5.6) as the effective energy-scale is decreased.

The 144 representation will be responsible for the highest energy condensate formed (as it must be to make sense of Figs. 5.1(b), 5.2(a) and 5.2(b)). The relevant representation products are

$$[144 \times 144]_s = 10 + 126 + 126^* + \dots \quad (5.7)$$

$$[10 \times 10]_s = 1 + 54 \quad (5.8)$$

$$[126 \times 126^*] = 1 + 45 + 210 + \dots \quad (5.9)$$

From (5.7) the highest scale four-fermion condensate formed satisfying (5.3) will be a product of $\langle 10 \rangle$ condensates. Note that since the singlet in (5.8) is the scalar product formed from the two "vector" 10's and the 54 is the symmetric tensor similarly formed from the two 10's it follows that in a gauge where the vector $\langle 10 \rangle$ has a single component $\langle 10 \rangle_0$ and (5.3) is satisfied with $\langle C \rangle = \langle 1 \rangle$ then (5.4) will be satisfied with $\langle G \rangle = \langle 54 \rangle$, that is

$$\langle 1 \rangle \equiv \langle 10 \rangle_0 \langle 10 \rangle_0 \sim \langle 54 \rangle_{00} \quad (5.10)$$

As we discussed above, the $\langle 54 \rangle$ can be defined in a general gauge by gauge-transforming (5.10).

Since

$$16^* \times 144 = 45 + 54 + \dots, \quad (5.11)$$

it follows that the $\langle 54 \rangle$ condensate will generate a mass for the 16^* --together with 16 components of the 144--through the instanton interaction of Fig. 5.2(b). Comparing (4.12) and (4.13) we see that it is the

$$(4,2,1) + (4^*,1,2) \quad (5.12)$$

part of the 144 which acquires a mass--leaving $R_{SU(4)} \times SU(2) \times SU(2)$ massless.

Next we come to the most fundamental and far-reaching assumption we make. Since the $\langle 10 \rangle$ and $\langle 54 \rangle$ condensates are gauge-dependent we can apply "large" gauge-transformations with non-trivial topology to them. This would generate, through the instanton interaction, a mass-term with non-trivial topology. In general, since there is no charge conjugation, we expect² such a "topological" mass-term to produce a shift of the zero-energy point in the fermion spectrum (relative to perturbation theory). That is one (or more) energy level(s) of the effective Dirac equation will cross zero because of the non-trivial topology of the mass-term.

An additional element in this discussion is that the presence of remaining massless fermions coupled to the instanton interaction implies that the topological classification of gauge fields breaks down. From the example of Manton's analysis¹ of the Schwinger model and his suggested generalization to four-dimensional gauge theories we anticipate that this breakdown will take the form that a non-integer expectation-value develops for the exponentiated winding-number operator

$$X = \exp \frac{i}{8\pi} \int \epsilon_{ijk} \text{Tr} \left(F_{ij} A_k - \frac{2}{3} A_i A_j A_k \right) d^3x . \quad (5.13)$$

(As a first approximation) this expectation value replaces sums over gauge field topology by a single non-integer winding number. In this case the effects of gauge-field topology on our present discussion would be summarized by giving the $\langle 54 \rangle$ condensate in (5.10) a non-trivial winding number.

For the 16^* the combination of the instanton interaction and condensates represented by Fig. 5.2(a) will be only the effective mass interaction we have discussed since the 16^* will not mix directly with the condensate through perturbative interactions (unlike the 144). Consequently the 16^* will feel the condensate topology only as a topological mass-term--which we assume therefore causes a net negative energy level to develop. That is the 16^* undergoes "topological heavy-fermion vacuum polarization" and is absorbed by the $\langle 54 \rangle$ topological condensate to produce a "five-fermion" scalar condensate

$$\langle 0 | \bar{\psi}_{16}^* \bar{\psi}_{144}^4 | 0 \rangle \sim \langle 16^* \times 54 \rangle \sim \langle 144^* \rangle . \quad (5.14)$$

Clearly such a scalar condensate implicitly carries a "fermionic" contribution through the non-trivial non-integer winding number of the background (condensate) gauge-field. Also it is clear that a vital part of our argument for its occurrence is the strong imbalance between the dimension of the right-handed fermion representation (effectively the 16) and the left-handed representation (the 144) that we have introduced.

The condensate (5.14) directly produces several dynamical features for us. Firstly the minimal symmetry breaking is now

$$144^* \xrightarrow{SU(5) \times U(1)} (24, 5) + \dots, \quad (5.15)$$

which directly gives

$$SO(10) \rightarrow SU(3) \times SU(2) \times U(1) \equiv SU(3) \times SU_L(2) \times U(1). \quad (5.16)$$

Secondly this condensate, in combination with the instanton interaction, generates further masses as illustrated in Fig. 5.3

Fig. 5.3

Since

$$(24, 5) \times (24, 5) = [(1, 10) \oplus 126^*] \quad (5.17)$$

$$+ [(24, 10)] + \dots, \quad (5.18)$$

$SU(5)$ invariant mass is generated for the $(24, -5)$ component of $R_{SO(10)}$ leaving $R_{SU(5)}$ as the massless sector--exactly as we wish. Note that (5.18) does give an $SU(5)$ breaking mass-term but the mass-term (3.23) is not

generated because the $SO(10)$ version of (3.22) would be

$$(5,7) \times (45^*,3) = (24,10) + \dots, \quad (5.19)$$

which has the wrong-sign $U(1)$ charge to give a singlet when combined with (5.18). Finally we note that a "five-fermion" condensate will remove the difficulty in deriving $SO(10)$ singlet fermion bound-states that we encountered because of (5.1).

The vital assumption of topological heavy fermion vacuum polarization has therefore shown how $R_{SO(10)}$ can generate a low-energy spectrum having $SU(5)$ symmetry while the gauge-interaction is broken to $SU(3) \times SU(2) \times U(1)$. Although the condensate (5.14) appears to be as gauge-dependent as any condensate we have introduced and therefore of doubtful significance we believe that the fermion vacuum polarization effect it signals should have a gauge-independent formulation and therefore when utilized in gauge-invariant quantities, such as the instanton interaction of Fig. 5.3, should be well-defined (to the extent this interaction is well-defined--as we discuss in the next Section).

After the $\langle 10 \rangle$ condensate the next to form (at what will ultimately be the electroweak scale) will be the $\langle 126 \rangle$ and $\langle 126^* \rangle$ condensates. These will produce a $\langle 210 \rangle$ four-fermion condensate via (5.9). (Note that the component of the 45 which is at least $SU(3) \times U(1)$ invariant but breaks $SU(2) \times SU(2)$ arises from an antisymmetric product of real components of the 126 and 126^* , and hence will not occur in a symmetrized four-fermion condensate.)

In discussing the $\langle 210 \rangle$ condensate it will be useful to at first ignore the condensate (5.14) and consider $R_{SU(4) \times SU(2) \times SU(2)}$ as interacting through

an $SU(4) \times SU(2) \times SU(2)$ gauge theory. In this case the most attractive channel rule will lead to the formation of condensates from $R_{20} \times R_{20}$, $R_{20^*} \times R_{20^*}$ and $R_{20} \times R_{20^*}$ at the highest scale. To discuss possible candidates we need in addition to (4.29) the product¹⁰

$$[20 \times 20]_S = 10 + 10^* + 64 + \dots \quad (5.20)$$

The candidate condensates which do not break $SU(4)$ down further than $SU(3)$ are (in $SU(4) \times SU(2) \times SU(2)$ notation)

$$\begin{aligned} & \text{i) } (1,2,2) , \quad \text{ii) } (15_1,2,2) , \quad \text{iii) } (15_2,2,2) , \quad \text{iv) } (10,3,1) , \\ & \text{v) } (10^*,3,1) , \quad \text{vi) } (10,1,3) , \quad \text{vii) } (10^*,1,3) . \end{aligned} \quad (5.21)$$

The latter six representations appear in the 126 and 126^* through

$$126 = (15,2,2) + (10,3,1) + (10^*,1,3) + \dots \quad (5.22)$$

$$126^* = (15,2,2) + (10^*,3,1) + (10,1,3) + \dots \quad (5.23)$$

The first representation in (5.21) contributes to the $\langle 10 \rangle$ condensate and indeed the Casimir rule would have such a condensate occur at a higher scale than the remaining six candidates. Since the 15 and 10 representations of $SU(4)$ have comparable Casimirs (4 and $4\frac{1}{2}$ respectively) we expect all of the latter six in (5.21) to form at a comparable scale, which from (5.22), (5.23) and (5.9) can also be regarded as the scale at which the $\langle 126 \rangle$ and $\langle 126^* \rangle$

condensates form and combine to give the four-fermion $\langle 210 \rangle$ condensates.

Products of (5.22) and (5.23) appear in the 210 representation through¹⁰

$$210 = (1,1,1) + (15,1,1) + (15,3,1) + (15,1,3) + (10,2,2) + (10^*,2,2) + \dots \quad (5.24)$$

Minimal symmetry breaking (beyond that produced by the $\langle 144^* \rangle$) is achieved by assuming that the condensates in (5.22) and (5.23) combine to give

$$\langle 210 \rangle = (10^*,2,2) , \quad (5.25)$$

which gives the symmetry breaking

$$SU(4) \times SU_L(2) \times SU_R(2) \rightarrow SU(3) \times SU_{L+R}(2) \times U(1) . \quad (5.26)$$

if we utilize

$$(10^*,2,2) \xrightarrow{SU(3) \times SU(2) \times SU(2) \times U(1)} (1,2,2,1) + \dots \quad (5.27)$$

Combined with (5.16), (5.26) gives the full symmetry breaking down to $SU(3) \times U(1)$.

Since the $(1,10)$ component of the 126^* appearing in (5.17) is a component of the $(10,1,3)$ representation appearing in (5.23), it follows that combining the $\langle 210 \rangle$ condensate with the $\langle 144^* \rangle$ condensate and the instanton interaction as illustrated in Fig. 5.4.

Fig. 5.4

will give the mass term

$$(10^*, 2, 2) \times (10, 1, 3) = (1, 2, 2) . \quad (5.28)$$

This is an $SU(4)$ invariant mass term which therefore couples only R_4 to R_4^* and R_{20} to R_{20}^* in (4.26).

We move on now to consider what can be said about the gauge-invariant physical states of the theory. The states must be $SO(10)$ invariant, but can we restrict them any further? We have observed that N_F is a conserved quantum number for the complete theory. If we allow general values of N_F we will have very peculiar constraints on the production processes that can occur. Consequently we assume (again in analogy with¹ the zero-charge of physical states in the Schwinger model) that physical states all have

$$N_F = 0 . \quad (5.29)$$

The simplest possible physical states therefore have the fermion content

$$|P\rangle = \bar{\psi}_{16}^* \bar{\psi}_{144}^{17} |0\rangle \quad (5.30)$$

$$\equiv |16^* 144^{17}\rangle \quad . \quad (5.31)$$

While these states are gauge-invariant we can, of course, describe them in a gauge-dependent manner. In particular if we extract from (5.31) the symmetry-breaking gauge-dependent condensates we have discussed then we obtain

$$|P\rangle = \langle 16^* 144^4 \rangle \langle 144^4 \rangle |144^9\rangle \quad (5.32)$$

$$= \langle 144^* \rangle \langle 210 \rangle |144^9\rangle \quad . \quad (5.33)$$

The $|144^9\rangle$ state will be a fermion and if we describe it as a "baryon" formed from three "quarks" then clearly the states we should identify as quarks have the form

$$\tilde{\Psi}_{144}^3 |0\rangle \equiv |144^3\rangle \quad . \quad (5.34)$$

Amongst such states are just those that we identified as giving three generations of quarks (and leptons) in (4.35). Comparing (5.21)-(5.23) with (4.29) and (4.30) we see that the three generations of states of the form (4.36) could be identified as

$$|144^3\rangle \equiv \langle 144^2 \rangle |144\rangle + \dots \quad (5.35)$$

$$\rightarrow \langle 10 \rangle |144\rangle, \langle 126 \rangle |144\rangle, \langle 126^* \rangle |144\rangle, \dots \quad (5.36)$$

(5.36) will give three sets of states with the quantum numbers of (4.33) if the $\langle 10 \rangle$, $\langle 126 \rangle$ and $\langle 126^* \rangle$ condensates are identified (respectively) with the first three condensates in (5.21).

These three bilinear condensates, if they were well-defined and independent, would be sufficient to break the $SU(4) \times SU(2) \times SU(2)$ symmetry of $R_{SU(4) \times SU(2) \times SU(2)}$ down to $SU(3) \times U(1)$ (with the $U(1)$ being a sub-group of $SU(4)$) independently of the $\langle 144^* \rangle$ condensate. In fact we have assumed that the bilinear condensates only contribute to symmetry breaking through the (four-fermion) $\langle 210 \rangle$ condensate of the form (5.25). (Although of course, additional components of the $\langle 210 \rangle$ could produce further symmetry breaking.)

In the next Section we shall effectively consider the limit in which $\langle 144^* \rangle$ becomes infinite and all the components of $\tilde{r}$ acquire infinite mass leaving only $R_{SU(5)}$ to form condensates. To consider the implications of this, we note that since the condensates in (5.21) are $SU(3) \times U(1)$ invariant we can appeal to (5.6) and take them to be formed from the three representations in R_{20} and R_{20}^* having the highest $SU(3) \times U(1)$ Casimirs, that is i)-iii) will be linear combinations of

$$(3^*, 2, 1, \frac{5}{6}) \times (3, 1, 2, -\frac{5}{6}); (6^*, 2, 1, \frac{1}{6}) \times (6, 1, 2, -\frac{1}{6}); (8, 2, 1, -\frac{1}{2}) \times (8, 1, 2, \frac{1}{2}). \quad (5.37)$$

Comparing with (4.23) and (4.25) we see that only the sextet condensate remains in its entirety if all components of $\tilde{r}$ (which are denoted by $[]_{24}$ in (4.23) and (4.25)) are decoupled from the theory. In this context therefore, the issue as to how much the gauge symmetry is broken by the three condensates utilized in (5.36) becomes irrelevant. We expect it to be important, however, that we can use all three condensates in constructing the states (5.33) in an

approximation in which $\langle 144^* \rangle$ can be taken to be small.

It seems therefore that the physical states of the theory are very complicated eighteen (at least) fermion states at high-energy, but that at low-energy a gauge-dependent description exists in which almost all degrees of freedom are frozen out via condensates. We anticipate that there will be many different equivalent descriptions of the low-energy dynamics. In particular we could rewrite the physical states (5.30) in the form

$$|P\rangle = \langle 16^* 144^2 \rangle [\langle 144^4 \rangle |144\rangle]^3, \quad (5.38)$$

if we assume the "five-fermion" condensate (5.14) can also be factorized into gauge-dependent condensates. That is

$$\langle 16^* 144^4 \rangle = \langle 16^* 144^2 \rangle \langle 144^2 \rangle \quad (5.39)$$

$$= \langle 16 \rangle \times \langle 10 \rangle. \quad (5.40)$$

If we also take as (an example of) three independent four-fermion condensates

$$\langle 144^4 \rangle = \langle 10 \rangle \langle 10 \rangle, \langle 10 \rangle \langle 126 \rangle, \langle 10 \rangle \langle 126^* \rangle, \quad (5.41)$$

with the two-fermion condensates again taken to be the first three in (5.21), we have three generations of the form

$$\langle 144^4 \rangle |144\rangle + \langle (1,2,2,0) \rangle \times \langle (1,2,2,0) \rangle \times [R_4 + R_4^*] \quad (5.42)$$

$$= (1,3,3,0) \times \left[(1,3,2, \frac{1}{2}) + (3^* 3,2, -\frac{1}{6}) + (1,2,3, -\frac{1}{2}) + (3,2,3, \frac{1}{6}) \right] \quad (5.43)$$

$$= (1,1,2, \frac{1}{2}) + (3^* 1,2, -\frac{1}{6}) + (1,2,1, -\frac{1}{2}) + (3,2,1, \frac{1}{6}) , \quad (5.44)$$

which is three conventional generations (without the interchange of $SU(2)_L$ and $SU_R(2)$ quantum numbers). Another alternative would be to take the three distinct bilinear condensates in (5.41) to be those with the $SU(3) \times U(1)$ quantum numbers of (5.37). We could also (as in the next Section) use products of distinct condensates rather than identical condensates.

The form (5.38) is appropriate for discussing baryons, for leptons we can utilize

$$|P\rangle = \langle 16^* 144^4 \rangle \langle 144^4 \rangle \langle 144^4 \rangle [\langle 144^4 \rangle |144\rangle] , \quad (5.45)$$

which is sufficient to show that the lepton components of (5.44) can be treated on the same footing as the baryons written in the form (5.38).

SECTION 6: SU(5) DYNAMICS--RENORMALON DISAPPEARANCE AND SKYRMION VACUUM POLARIZATION

In this Section we give a preliminary outline of how the dynamical solution of $R_{SO(10)}$ developed in the last Section may generate the low-energy physical solution of $R_{SU(5)}$ that we have been seeking.

We begin by noting that the instanton interactions we have exploited are not actually well-defined in $R_{SO(10)}$ because of infra-red divergences^{9,18} associated with renormalons. In the Borel transform b -plane, defined by representing any Greens function $A(g^2)$ according to

$$A(g^2) = \int_0^{\infty} db \, e^{-b/g^2} \tilde{A}(b) , \quad (6.1)$$

the first renormalon singularity, arising from the (wild) divergence of the perturbation expansion, occurs for⁹

$$b_R = \frac{24\pi}{B_{SO(10)}} = \frac{24\pi}{44-34-2} = \frac{24\pi}{8} = 3\pi , \quad (6.2)$$

whereas the first instanton/anti-instanton singularity occurs at

$$b_I = 4\pi . \quad (6.3)$$

As a consequence instanton interactions are less important non-perturbatively than renormalons and do not give a well-defined contribution to the Borel integral (6.1). (This ambiguity is directly related to the infra-red divergence over scale-size²⁰ of instanton functional integral contributions.)

If we consider $R_{SU(5)}$ directly, then the saturation property $v)$ of Section 2 produces

$$b_R = \frac{24\pi}{B_{SU(5)}} = \frac{24\pi}{55-54} = 24\pi , \quad (6.4)$$

immediately moving the first renormalon singularity far away from B_I (which remains given by (6.3)). In addition the nature of the b_R singularity is²¹ that of

$$\frac{4B'}{(b-b_R)^{B^2}} - 1 , \quad (6.5)$$

where the β -function is denoted as

$$\beta(\alpha) = B\alpha^2 + B'\alpha^3 + \dots . \quad (6.6)$$

Since B^{-1} is an integer, (6.5) is analytic and there is in fact no renormalon singularity. Indeed the existence of a zero in the β -function of $R_{SU(5)}$ and the consequent lack of growth of the gauge-coupling may imply there are no renormalons at all in the $R_{SU(5)}$ theory. As a consequence instanton interactions within $R_{SU(5)}$ will be well-defined and will be the non-perturbative problem of the theory.

We may therefore ask whether any purpose is served by embedding $R_{SU(5)}$ in $R_{SO(10)}$, since the instanton interactions we wish to exploit are better defined in $R_{SU(5)}$. However, we believe the $SU(5)$ problem is too difficult to solve directly. $R_{SU(5)}$ has subtle reality properties and the instanton interaction is very complicated--conserving the fermion numbers

$$22N_{45}^* - 24N_{40}, \quad 7N_{45}^* - 24N_{15}, \quad N_{45}^* - 24N_5 \quad (6.7)$$

--the simplest interaction conserving all these numbers is illustrated in Fig. 6.1

Fig. 6.1

A-priori it is certainly unclear how this interaction can be exploited to obtain a sensible regularization of the $R_{SU(5)}$ fermion seas and consequent mass generation within (and presumably only within) appropriate composite states.

By embedding $R_{SU(5)}$ in $R_{SO(10)}$ we have added a high-energy real ($SU(5)$) representation and as a result introduced a (presumably) well-defined ultra-violet regularization of the fermion seas. We would like to regard this as effectively the first-stage of a regularization of the $R_{SU(5)}$ theory. The second-stage being the analysis of the $SO(10)$ instanton interaction with an infra-red cutoff and the final stage the removal of the infra-red cutoff producing, we suspect,

$$\langle 144^* \rangle \rightarrow \infty, \quad (6.8)$$

with all fermions in $R_{SU(5)}$ becoming infinitely massive and decoupling from the theory.

Independently of whether the $R_{SO(10)}$ theory can be rigorously used as a regulator of the $R_{SU(5)}$ theory we can look directly for a solution of the $SU(5)$ theory which reproduces the salient features of the $SO(10)$ dynamics. This will be the purpose of the following discussion. However, we also note the argument²² of D'Hoker and Farhi that infinitely massive fermions coupled to a scalar Higgs sector will be replaced by Higgs field solitons if constraining anomaly cancellations are operative. Such solitons also enable instanton interactions to survive the decoupling of the fermions. In our case the effective Higgs sector will be the sextet pion field and as we implied in Section 3, and will discuss further shortly, we expect sextet pion Skyrmions to absorb octet and (exotic) triplet quarks to produce soliton descriptions of the conventional "quark-lepton" spectrum. We therefore suspect that a full discussion of the $R_{SO(10)} \rightarrow R_{SU(5)}$ decoupling, exploiting the D'Hoker and Farhi argument, could produce the dynamical solution we now elaborate.

We note first that the 5-representation relates to the instanton interaction of Fig. 6.1 in a manner closely analogous to the 16^* appearance in the $SO(10)$ instanton interaction. That is, it is vastly outnumbered by multiple higher-dimensional representations which we expect to form the highest energy condensates. We therefore expect the 5 to acquire a large mass from the instanton interaction and undergo heavy fermion vacuum polarization. Noting that (in terms of $SU(5) \times U(1)$ quantum numbers)

$$(5,7) \times (40,-1) \times (40,-1) = (24,5) + \dots \quad (6.9)$$

we see that the 5 could be vacuum-polarized by a gauge-dependent $\langle 40 \times 40 \rangle$ condensate to give a condensate with the same quantum numbers as the $\langle 144^* \rangle$ induced by the vacuum polarization of the 16^* . Equivalently a three fermion state of the form (6.9) could be vacuum polarized by a $(24,0)$ condensate which is gauge-equivalent 9 (as discussed in the last Section) to a gauge-invariant condensate.

A gauge-invariant four-fermion condensate of the appropriate form is given by

$$(40, -1)^3 \times (45^*, 3) = (1, 0) + (24, 0) + \dots \quad (6.10)$$

We can also form a six-fermion condensate with the same quantum numbers as a product of $\langle 144^* \rangle$'s, that is

$$(40, -1)^2 \times (45^*, 3)^4 = (1, 10) + (24, 10) + \dots \quad (6.11)$$

Since they have the same pentalty, 15's can be substituted for 40's in each of the above. Note that to proceed as in the last Section and reduce the instanton interaction of Fig. 6.1 to a mass-term for the three-fermion state (6.9) using condensates, we have to use some 15's. We also have to use a mixture of four and six-fermion condensates. We can relate this to the analysis of $R_{SO}(10)$ as follows.

First we note that three-fermion states of the form

$$40 \times (45^*)^2 = 1 + 24 + \dots \quad (6.12)$$

can directly (at the $SU(5)$ level) replace the $[1]_{24}$ components of $SU(4) \times SU(2) \times SU(2)^*$ listed in (4.8)-(4.25), which are absent in $R_{SU(5)}$. Utilizing this replacement the six-fermion condensates (6.11) can be thought of as replacing four-fermion condensates formed from elements of $SU(4) \times SU(2) \times SU(2)^*$. In particular the decoupled color triplet and octet states can be replaced by, for example (using $SU(3) \times SU(2) \times U(1)$ notation)

$$[3^*, 2, \frac{5}{6}]_{24} \equiv (3^*, 1, \frac{1}{3})(8, 2, -\frac{1}{2})(8, 1, 1) \quad (6.13)$$

$$[8, 1, 0]_{24} \equiv (3^*, 1, \frac{1}{3})(3, 1, -\frac{4}{3})(8, 1, 1) . \quad (6.14)$$

In this case a four-fermion condensate such as, for example,

$$(8, 2, -\frac{1}{2})(8, 1, 1)(3^*, 2, \frac{5}{6})(3, 1, -\frac{4}{3}) > \sim <(8, 2, -\frac{1}{2})(8, 1, 1) > <(3^*, 2, \frac{5}{6})(3, 1, -\frac{4}{3}) > \quad (6.15)$$

will be replaced by the six-fermion condensate

$$<(8, 2, -\frac{1}{2})^2(8, 1, 1)^2(3^*, 1, \frac{1}{3})(3, 1, -\frac{4}{3}) > . \quad (6.16)$$

As we noted in the last Section, the sextet quark condensates remain intact within $R_{SU(5)}$ (by construction, of course) and we would now like to associate the breaking of the sextet quark chiral symmetry directly with the breaking of $SU(2)$ gauge symmetry. Although this may be an over-simplification it will enable us to have a clear separation of the dynamics of $SU(5)$ and $SU(2)$ symmetry breaking. Since the sextet quarks are distinguished from the octet and $SU(2)$ singlet, color triplet quarks by the protection of this chiral

symmetry, they can naturally have a much lower constituent mass than such quarks. As we shall see we obtain a consistent picture if we describe the dynamics of SU(5) symmetry breaking entirely in terms of octet and triplet quarks.

In the following discussion of states and condensates we assume we obtain the highest mass or energy-scale by maximizing the octet content (versus the triplet content). We therefore take the composite (6.9) to be

$$(1,1,0) \equiv (3,1,-\frac{1}{3})(3^*,1,-\frac{2}{3})(3,1,1) \quad (6.17)$$

and the four-fermion condensate (6.10) to be

$$\langle (3,1,-\frac{4}{3})(3,1,1)^2(3^*,1,-\frac{2}{3}) \rangle = \langle (1,1,0) \rangle. \quad (6.18)$$

Note that if we take the partner to (6.17), which also acquires mass, to be

$$(3^*,1,-\frac{1}{3})(3,1,-\frac{4}{3})(3,1,1) = (1,1,0) + \dots \quad (6.19)$$

and assume the condensate (6.18) factorizes appropriately then a mass-term coupling (6.17) and (6.19) is equivalent to a (gauge-dependent) mass-term of the first form appearing in (3.23).

To accommodate the 15's in the instanton interaction in condensates, the sextets can be absorbed in condensates in a manner which does not break the sextet chiral symmetry, for example

$$\langle (3,1,1)(3^*,1,-\frac{2}{3})(\bar{3},1,\frac{1}{3})(\bar{3},1,-\frac{2}{3}) \rangle \quad \langle 40^2 45^* 15 \rangle \quad (6.19)$$

and

$$\langle (8, 2, -\frac{1}{2})^2 (8, 1, 1) (3^*, 1, \frac{1}{3}) (6, 1, \frac{1}{3}) (6, 1, -\frac{2}{3}) \rangle \quad \langle (45^*)^4 40 \ 15 \rangle$$

However, we assume that (6.18) and (6.16) are respectively much larger than (6.19) and (6.20), and that (6.18) is in fact the largest condensate. In this case SU(5) symmetry breaking can be understood as vacuum polarization of the composite (6.17), by a gauge-dependent topological condensate (6.18), via the instanton interaction. The topological character of the condensate is, as we discussed in the last Section, induced by the presence of remaining massless fermions coupled to the instanton interaction. If the condensate (6.18) does factorize, then (6.17) could be interpreted directly as giving a three-fermion condensate obtained by $\langle (3^*, 1, -\frac{2}{3}) (8, 1, 1) \rangle$ absorbing $(3, 1, -\frac{1}{3})$ as discussed after (6.9).

If the physical states again have zero-value of the fermion numbers conserved by the instanton interaction then the simplest gauge-invariant state contains 54 fermions. Clearly a very large proportion of the degrees of freedom of such states are now to be envisaged as (in a gauge-dependent description) frozen out at low-energy. To identify three generations of constituents, in parallel with the last Section, we proceed as in (5.38) onwards by first listing the three independent products of condensates that will be analogous to (5.41) in our discussion. We choose (in SU(4) $\times$ SU(2) $\times$ SU(2) notation)

$$\langle (8, 2, 1, -\frac{1}{2}) (8, 1, 2, \frac{1}{2}) \rangle \langle (8, 2, 1, -\frac{1}{2}) (8, 1, 2, \frac{1}{2}) \rangle \quad (6.21)$$

$$\langle (8, 2, 1, -\frac{1}{2}) (8, 1, 2, \frac{1}{2}) \rangle \langle (3^*, 2, 1, \frac{5}{6}) (3, 1, 2, -\frac{5}{6}) \rangle \quad (6.22)$$

$$\langle (3^*, 2, 1, \frac{5}{6}) (3, 1, 2, -\frac{5}{6}) \rangle \langle (3^*, 2, 1, \frac{5}{6}) (3, 1, 2, -\frac{5}{6}) \rangle \quad (6.23)$$

since these three products do not involve the color sextets. Consequently when physical states are formed through (3.28) there will be no complications in their transformation properties under the physical $SU_W(2)$.

Utilizing (6.13) and (6.14), together with

$$[3, 2, -\frac{5}{6}]_{24} = (3, 2, \frac{1}{6}) (8, 2, -\frac{1}{2}) (8, 2, -\frac{1}{2}) \quad (6.24)$$

(to replenish R_4) we can use the analogue of (5.42) involving (6.21)-(6.23) to form three generations of quarks and leptons. If we were to treat (6.21)-(6.23) as simple condensates it would, for example, be straightforward to pick out that component of (6.21) which converts the $(1, 2, -\frac{3}{2})$ element of $R_4 + R_4^*$ to a conventional lepton. The result is

$$(1, 2, -\frac{1}{2}) = \langle (8, 2, -\frac{1}{2}) (8, 1, 1) \rangle \langle (3, 2, -\frac{1}{2}) (8, 1, 1) \rangle (1, 2, -\frac{3}{2}) . \quad (6.25)$$

Similarly picking out the element of (6.21) which converts $(1, 2, \frac{1}{2})$ to a conventional lepton would give (utilizing (6.14))

$$\begin{aligned}
 (1, 2, -\frac{1}{2}) &= \langle (8, 2, -\frac{1}{2})(8, 1, 1)(3^*, 1, \frac{1}{3})(3, 1, -\frac{4}{3}) \rangle \\
 &\quad \langle (8, 2, -\frac{1}{2})(8, 1, 1)(3^*, 1, \frac{1}{3})(3, 1, -\frac{4}{3}) \rangle (1, 2, \frac{1}{2})
 \end{aligned}
 \tag{6.26}$$

In fact if (6.21)-(6.23) existed as independent well-defined condensates then, as discussed in the last section, we would expect to see further effective symmetry breaking at low-energy. (In particular there would be no electric charge conservation.) Instead we propose that while the analogous condensates would naturally exist in $R_{SO(10)}$ and be appropriate for describing physical states when $\langle 144^* \rangle \sim 0$, they are instead replaced by the mechanism of Skyrmon absorption of heavy fermions in $R_{SU(5)}$.

Each of the fermion pairs (treating elementary and composite fermions alike) appearing in (6.21)-(6.23) has the appropriate quantum numbers to acquire mass from the sextet chiral condensate. Consequently we suppose that the U_6 operator which transforms from $SU(2)$ gauge-dependent fermion states to "physical states", as in (3.28), has a short-distance topological structure which allows us to write

$$U_6 \sim S\bar{S} , \tag{6.27}$$

where $S(\bar{S})$ is a Skyrmon ("anti-Skyrmon") configuration. In this case the Skyrmon can absorb the fermions appropriate to one condensate pair in (6.21), (6.22) or (6.23) while the anti-Skyrmon configuration absorbs the second pair.

A vital feature of Skyrmon absorption of heavy fermions is that the energy of the configuration is actually lowered by the process. This is

because the heavy fermions occupy negative energy levels as a result of the topology of the Skyrmon configuration. Consequently a sextet pion Skyrmon configuration which would initially have the (large) energy-scale of a sextet-quark baryon can have a much lower-energy after absorbing the heavy octets and triplets contained in (6.25) and (6.26), for example. Of course, we must also assume that the complete Skyrmon/anti-Skyrmon configuration absorbing heavy fermion sets such as (6.25) and (6.26) is stabilized by the presence of the elementary lepton. Our assumption is that it is the existence of the anomalies controlling the low-energy spectrum which can be understood as producing this stability. In particular we hope that following through the $R_{SO(10)} \rightarrow R_{SU(5)}$ decoupling would show that light states involving elementary fermions in appropriate condensates such as (6.21)-(6.23) become, during the decoupling, light states involving Skyrmon/anti-Skyrmon configurations. This replacement of fermions in condensate configurations by soliton configurations would be very close in spirit to the D'Hoker and Farhi argument.²²

For states containing R_4 components we can distinguish distinct generations by the elementary quark or lepton they contain--for example, the lepton doublets of (6.25) and (6.26) can be so distinguished. The combination of octet and triplet quarks which must be buried in the state at short distances is then determined by adding the contributions from (6.21)-(6.23) as in (6.25) and (6.26). For states containing R_4^* components the generations are not distinguished in this way at the elementary level--for example, the (1,3,1) component of R_4^* must produce three generations of lepton singlets (the triplet antiquarks $(3^*, 3, \frac{1}{3})$ and $(3^*, 3, -\frac{2}{3})$ must behave similarly). In this case the octet and triplet quark content implied by (6.21)-(6.23) can be

thought of as distinguishing the three generations. Of course, the full mass-matrix has to be diagonalized to properly designate generation components.

Finally we note that we could attempt to build the complete set of 54-component $SU(5)$ singlet states utilizing the condensates and fermion vacuum polarization effects we have identified, in analogy with our discussion of $R_{SO}(10)$ in the last Section. Although we believe this is straightforward in principle, it is a complicated exercise and we will not attempt it here. For the moment we are satisfied to have at least outlined how the $SU(5)$ symmetry breaking is achieved and how three generations of physical states can be identified.

REFERENCES

1. In particular N. S. Manton's exposition of the Schwinger model in terms of the anomaly and fermion spectral flow, Ann. Phys. 159, 220 (1985).
2. R. Mackenzie, F. Wilczek and A. Zee, Phys. Rev. Lett. 53, 2203 (1984);
R. Mackenzie, Proceedings of the Symposium on Anomalies, Geometry and Topology (1985), edited by W. A. Bardeen and A. R. White and published by World Scientific Publishing Co., Singapore, and further references in these proceedings.
3. L. Susskind, Phys. Rev. D20, 2619 (1979).
4. W. Marciano, Phys. Rev. D21, 2425 (1980), E. Braaten, C. R. Willcox, and A. R. White, to be published in Journal of Modern Physics A - published by World Scientific Press.
5. A. R. White, Phys. Rev. D29, 1435 (1983) and High Energy Hadron Scattering in QCD, Argonne Lectures (1985), in preparation as an Argonne preprint.
6. This is also commented on by M. B. Green, Proceedings of the Symposium on Anomalies, Geometry and Topology (1985).
7. A. A. Migdal, A. M. Polyakov and K. A. Ter Martirosyan, Zh. Eksp. Teor. Fiz. 67, 84 (1974), H. D. I. Abarbanel and J. B. Bronzan, Phys. Rev. D9, 2397 (1974).
8. A. R. White, Proceedings of the Workshop on Elastic and Diffractive Scattering (1985), edited by B. Nicolescu and J. Tran Thanh Van, published by World Scientific Publishing Co., Singapore.
9. G. Parisi, Nucl. Phys. B150, 163 (1979), G. 't Hooft, in the Whys of Subnuclear Physics (1977) edited by A. Zichichi--Erice Lectures,

published by Plenum Press, New York.

10. R. Slansky, Phys. Reports 79, 1 (1981).
11. For example, H. Georgi, Nucl. Phys. B156, 126 (1972); P. H. Frampton, Phys. Lett. 89B, 352 (1980); K. Kang and I. G. Koh, Phys. Rev. D25, 1700 (1982); K. Kang, C. K. Kim, and J. K. Kim, Phys. Rev. D33, 260 (1986).
12. R. C. King, Nucl. Phys. B185, 133 (1981).
13. E. Eichten, K. Kang, I. G. Koh, J. Math. Phys. 23, 2529 (1982).
14. D. Gross and F. Wilczek, Phys. Rev. D8, 3633 (1973), T. Banks and A. Zaks, Nucl. Phys. B196, 189 (1982).
15. G. Parisi, Proceedings of XX International Conference, Madison (1980), AIP Conference Proceedings No. 68, Particles and Fields Subseries No. 22.
16. G. 't Hooft, in Recent Developments in Gauge Theories (1979), edited by 'G. 't Hooft et al., published by Plenum Press, New York.
17. O. Napoly, Z. Phys. C 28, 195 (1985).
18. F. David, Nucl. Phys. B234, 237 (1984).
19. S. Raby, S. Dimopoulos and L. Susskind, Nucl. Phys. B169, 373 (1980).
20. G. 't Hooft, Phys. Rev. D14, 3432 (1976), C. G. Callan, R. Dashen and D. J. Gross, Phys. Rev. D17, 2717 (1978).
21. A. H. Mueller, Nucl. Phys. B250, 327 (1985).
22. E. D'Hoker and E. Farhi, Nucl. Phys. B248, 77 (1984).