Spectral Estimates of a Wind Fluctuation Statistic Pertaining to Wind Energy Generators

T. C. Kerrigan
Atmospheric Physics Section

September 1978

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SUMMARY

The object of this paper is to develop an estimate of the frequency with which a volume average of the longitudinal component of wind velocity changes by a given amount in a given time. First, a general stochastic model for the longitudinal component of atmospheric turbulence is explicitly constructed. Second, the mathematical foundations on which this paper rests are stated. They include Rice's theorem on level crossings and a modification of the power spectrum for the longitudinal wind component to account for the volume averaging induced by a windmill rotor. Third, these mathematical results are applied to the general wind model to obtain an estimate of the frequency described above in terms of general spectral information. Fourth, attention is restricted to a stochastic wind model based on the theory of isotropic turbulence in order to obtain this estimate in terms of commonly available spectral information. And fifth, a summary and sample calculations are provided.

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I. INTRODUCTION

Atmospheric turbulence is generally thought to complicate the operation and design of wind energy generators^[1]. Rapid changes in the wind in the direction of the mean flow degrade power output and may threaten structural integrity. Thus, an estimate of the frequency with which this wind component changes by a given amount in a given time is regarded as important.

This paper develops such an estimate in the context of two mathematically explicit spectral models for the wind. Each model represents the longitudinal component of wind velocity as a normal stationary stochastic process which satisfies the hypotheses of the spectral theorem. This estimate then emerges as a corollary of a theorem on level crossings originally given by S. O. Rice^[2]. He computes the expected number of times a normal scalar process crosses a level u in the interval $[0, T]$ in terms of its first several spectral moments. We derive a process from the stochastic model for the longitudinal wind component such that a level crossing for the derived process may be interpreted as a change in this wind component by an amount Δ in time 2τ . Rice's result then yields the expected number of such changes in $[0, T]$ in terms of the first several spectral moments of the derived process.

Rice's result does not apply directly to this estimation problem, however. The obstacle is that most popular characterizations of the power spectral density for the longitudinal

component of atmospheric turbulence fall off according to a $5/3$ -power law^[3]. This asymptotic behavior implies the divergence of the second spectral moment for the derived process, which in turn implies an infinite expected number of wind events of the kind described above.

These characterizations fail to account for two significant physical effects, however. The first is viscous depletion of kinetic energy in the higher spectral frequencies^[4]. Although including viscous depletion results in the convergence of the second spectral moment for the derived process, the specific numerical value of the result is sensitive to the distribution of energy through the entire range of frequencies. The second effect is the virtual volume averaging of the wind field by the presence of the windmill rotor. Including this averaging also results in the convergence of the second spectral moment for the derived process, since it effectively diminishes the role of the higher spectral frequencies. In the course of this paper, we shall modify our spectral models by including this second effect. Besides accounting for windmill geometry and providing for the convergence of the second spectral moment, this volume averaging dominates the less assessable influence of viscous depletion.

Finally, we observe the following outline of the derivation of this spectral estimate. First, we explicitly construct a general stochastic model for the longitudinal component of atmospheric turbulence. Second, we state the mathematical foundations on which this paper rests. They consist of Rice's theorem

on level crossings, a construction of the spectrum for the derived process, and a modification of this spectrum to account for volume averaging. Third, we apply these mathematical results to the general wind model to obtain an estimate of the statistic we seek in terms of general spectral information. Fourth, we restrict attention to a stochastic model based on the theory of isotropic turbulence in order to obtain this estimate in terms of commonly available spectral information. And fifth, we provide a summary of the results which are most useful in computing this estimate and some sample calculations which serve to illustrate these results.

II. THE GENERAL WIND MODEL

The object of this section is to construct a stochastic process which will serve as the general model for the wind component in the direction of the mean flow.

Let t denote time and x denote position. We write $x = (x_1, x_2, x_3)$ for a general element in the three-dimensional Euclidean space E_3 . We specify that x_1 correspond to the longitudinal direction (direction of the mean flow), x_2 to the lateral direction, and x_3 to the vertical direction. We take the earth's surface to be a plane normal to the vertical axis, center the coordinate system above this surface and assume that the wind speed is identically zero in the opposite half-space.

We assume that the longitudinal component of wind velocity is modeled by a stochastic process $u(t, x)$ which is stationary in t , homogeneous in (x_1, x_2) , and continuous in quadratic mean. Let E denote expectation and define the mean U and the covariance function R by

$$U(t, x) \equiv E u(t, x) \quad ,$$

and

$$R(t, x, y) \equiv E(u(t, x) - U(t, x))(u(t, y) - U(t, y)) \quad .$$

Stationarity and homogeneity imply that U depends on just x_3 , and stationarity implies that R is independent of t . Unfortunately, the spectral theorem does not apply directly to the

spatial analysis of the u -process since R lacks translation invariance in x_3 . We proceed to replace u by an associated homogeneous process u' , however, in order to take advantage of the spectral approach.

Let k denote a function on E_3 having the following properties:

- 1) $k(x) \geq 0$ for all x in E_3 ,
- 2) $k(x) = 0$ for $|x| \geq r$ for some r ,
- 3) $\int_{E_3} k(x) dx = 1$.

k serves to define the region of space in which the wind has an influence on the windmill rotor. A precise specification of the actual region of influence is quite beyond the scope of the present development. Two preliminary recommendations are offered in Appendix B, however. They correspond to shapes associated with popular rotor geometries, namely, a disk and a sphere.

Now set

$$R'(x) \equiv \int_{E_3} \frac{1}{2} [R(x+y, y) + R(-x+y, y)] k(y) dy \quad .$$

R' is just the symmetrized volume average of the covariance function over the volume defined by k . Let $\{z_1, \dots, z_n\}$ denote a set of n complex numbers and $\{w_1, \dots, w_n\}$ a set of n vectors in E_3 . Then

$$\begin{aligned}
& \sum_{i,j=1}^n R'(w_i - w_j) z_i \overline{z_j} \\
&= \frac{1}{2} \int_{E_3} \left[\sum_{i,j=1}^n R(w_i + y, w_j + y) z_i \overline{z_j k(w_j + y)} \right] dy \\
&+ \frac{1}{2} \int_{E_3} \left[\sum_{i,j=1}^n R(w_j + y, w_i + y) z_i k(w_i + y) \overline{z_j} \right] dy \\
&\geq 0
\end{aligned}$$

since the quantities in square brackets are nonnegative. That is, R' inherits nonnegative definiteness from R . Now since R' is also symmetric, it is the covariance function for a unique (up to equivalence) mean-zero normal homogeneous stochastic process $u'(x)$ [2].

In the spirit of Taylor's hypothesis [5], we invest u' with a time dependence by taking

$$u'(t, x_1, x_2, x_3) \equiv u'(x_1 - U't, x_2, x_3) ,$$

where

$$U' \equiv \int_{E_3} U(x) k(x) dx .$$

We shall see that accounting for time in this way is actually just a convenient mathematical device. In practice, we shall rely on a much weaker version of this sample path assertion, i.e.,

$$E u'(s+t, x) u'(s, x) = E u'(s, x+y) u'(s, x) ,$$

where

$$y = (-U't, 0, 0) \quad .$$

Henceforth, we shall regard $U' + u'(t, x)$ as the general wind model. In some sense, $u'(t, x)$ is the normal stationary-homogeneous process most closely associated with the longitudinal component of velocity fluctuations in the volume affecting the windmill rotor. We now proceed to the next section, where we use this model in analyzing these fluctuations.

III. AN ESTIMATE BASED ON THE GENERAL WIND MODEL

The object of this section is to estimate the expected number of times t in $[0, T]$ that a volume average of the longitudinal wind component changes by an amount Δ between $t-\tau$ and $t+\tau$. This estimate is based on the general wind model constructed in Section II. We may restrict attention to u' at the onset, however, since the remaining term in the model is time-independent.

We begin by recalling the definitions of the operators D_τ and K given in Appendix A:

$$D_\tau \xi(t) \equiv \xi(t+\tau) - \xi(t-\tau)$$

and

$$K\psi(x) \equiv \int_{E_3} k(x+y) \psi(y) dy \quad ,$$

where we identify this function k with that given in Section II. We next define $v(t)$ by

$$v(t) \equiv D_\tau K u'(t, 0)$$

for each sample path in the $u'(t, x)$ -process. Clearly, the $v(t)$ -process is normal, stationary and such that $\bar{E}v(t) = 0$ for each real t . In addition, we assume sufficient conditions on $R(x, y)$ to make $v(t)$ continuous both almost surely and in quadratic mean. Thus, the results of Appendix A apply to the $v(t)$ -process.

Let $D_{\Delta, \tau}(0, T)$ denote the number of times t in $[0, T]$ that $v(t) = \Delta$. Then $E\{D_{\Delta, \tau}(0, T)\}$ is just the expected number of times t in $[0, T]$ that the volume average of u' associated with k changes by an amount Δ between $t-\tau$ and $t+\tau$. According to the version of Rice's theorem stated in Proposition A1,

$$E\{D_{\Delta, \tau}(0, T)\} = \frac{T}{\pi} \left(\frac{\lambda_2'}{\lambda_0'} \right)^{1/2} \exp\left(-\frac{\Delta^2}{2\lambda_0'}\right),$$

where λ_n' denotes the n^{th} spectral moment for the $v(t)$ -process. Thus, obtaining the estimate we seek is now simply a matter of computing λ_0' and λ_2' .

Set

$$\phi'(\kappa) = \frac{1}{(2\pi)^3} \int_{E_3} e^{-i\kappa \cdot x} R'(x) dx,$$

the spectral density for the $u'(x)$ -process. First, Remark A2 implies that

$$(2\pi)^6 |Fk(\kappa)|^2 \phi'(\kappa)$$

is the spectral density for the $Ku'(x)$ -process. Next, remark A1 and the stochastic equivalence of time and space hypothesized in Section II imply that

$$4(2\pi)^6 \sin^2(U'\tau\kappa_1) |Fk(\kappa)|^2 \phi'(\kappa)$$

is the spectral density for the $D_{\tau} Ku'(t, x)$ -process. Continuing to regard t as a parameter, we have that

$$4(2\pi)^6 \sin^2(U'\tau\kappa_1) \int_{E_2} |Fk(\kappa)|^2 \phi'(\kappa) d\kappa_2 d\kappa_3$$

is the spectral density for the $D_{\tau} Ku'(t, x_1, 0, 0)$ -process. Finally, returning to the temporal point of view, we have that

$$4(2\pi)^6 \frac{\sin^2(\tau\lambda)}{U'} \int_{E_2} \left| Fk\left(\frac{-\lambda}{U'}, \kappa_2, \kappa_3\right) \right|^2 \phi'\left(\frac{-\lambda}{U'}, \kappa_2, \kappa_3\right) d\kappa_2 d\kappa_3$$

is the spectral density for the $v(t)$ -process. We obtain λ_0' and λ_2' from this density according to the definition of spectral moments given in Appendix A. Thus, given $R(x, y)$, it is now a simple matter in principle to compute $E \left\{ D_{\Delta, \tau}(0, T) \right\}$.

IV. AN ESTIMATE BASED ON AN ISOTROPIC TURBULENCE MODEL

Even if ϕ' were generally available, estimating λ_0' and λ_2' would still be encumbered by the double integration in the representation of the spectral density for the $v(t)$ -process. We now turn to a stochastic model for the wind which is based on the theory of isotropic turbulence. Although this model fails to account for statistical inhomogeneities in the flow field, it compensates by furnishing tractable estimates for λ_0' and λ_2' in terms of more readily available spectral information.

At the onset, we model the longitudinal component of wind velocity by a normal, stationary and homogeneous stochastic process $u(t, x)$. We set $U \equiv Eu(t, x)$ and assume that the version of Taylor's hypothesis described in Section II holds. We further assume that the sample paths are continuous both almost surely and in quadratic mean. Now define the $w(t)$ -process by taking

$$w(t) \equiv D_{\tau} K u(t, 0)$$

for each sample path $u(t, x)$. Let $D_{\Delta, \tau}(0, T)$ denote the number of times t in $[0, T]$ that $w(t) = \Delta$. As in Section III,

$$E \left\{ D_{\Delta, \tau}(0, T) \right\} = \frac{T}{\pi} \left(\frac{\lambda_2'}{\lambda_0'} \right)^{1/2} \exp \left(- \frac{\Delta^2}{2\lambda_0'} \right) ,$$

where λ_n' denotes the n^{th} spectral moment for the $w(t)$ -process.

We proceed to compute estimates for λ_0' and λ_2' based on the theory of isotropic turbulence. This section essentially repeats much of Section III. As in Section III, we find it convenient to suppress the explicit dependence of u on time during the computation of the spectral density for the $w(t)$ -process.

Let u_1 , u_2 and u_3 denote the longitudinal, lateral and vertical components of the turbulence model. (We have identified u_1 and $u-U$). Assume that this vector process is isotropic. Define the covariance tensor by

$$R_{ij}(x) \equiv E u_i(x+y) u_j(y) \quad ,$$

and the spectral density tensor by

$$\Phi_{ij}(\kappa) \equiv \frac{1}{(2\pi)^3} \int_{E_3} e^{-i\kappa \cdot x} R_{ij}(x) dx \quad .$$

Let ϕ denote the spectral density for the $u_1(x_1, 0, 0)$ -process:

$$\phi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x_1} R_{11}(x_1, 0, 0) dx_1 \quad .$$

According to the theory of isotropic turbulence^[5], Φ_{ij} is given by

$$\Phi_{ij}(\kappa) = \frac{E(|\kappa|)}{4\pi|\kappa|^4} \left(|\kappa|^2 \delta_{ij} - \kappa_i \kappa_j \right) \quad ,$$

where

$$E(\lambda) \equiv \lambda^3 \frac{d}{d\lambda} \left[\frac{1}{\lambda} \frac{d\phi}{d\lambda}(\lambda) \right] .$$

As in Section III, the spectral density for the $w(t)$ -process is given by

$$\begin{aligned} & 4(2\pi)^6 \frac{\sin^2(\tau\lambda)}{U} \int_{E_2} \left| Fk \left(\frac{-\lambda}{U}, \kappa_2, \kappa_3 \right) \right|^2 \phi_{11} \left(\frac{-\lambda}{U}, \kappa_2, \kappa_3 \right) d\kappa_2 d\kappa_3 \\ &= 2(2\pi)^5 \frac{\sin^2(\tau\lambda)}{U} \int_{E_2} \left| Fk \left(\frac{-\lambda}{U}, \kappa_2, \kappa_3 \right) \right|^2 (\kappa_2^2 + \kappa_3^2) \\ & \quad \left[\left(\frac{1}{\sigma} \frac{d}{d\sigma} \right)^2 \phi(\sigma) \right] \bigg|_{\sigma = \sqrt{\left(\frac{\lambda}{U} \right)^2 + \kappa_2^2 + \kappa_3^2}} d\kappa_2 d\kappa_3 . \end{aligned}$$

It is now a simple matter in principle to compute λ_0' and λ_2' , from which we obtain the estimate $E \left\{ D_{\Delta, \tau}(0, T) \right\}$ in the context of isotropic turbulence.

The assumption of isotropy allows us to make a considerable simplification at this point. We refer to Appendix B for the computation of Fk for k corresponding to the geometry of two basic windmill rotor configurations--the disk and the sphere. In both cases $|Fk(\kappa)|^2$ assumes the form $G(\kappa_1, \kappa_2^2 + \kappa_3^2)$. We substitute this form and proceed to compute simplified expressions for λ_0' and λ_2' :

$$\begin{aligned}
& \int_{E_2} G\left(\frac{-\lambda}{U}, \kappa_2^2 + \kappa_3^2\right) (\kappa_2^2 + \kappa_3^2) \left[\left(\frac{1}{\sigma} \frac{d}{d\sigma} \right)^2 \phi(\sigma) \right] \Bigg|_{\sigma = \sqrt{\left(\frac{\lambda}{U}\right)^2 + \kappa_2^2 + \kappa_3^2}} d\kappa_2 d\kappa_3 \\
&= \int_0^{2\pi} d\theta \int_0^\infty dr r^3 G\left(\frac{-\lambda}{U}, r^2\right) \left[\left(\frac{1}{\sigma} \frac{d}{d\sigma} \right)^2 \phi(\sigma) \right] \Bigg|_{\sigma = \sqrt{\left(\frac{\lambda}{U}\right)^2 + r^2}} \\
&= 2\pi \int_{\frac{\lambda}{U}}^\infty \left(\sigma^2 - \left(\frac{\lambda}{U}\right)^2 \right) G\left(\frac{-\lambda}{U}, \sigma^2 - \left(\frac{\lambda}{U}\right)^2\right) \frac{d}{d\sigma} \left(\frac{1}{\sigma} \frac{d\phi}{d\sigma}(\sigma) \right) d\sigma.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lambda_O' &= 4(2\pi)^6 \int_0^\infty d\lambda \frac{\sin^2(\tau\lambda)}{U} \int_{\frac{\lambda}{U}}^\infty d\sigma \left(\sigma^2 - \left(\frac{\lambda}{U}\right)^2 \right) G\left(\frac{-\lambda}{U}, \sigma^2 - \left(\frac{\lambda}{U}\right)^2\right) \\
&\quad \frac{d}{d\sigma} \left(\frac{1}{\sigma} \frac{d\phi}{d\sigma}(\sigma) \right) \\
&= 4(2\pi)^6 \int_0^\infty d\sigma \frac{d}{d\sigma} \left(\frac{1}{\sigma} \frac{d\phi}{d\sigma}(\sigma) \right) \int_0^\sigma d\lambda \sin^2(U\tau\lambda) (\sigma^2 - \lambda^2) G(-\lambda, \sigma^2 - \lambda^2) \\
&= -4(2\pi)^6 \int_0^\infty d\sigma \frac{d\phi}{d\sigma}(\sigma) \frac{1}{\sigma} \int_0^\sigma d\lambda \sin^2(U\tau\lambda) \frac{d}{d\sigma} \left[(\sigma^2 - \lambda^2) \right. \\
&\quad \left. G(-\lambda, \sigma^2 - \lambda^2) \right]
\end{aligned}$$

$$\begin{aligned}
&= 4 (2\pi)^6 \int_0^\infty d\sigma \frac{d\phi}{d\sigma}(\sigma) \frac{1}{\sigma} \int_0^\sigma d\lambda \frac{1}{2} (1 - \cos(2U\tau\lambda)) \\
&\quad \frac{d}{d\lambda} \left[(\sigma^2 - \lambda^2) G(-\delta, \sigma^2 - \lambda^2) \right] \Big|_{\delta=\lambda} \\
&= -4 \int_0^\infty d\sigma \frac{d\phi}{d\sigma}(\sigma) \frac{1}{\sigma} \int_0^\sigma d\lambda (\sigma^2 - \lambda^2) U\tau \sin(2U\tau\lambda) \\
&\quad (2\pi)^6 G(-\lambda, \sigma^2 - \lambda^2) .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lambda_2' &= -4 \int_0^\infty d\sigma \frac{d\phi}{d\sigma}(\sigma) \frac{1}{\sigma} \int_0^\sigma d\lambda (\sigma^2 - \lambda^2) \left[2U^2\lambda \sin^2(U\tau\lambda) \right. \\
&\quad \left. + U^3\tau\lambda^2 \sin(2U\tau\lambda) \right] (2\pi)^6 G(-\lambda, \sigma^2 - \lambda^2) .
\end{aligned}$$

Therefore, given ϕ , τ and G , it is now a simple matter to compute $E \left\{ D_{\Delta, \tau}(0, T) \right\}$ using these expressions for λ_0' and λ_2' , and the version of Rice's theorem stated earlier in this section.

V. SUMMARY AND SAMPLE CALCULATIONS

This section consists of a statement of the most useful results developed in this paper and some sample computations which serve to illustrate these results. Recall that $E \left\{ D_{\Delta, \tau}(0, T) \right\}$ denotes the expected number of times t in $[0, T]$ that a volume average of the longitudinal wind component changes by an amount Δ between $t-\tau$ and $t+\tau$. An estimate of this expected number based on Rice's theorem is given by

$$E \left\{ D_{\Delta, \tau}(0, T) \right\} = \frac{T}{\pi} \left(\frac{\lambda_2'}{\lambda_0'} \right)^{1/2} \exp \left(- \frac{\Delta^2}{2\lambda_0'} \right) ,$$

where λ_0' and λ_2' denote spectral moments of a modified power spectrum for atmospheric turbulence. Assuming an isotropic turbulence model, we have that

$$\lambda_0' = -4 \int_0^\infty d\sigma \frac{d\phi}{d\sigma}(\sigma) \frac{1}{\sigma} \int_0^\sigma d\lambda (\sigma^2 - \lambda^2) U \tau \sin(2U\tau\lambda) (2\pi)^6 G(-\lambda, \sigma^2 - \lambda^2)$$

and

$$\begin{aligned} \lambda_2' = -4 \int_0^\infty d\sigma \frac{d\phi}{d\sigma}(\sigma) \frac{1}{\sigma} \int_0^\sigma d\lambda (\sigma^2 - \lambda^2) & \left[2U^2\lambda \sin^2(U\tau\lambda) \right. \\ & \left. + U^3\tau\lambda^2 \sin(2U\tau\lambda) \right] (2\pi)^6 G(-\lambda, \sigma^2 - \lambda^2) , \end{aligned}$$

where ϕ denotes the (whole-line) power spectral density for the longitudinal component of atmospheric turbulence, U denotes the

mean wind speed and G is a function associated with windmill geometry. By Appendix B,

$$(2\pi)^6 G(-\lambda, \sigma^2 - \lambda^2) = 4 \left[\frac{\sin a\lambda}{a\lambda} \frac{J_1(b\sqrt{\sigma^2 - \lambda^2})}{b\sqrt{\sigma^2 - \lambda^2}} \right]^2$$

for a disk of thickness $2a$ and radius b , where the a -axis is oriented parallel to the mean wind velocity and J_1 denotes the first order Bessel function, and

$$(2\pi)^6 G(-\lambda, \sigma^2 - \lambda^2) = 9 \left[\frac{j_1(c\lambda)}{c\lambda} \right]^2$$

for a sphere of radius c , where j_1 denotes the first order spherical Bessel function.

We have performed several calculations of $E \{ D_{\Delta, \tau}(0, T) \}$ based on the theory summarized in this section. We confine attention to the geometry of a disk and to the Von Karman spectrum, which is given by

$$\phi(\sigma) \equiv \frac{L(k_v U)^2}{\pi} \left[1 + (1.339 L\sigma)^2 \right]^{-5/6},$$

where L denotes turbulent length scale and k_v denotes turbulence intensity^[3]. For definiteness, we take

$$a = 0.15 \text{ m} \quad ,$$

$$b = 4.5 \text{ m} \quad ,$$

$$k_v = 0.2 \text{ and } 0.3 \quad ,$$

$$L = 60.0 \text{ m} ,$$

$$T = 1.0 \text{ hours} ,$$

$$U = 6.7 \text{ m/sec (15.0 mi/hr)} .$$

The results of these calculations are given in Tables 1 and 2. Table 1 presents values of $E \{ D_{\Delta, \tau}(0, T) \}$ for various values of Δ and τ , and $k_v = 0.2$; Table 2 presents values of $E \{ D_{\Delta, \tau}(0, T) \}$ for various values of Δ and τ , and $k_v = 0.3$.

Table 1. $E \{ D_{\Delta, \tau}(0, T) \}$ for Various Values
of Δ and τ , and $k_v = 0.2$

$\Delta \backslash \tau$	0.125	0.25	0.5	1.0	2.0
2.0	1.3×10^{-53}	1.0×10^{-13}	4.2×10^{-3}	5.2	5.3×10^1
4.0	~ 0	4.4×10^{-63}	3.1×10^{-20}	2.4×10^{-7}	6.9×10^{-3}
6.0	~ 0	~ 0	8.9×10^{-49}	1.4×10^{-19}	2.3×10^{-9}
8.0	~ 0	~ 0	9.8×10^{-89}	1.1×10^{-36}	2.0×10^{-18}
10.0	~ 0	~ 0	~ 0	1.0×10^{-58}	4.3×10^{-30}

Table 2. $E \{ D_{\Delta, \tau}(0, T) \}$ for Various Values
of Δ and τ , and $k_v = 0.3$

$\Delta \backslash \tau$	0.125	0.25	0.5	1.0	2.0
2.0	3.2×10^{-22}	1.5×10^{-4}	6.2	1.2×10^2	2.8×10^2
4.0	~ 0	1.7×10^{-26}	1.5×10^{-7}	6.5×10^{-2}	5.2
6.0	~ 0	4.4×10^{-63}	3.1×10^{-20}	2.4×10^{-7}	6.9×10^{-3}
8.0	~ 0	~ 0	5.4×10^{-38}	5.9×10^{-15}	6.4×10^{-7}
10.0	~ 0	~ 0	7.9×10^{-61}	9.7×10^{-25}	4.2×10^{-12}

VI. CONCLUDING REMARKS

The limits of validity for these estimates are presently unknown. Homogeneity is evidently necessary for the application of Rice's theorem. A paper examining the Gaussian assumption should be out shortly. A Battelle report comparing this estimation method with others should also be out shortly. The experimental measurements presented in it will hopefully shed some light on this issue.

Nevertheless, the estimates represented by $E \left\{ D_{\Delta, \tau}(0, T) \right\}$ are possibly the best presently available. Any such estimates ultimately rest on the theory of sample path properties for stationary stochastic processes. Better estimates would correspond to deeper mathematical theorems. For example, an important generalization might be to estimate the expected number of times t in $[0, T]$ that the v or w -process changes from a to b between $t-\tau$ and $t+\tau$. The corresponding mathematical problem is unsolved, however.

Finally, we are fortunate in that the choices of k corresponding to a disk and a sphere lead to considerable mathematical simplification. We obtain this same simplification for choices of k corresponding to an annular solid and a spherical shell. In fact, Fk for these cases can be obtained from the expressions given in Appendix B with little additional effort.

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APPENDIX A
MATHEMATICAL FOUNDATIONS

This section consists of a precise statement of Rice's theorem and two remarks about linear operators on spaces of sample paths. It forms the mathematical basis for calculating the expectation described in the introduction. A detailed exposition of these topics may be found in [2].

First, let $\xi(t)$ denote a normal stationary stochastic process and E denote expectation. We specify that $\xi(t)$ be real valued, continuous in quadratic mean and such that $E\xi(t) = 0$ for all real valued t . Thus, the spectral theorem holds. Let $\zeta(\lambda)$ denote the orthogonal process dual to $\xi(t)$ and $dF(\lambda)$ its spectral measure, and define spectral moments λ_n by

$$\lambda_n = \int_{-\infty}^{\infty} \lambda^n dF(\lambda) , \quad (n \geq 0) .$$

Now let $C_u(0,T)$ denote the number of times $\xi(t)$ crosses the level u in the interval $[0,T]$. Although the finiteness of λ_2 guarantees that $\xi(t)$ is equivalent to a process with continuous sample paths, we assume that $\xi(t)$ has continuous sample paths almost surely in any case. Then

Theorem A1 (Rice):

$$E \left\{ C_u(0,T) \right\} = \frac{T}{\pi} \left(\frac{\lambda_2}{\lambda_0} \right)^{1/2} \exp \left(- \frac{u^2}{2\lambda_0} \right) .$$

The case $\lambda_2 = \infty$ is included.

We now state the modified version of this theorem which applies to our situation. Let $N_u(0,T)$ denote the number of times that $\xi(t) = u$ in $[0,T]$. Then

Proposition A1:

$$E \left\{ N_u(0,T) \right\} = E \left\{ C_u(0,T) \right\} .$$

The case $\lambda_2 = \infty$ is included.

Next, define the operator D_τ on the sample path $\xi(t)$ by

$$D_\tau \xi(t) \equiv \xi(t+\tau) - \xi(t-\tau) .$$

The $D_\tau \xi(t)$ -process is clearly stationary, continuous in quadratic mean and such that $ED_\tau \xi(t) = 0$ for all real valued t . As above, the spectral theorem holds. In particular, $D_\tau \xi(t)$ enjoys the following representation:

$$\begin{aligned} D_\tau \xi(t) &= \int_{-\infty}^{\infty} e^{i(t+\tau)\lambda} d\zeta(\lambda) - \int_{-\infty}^{\infty} e^{i(t-\tau)\lambda} d\zeta(\lambda) \\ &= \int_{-\infty}^{\infty} 2i \sin(\tau\lambda) e^{it\lambda} d\zeta(\lambda) . \end{aligned}$$

Therefore,

Remark A1:

The spectral measure for the $D_\tau \xi(t)$ -process is given by

$$E \left| 2i \sin(\tau\lambda) d\zeta(\lambda) \right|^2 = 4 \sin^2(\tau\lambda) dF(\lambda) .$$

Similarly, let $\psi(x)$ denote a homogeneous stochastic process such that the parameter x takes values in the n -dimensional Euclidean space E_n . We specify that $\psi(x)$ be real-valued, continuous in quadratic mean and such that $E\psi(x) = 0$ for all x in E_n . Then an n -dimensional version of the spectral theorem holds. Let $\eta(\kappa)$ denote the orthogonal process dual to $\psi(x)$ and $d\Phi(\kappa)$ its spectral measure.

Further, let k denote an integrable function with compact support in E_n [6], and assume that the $\psi(x)$ -process has continuous sample paths almost surely. Then the operator K defined by

$$K\psi(x) \equiv \int_{E_n} k(x+y) \psi(y) dy$$

is well-defined almost surely. In addition,

$$\begin{aligned} K\psi(x) &= \int_{E_n} k(x+y) \left(\int_{E_n} e^{iy \cdot \kappa} d\eta(\kappa) \right) dy \\ &= \int_{E_n} \left(\int_{E_n} k(x+y) e^{i(x+y) \cdot \kappa} dy \right) e^{-ix \cdot \kappa} d\eta(\kappa) \\ &= \int_{E_n} (2\pi)^n \overline{\left(\frac{1}{(2\pi)^n} \int_{E_n} e^{-i\kappa \cdot y} k(y) dy \right)} e^{-ix \cdot \kappa} d\eta(\kappa) \\ &= \int_{E_n} (2\pi)^n \overline{(Fk)(\kappa)} e^{-ix \cdot \kappa} d\eta(\kappa) \quad , \end{aligned}$$

where Fk denotes the Fourier transform of k . Note that the integrability of k legitimizes interchanging the order of integration^[7]. Therefore,

Remark A2:

The spectral measure for the $K\psi(x)$ - process is given by

$$E \left| (2\pi)^n \int Fk(\kappa) d\eta(\kappa) \right|^2 = (2\pi)^{2n} \int |Fk(\kappa)|^2 d\Phi(\kappa) \quad .$$

APPENDIX B

FOURIER TRANSFORMS OF FUNCTIONS DESIGNATING
AFFECTED VOLUME FOR TWO CANONICAL ROTOR GEOMETRIES

The object of this section is to compute Fk , the Fourier transform of the function specifying affected volume, for two choices of k .

The first choice for k corresponds to a disk of thickness $2a$ and radius b . Set

$$k_d(x) \equiv \frac{1}{2\pi ab^2} \chi_{[-a,a]}^{(x_1)} \chi_{[0,b^2]}^{(x_2^2 + x_3^2)} ,$$

where

$$\chi_{[\alpha,\beta]}^{(\gamma)} \equiv \begin{cases} 1 & \text{in case } \alpha \leq \gamma \leq \beta \\ 0 & \text{otherwise} \end{cases} .$$

The orientation of the disk relative to the mean wind velocity should be clear from this representation. Let J_n denote the Bessel function of order n ^[8]. Then

$$\begin{aligned} Fk_d(\kappa) &= \frac{1}{2\pi ab^2} \frac{1}{2\pi} \int_{-a}^a dx_1 e^{-i\kappa_1 x_1} \frac{1}{(2\pi)^2} \int_{E_2} dx_2 dx_3 e^{-i(\kappa_2 x_2 + \kappa_3 x_3)} \\ &\quad \chi_{[0,b^2]}^{(x_2^2 + x_3^2)} \\ &= \frac{1}{2\pi ab^2} \frac{1}{2\pi} \int_{-a}^a dx_1 \frac{1}{\kappa_1} \frac{d}{dx_1} \sin \kappa_1 x_1 \frac{1}{(2\pi)^2} \int_0^b dr r \int_0^{2\pi} d\theta \\ &\quad e^{-i\sqrt{\kappa_2^2 + \kappa_3^2} r \cos \theta} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi ab^2} \frac{1}{2\pi} \frac{2 \sin a\kappa_1}{\kappa_1} \frac{1}{2\pi} \int_0^b dr \, r \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{i\sqrt{\kappa_2^2 + \kappa_3^2} r \sin \theta} \\
&= \frac{2}{(2\pi)^3 ab^2} \frac{\sin a\kappa_1}{\kappa_1} \int_0^b dr \, r J_0\left(\sqrt{\kappa_2^2 + \kappa_3^2} r\right) \\
&= \frac{2}{(2\pi)^3 ab^2} \frac{\sin a\kappa_1}{\kappa_1(\kappa_2^2 + \kappa_3^2)} \int_0^b \sqrt{\kappa_2^2 + \kappa_3^2} \, dr \, \frac{d}{dr} (r J_1(r)) \\
&= \frac{2}{(2\pi)^3} \frac{\sin a\kappa_1}{a\kappa_1} \frac{J_1(b\sqrt{\kappa_2^2 + \kappa_3^2})}{b\sqrt{\kappa_2^2 + \kappa_3^2}} .
\end{aligned}$$

That is,

Proposition B1:

$$Fk_d(\kappa) = \frac{2}{(2\pi)^3} \frac{\sin a\kappa_1}{a\kappa_1} \frac{J_1(b\sqrt{\kappa_2^2 + \kappa_3^2})}{b\sqrt{\kappa_2^2 + \kappa_3^2}} .$$

The second choice for k corresponds to a sphere of radius c .

Set

$$k_s(x) \equiv \frac{3}{4\pi c^3} \chi_{[0,c]}(|x|) ,$$

and let j_n denote the spherical Bessel function of order n . Then

$$Fk_s(\kappa) = \frac{3}{4\pi c^3} \frac{1}{(2\pi)^3} \int_0^c dr \, r^2 \int_{S^2} d\eta \, \cos(\alpha \hat{k} \cdot \eta) ,$$

where $\alpha \equiv r|\kappa|$, $\hat{\kappa} \equiv \kappa/|\kappa|$ and S^2 denotes the subset of vectors in E_3 with unit magnitude. $\int_{S^2} d\eta \cos(\alpha \hat{\kappa} \cdot \eta)$ is clearly invariant under rotations and so is constant with respect to the argument $\hat{\kappa}$. Thus, we may take $\hat{\kappa} \equiv (0,0,1)$ for convenience. Coordinatizing S^2 by taking $\eta \equiv (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, we have that

$$\begin{aligned} \int_{S^2} d\eta \cos(\alpha \hat{\kappa} \cdot \eta) &= \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta \cos(\alpha \cos \theta) \\ &= -\frac{2\pi}{\alpha} \int_0^\pi d\theta \frac{d}{d\theta} (\sin(\alpha \cos \theta)) \\ &= \frac{4\pi \sin \alpha}{\alpha} . \end{aligned}$$

Thus,

$$\begin{aligned} Fk_s(\kappa) &= \frac{1}{(2\pi)^3} \frac{3}{4\pi c^3} \int_0^c dr r^2 4\pi \frac{\sin(|\kappa|r)}{|\kappa|r} \\ &= \frac{1}{(2\pi)^3} \frac{3}{(c|\kappa|)^3} \int_0^{c|\kappa|} dr r \sin r \\ &= \frac{1}{(2\pi)^3} \frac{3}{(c|\kappa|)^3} \int_0^{c|\kappa|} dr \frac{d}{dr} (r^2 j_1(r)) \\ &= \frac{3}{(2\pi)^3} \frac{j_1(c|\kappa|)}{c|\kappa|} . \end{aligned}$$

That is,

Proposition B2:

$$Fk_s(\kappa) = \frac{3}{(2\pi)^3} \frac{j_1(c|\kappa|)}{c|\kappa|} .$$

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