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MASTER

ON THE TRANSLATION OPERATOR FOR FINITE
DIMENSIONAL ELECTROMAGNETIC FIELDS

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1. ABSTRACT

Computation of electromagnetic fields in particular applications is usually accompanied by the adhoc assumption that the field contains a finite number of degrees of freedom. Herein, this assumption is made at the outset. It is shown that if an annular region between two closed surfaces contains no sources or sinks and is isotropic, lossless and homogeneous, a unique translation operator can be defined algebraically. Conservation of energy defines the translation operator T to within an arbitrary unitary transformation. The conditions of causality, unitarity and energy conservation are shown to uniquely determine T . Both scalar and vector fields are treated. In both of these cases, frequency and time domain transforms are computed. The transform T is compared with the analogous one as derived from the time domain Stratton-Chu Formulation. The application to a radiation condition boundary constraint on finite difference and finite element computations is discussed.

2. INTRODUCTION

In this monograph we examine the case for the existence, uniqueness and construction of a translation operator T for finite dimensional electromagnetic fields. Because uniqueness properties of the electromagnetic fields are associated with integrations over surfaces, it is not surprising that T is obtained as an isomorphism (one-to-one and onto) mapping of fields from one closed surface S_1 to another closed surface S_2 which encloses S_1 .

In finite element and finite difference schemes in open region problems, it becomes necessary to truncate the solution on some finite extent computational surface. At this boundary, it is necessary to employ a radiation type boundary condition which eliminates spurious reflections. The ability of a technique to do this becomes evident at late times in a transient problem when the reflections have time to get back to the center of the computational region. Many authors have written on this general subject. A review of their work is beyond the scope of this report. Key contributions are listed in the references. Suffice it to say that none of their results are directly applicable to the case of time domain vector electromagnetic fields.

An important related method in the general area of hybrid field computation is Mei's Unimoment Method [3,4]. In particular, the analytical methods given to matching finite element solutions to mode or integral equation solutions for the case of a spherical surface buried in a planar half-space are most useful.

The concept of hybrid computation using simultaneous partial differential equations and integral equation formulating allows one to treat open region problems by decomposing the total space into two regions. In the finite "bad" region where all types of inhomogeneities and field variations are allowed, one solves the problem using the more flexible direct partial differential equation

solutions of finite differences or finite elements. In the complementary open region, one sets up an appropriate integral equation formulation of the problem. A good reference on this hybrid approach is discussed by McDonald and Wexler [2,3].

Related to these techniques are methods which directly address the problem of determining appropriate boundary conditions on finite closed surfaces such that no waves are reflected back into the interior of a computational region. This allows one to terminate a finite difference mesh on a computational region without generating spurious reflections. An example of this approach which has found use in plasma physics problems is given by Lindman [10].

Application of hybrid matching techniques to geo-electromagnetic subsurface prospecting is expected to produce breakthroughs in our ability to model the response of subsurface anomalies and the related interpretation of field measurements. The need for such an approach has been documented [9].

3. SCALAR THEORY

In hybrid computation of electromagnetic fields in open region problems using finite element or finite difference methods, it is necessary to enforce a radiation boundary condition at the outermost finite surface in the computational scheme. The problem is posed as follows: There is a closed computational region A_1 with boundary surface S_1 . There is an annular region B surrounding the surface S_1 which has a surface S_2 as shown in Fig. 3.1.

Fig. 3.1. Geometry

If there are no sources in B or exterior to S_2 then by Maxwell's theory there exists a flux conservation theorem between surfaces S_1 and S_2 .

In the frequency domain when there are no losses occurring in the annular region B, this conservation theorem takes the familiar form

$$\operatorname{Re} \oint_{S_1} \vec{E} \times \vec{H}^* \cdot d\vec{S}_1 = \operatorname{Re} \oint_{S_2} \vec{E} \times \vec{H}^* \cdot d\vec{S}_2. \quad (3.1)$$

In equation (3.1), $d\vec{S}_i = \hat{n}_i dS_i$, $i = 1, 2$ where $\hat{n}_i$ is the local outward pointing unit normal on the surface S_i . The electromagnetic fields in (3.1) are the total fields. For the sake of simplicity, let us first consider two-dimensional fields. Thus let the electric field $\vec{E}$ be a single component in the z direction, i.e.,

$$\vec{E} = \hat{z} \Psi, \quad \partial/\partial z \equiv 0 \quad (3.2)$$

Then from Maxwell's equations, and for a time factor $e^{-i\omega t}$,

$$\vec{H}^* = \frac{i}{\omega\mu_0} \left(\hat{x} \frac{\partial \Psi^*}{\partial y} - \hat{y} \frac{\partial \Psi^*}{\partial x} \right)$$

Thus

$$\vec{E} \times \vec{H}^* \cdot \hat{n} = \frac{i}{\omega\mu_0} \Psi \frac{\partial \Psi^*}{\partial n} \quad \text{where } \partial/\partial n = \hat{n} \cdot \vec{\nabla} \quad (3.3)$$

is the normal derivative. To discretize the problem or equivalently to assume a finite number of degrees of freedom in the fields we assume that

$$\Psi \approx \sum_{n=1}^N a_n \phi_n(\vec{x}) \quad (3.4)$$

The interpretation of expansion (3.4) in the finite element method is of an interpolation series such that each function ϕ_n is unity at the n^{th} node and zero at all other nodes

$$\phi_n(\vec{x}_m) = \delta_{nm} \quad (3.5)$$

so that a_n becomes the value of the electric field at the n^{th} node. Alternatively expansions (3.4) could be a multipole series or a modal expansion of the fields where, in the latter case, each function $\phi_n(x)$ is a solution to the wave equation. In what follows, the actual interpretation of the terms in (3.4) are at least formally not significant. All that we require is that $\phi_n \in C^1$ where C^1 is the class of once differentiable functions. Under these conditions, substitution of (3.4) into (3.1) yields the discrete relation

$$\text{Re} \left[\sum_{i,j=1}^N L_{ij}^{(1)} x_i x_j^* \right] = \text{Re} \left[\sum_{i,j=1}^N L_{ij}^{(2)} y_i y_j^* \right] \quad (3.6)$$

where

$$L_{ij}^{(p)} = i/\omega\mu_0 \oint_{S_p} \phi_i(\vec{x}) \frac{\partial \phi_j^*}{\partial n}(x) dS_p \quad (3.7)$$

$p = 1, 2$

In (3.6) we have denoted 'nodal' values on S_1 as $\{x_i\}_{i=1}^N$ and similarly

the values on S_2 as $\{y_i\}_{i=1}^N$

The algebraic Green's theorem that we seek is nothing more than the statement of the fact that knowledge of the tangential electric field over a closed surface (which contains all sources and material inhomogeneities) is sufficient to determine the electric field at an arbitrary point outside S_1 . In particular, if $\{y_i\}_{i=1}^N$ are the 'nodal' values on S_2 outside of S_1 (see Fig. 3.1) then

there exists a linear transformation such that

$$y_i = \sum_{j=1}^N A_{ij} x_j + b_i \quad (3.8)$$

$i = 1, 2, 3 \dots N$

where again $\{x_j\}_{j=1}^N$ are the 'nodal' values on S_1 . In our application, all sources are inside S_1 ; therefore, a null field on S_1 must map into a null field on S_2 . This requires

$$b_i = 0, \quad i = 1, 2, 3 \dots N \quad (3.9)$$

Substitution of transformation (3.8), as modified by (3.9), into (3.6) yields the system

$$\text{Re} \left\{ \sum_{i,j=1}^N \left(L_{ij}^{(1)} - \sum_{m,n=1}^N L_{mn}^{(2)} A_{mi} A_{nj}^* \right) x_i x_j^* \right\} = 0 \quad (3.10)$$

Since this conservation law holds for arbitrary fields on S_1 , it follows that the conservation law can be written independent of the field coordinates x_i .

To see this, let us define the auxiliary matrix B as

$$B_{ij} = L_{ij}^{(1)} - L_{mn}^{(2)} A_{mi} A_{nj}^* \quad (3.11)$$

where in (3.11) and henceforth, we will use the summation convention on repeated subscripts, i.e.,

$$\alpha_i \beta_i \stackrel{\text{Def.}}{=} \sum_{i=1}^N \alpha_i \beta_i \quad (3.12)$$

In terms of the auxiliary matrix B , the conservation law takes the form

$$\text{Re} (B_{ij} x_i x_j^*) = 0 \quad (3.13)$$

Expanding (3.13) and rearranging subscripts yields the equivalent form

$$(B_{ij} + B_{ji}^*) x_i x_j^* = 0 \quad (3.14)$$

Because of the linear independence of the field coordinates, we determine the coordinate free form of the conservation law:

$$\begin{aligned} B_{ij} + B_{ij}^* &= 0 \\ i, j &= 1, 2, 3 \dots N \end{aligned} \quad (3.15)$$

To proceed, we define the matrix $C_{ij}^{(P)}$ as

$$\begin{aligned} C_{ij}^{(P)} &= L_{ij}^{(P)} + L_{ji}^{(P)*}, \quad P = 1, 2. \\ i, j &= 1, 2, 3 \dots N \end{aligned} \quad (3.16)$$

Substitute (3.11) into symmetry relation (3.15) and rearrange to obtain

$$C_{ij}^{(1)} = C_{mn}^{(2)} A_{mi} A_{nj}^* \quad (3.17)$$

Note that by the definition (3.16),

$$C_{ij}^{(P)*} = C_{ji}^{(P)}, \quad P = 1, 2 \quad (3.18)$$

Let

$T = \tilde{A}$, then (2.36) can be written as

$$T C^{(2)} T^\dagger = C^{(1)} \quad (3.19)$$

where $T_{ij}^\dagger = T_{ji}^*$ is the complex conjugate transpose of T . Equation (3.19) is the mathematical statement of energy conservation between surfaces S_1 and S_2 .

This completes the scalar theory.

4. VECTOR THEORY

We again seek an algebraic means of relating fields on concentric closed surfaces. In order to do this, it is convenient to introduce complex vector fields. This technique is discussed by Stratton [6] (pp. 32-34). Thus we define the complex vector $\bar{P}$ as.

$$\bar{P} = \bar{E} + i\eta \bar{H} \quad , \quad \eta = (\mu/\epsilon)^{1/2} \quad (4.1)$$

Maxwell's four equations become

$$\begin{aligned} \nabla \times \bar{P} &= i\eta \bar{J} + i \partial/\partial \tau \bar{P} \quad , \quad \tau = ct \\ \nabla \cdot \bar{P} &= \rho/\epsilon \end{aligned} \quad (4.2)$$

The analogue of the Poynting theorem which is a vector differential relation which follows from (4.1) and (4.2) is:

$$\nabla \cdot (\bar{P} \times \bar{P}^*) = 2i\eta \text{Re}(\bar{P}) \cdot \bar{J} + i \partial/\partial \tau (\bar{P} \cdot \bar{P}^*) \quad (4.3)$$

In fact, substitution of (4.1) into (4.3) and subsequent use of (4.2) shows that (4.3) is equivalent to

$$\nabla \cdot (\bar{E} \times \bar{H}^*) = -\bar{E} \cdot \bar{J} - 1/2 \partial/\partial \tau (\mu \bar{H}^2 + \epsilon \bar{E}^2) \quad (4.4)$$

which is recognized as the familiar form of Poynting's theorem.

In a source free, lossless region (4.3) becomes

$$\nabla \cdot (\bar{P} \times \bar{P}^*) = i \partial/\partial \tau (\bar{P} \cdot \bar{P}^*) \quad (4.5)$$

Again, let us apply result (4.5) over the annular region as depicted in Fig. 4.1.

Fig. 4.1. Integration Geometry

The flux conservation law equivalent to (4.5) is given by

$$\int_{S_2} \bar{p} \times \bar{p}^* \cdot \hat{n}_2 \, dS_2 = \int_{S_1} \bar{p} \times \bar{p}^* \cdot \hat{n}_1 \, dS_1 \quad (4.6)$$

$$+ i\partial/\partial\tau \int_B \bar{p} \cdot \bar{p}^* \, dv$$

where the subscripts and notation in Eq. (4.6) are defined in Fig. 4.1. We now again assume the vector field is finite dimensional, i.e.,

$$\bar{p} = \sum_{n=1}^N \bar{p}^v \phi^v \quad (4.7)$$

$$\text{where } p^v = \bar{e}^v + i\bar{n}^v \quad (4.8)$$

Similar to the scalar case treated previously, the interpretation of (4.7) is that for the finite element method \$(\bar{e}^v, \bar{n}^v)\$ is the total electromagnetic field evaluated at the $\sqrt{t_0}$ node. The known expansion functions \$\phi^v\$ are functions of

position which, in the finite element method, are conveniently chosen to be interpolation functions such that

$$\phi^{\nu}(\bar{x}^{\mu}) = \delta^{\nu\mu} \quad (4.9)$$

Actually, on the surfaces S_1 and S_2 it will be useful to introduce interpolation functions ϕ^{ν} a posteriori to increase the accuracy of the integrations required (4.6). Using expansion (4.7) in a flux integral in (4.6) we obtain

$$\begin{aligned} \oint_{S_p} \bar{p} \times \bar{p}^* \cdot \hat{n}_p \, dS_p &= \sum_{\mu, \nu} (\bar{p}^{\mu} \phi^{\mu}) \times (\bar{p}^{\nu*} \phi^{\nu}) \cdot \hat{n}_p \, dS_p \\ P &= 1, 2. \\ &= \sum_{\mu, \nu} \bar{L}^{(P)}{}^{\mu\nu} \cdot (\bar{p}^{\mu} \times \bar{p}^{\nu*}) \end{aligned} \quad (4.10)$$

$$\text{where} \quad \bar{L}^{(P)}{}^{\mu\nu} = \oint_{S_p} \phi^{\mu} \phi^{\nu} \hat{n}_p \, dS_p, \quad P = 1, 2. \quad (4.11)$$

It is important to notice that the matrix element vector $\bar{L}^{(P)}{}^{\mu\nu}$ is independent of time. Before proceeding with the general time dependent transformation, let us consider the time harmonic vector problem. The time factor $e^{-i\omega t}$ is assumed. Of course, in so doing, care must be taken to distinguish between the notation i as used in (4.1) and the i introduced in $e^{-i\omega t}$. The two must be distinguished and would require something equivalent to "type other" arithmetic on, for example, a CDC computer.

Again, we shall make use of the summation convention on repeated indices. Then the time harmonic conservation law reads

$$\bar{L}^{(2)}{}^{\mu\nu} \cdot (\bar{y}^{\mu} \times \bar{y}^{\nu*}) = \bar{L}^{(1)}{}^{\lambda\gamma} \cdot (\bar{x}^{\lambda} \times \bar{x}^{\gamma*}) \quad (4.12)$$

In (4.12) we have introduced the nodal vectors $\bar{y}^\mu$ and $\bar{x}^\lambda$ on surfaces S_2 and S_1 , respectively, as shown in Fig. 4.1.

From the uniqueness theorem of electromagnetics, it follows that knowledge of the finite dimensional electromagnetic field on a closed surface uniquely determines the fields exterior to this provided no sources or sinks are between S_1 and S_2 . Furthermore, this relation is linear. Thus a nodal field value $\bar{y}^\mu$ is related to values on the surface S_1 through the relationship

$$\bar{y}^\mu = \bar{A}^{\mu\lambda} \cdot \bar{x}^\lambda + \bar{y}^\mu. \quad (4.13)$$

Again by uniqueness, when no sources or sinks exist in V , a null field on S_1 must map into a null field on S_2 . Consequentially,

$$\bar{y}^\mu = 0 \quad (4.14)$$

We now show that (4.12), (4.13), and (4.14) can be used to determine the transform dyadic $\bar{A}^{\mu\lambda}$. Direct substitution of (4.13) and (4.14) into (4.12) yields

$$\bar{L}^{(2)\mu\nu} \cdot (\bar{A}^{\mu\lambda} \cdot \bar{x}^\lambda) \times (\bar{A}^{\nu\gamma} \cdot \bar{x}^\gamma) = \bar{L}^{(1)\lambda\gamma} \cdot (\bar{x}^\lambda \times \bar{x}^\gamma) \quad (4.15)$$

In order to obtain (4.15) in a form which is independent of the field coordinates $\bar{x}^\lambda$, it is useful to calculate vector cross products in tensor notation, i.e., for two rank 1 tensors, i.e., vectors $\bar{C}$ and $\bar{D}$

$$\bar{C} \times \bar{D} = \epsilon_{ijk} \hat{e}_i C_j D_k \quad (4.16)$$

where in (4.16), and hereafter, summation on Roman indices is assumed to be 1 through 3 i.e.,

$$\bar{a}_i \bar{b}_i \equiv \sum_{i=1}^3 a_i b_i \quad (4.17)$$

In (4.16) $\hat{e}_i$ is the i^{th} Cartesian unit vector so that

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

and the third rank tensor ϵ_{ijk} is completely antisymmetric and hence totally determined by the additional constraint

$$\epsilon_{123} = 1 \quad (4.18)$$

Finally, in (4.16) and in what follows, subscripts on the transformation matrices and field coordinates will be associated with Cartesian coordinates. Thus, for example, we can expand the right hand side of (4.15) using (4.16) to obtain

$$\epsilon_{ijk} L^{(1)\lambda\gamma}_i x_j^\lambda x_k^{\gamma*} \equiv \bar{L}^{(1)} \cdot (\bar{x}^\lambda \times \bar{x}^{\gamma*}) \quad (4.19)$$

Similarly, the left hand side of (4.15) becomes

$$\epsilon_{ijk} L^{(2)\mu\nu} A_{ij}^{\mu\lambda} A_{km}^{\nu\gamma*} \chi_l^\lambda \chi_m^{\gamma*} \quad (4.20)$$

Now let us define the tensor

$$M_{jk}^{(P)\mu\nu} = \epsilon_{ijk} L_i^{(P)\mu\nu} \quad (4.21)$$

Then the conservation law (4.15) becomes in coordinate free form

$$M_{jk}^{(2)\mu\nu} A_{jl}^{\mu\lambda} A_{km}^{\nu\gamma*} = M_{lm}^{(1)\lambda\delta} \quad (4.22)$$

Now let D be the transpose of A in both sets of indices, i.e.,

$$D_{jl}^{\mu\lambda} = A_{lj}^{\lambda\mu} \quad (4.23)$$

Then (4.22) is written as

$$D_{lj}^{\lambda\mu} M_{jk}^{(2)\mu\nu} (D_{km}^{\nu\gamma})^\dagger = M_{lm}^{(1)\lambda\gamma} \quad (4.24)$$

where the conjugate transpose operator † is defined as usual, i.e.,

$$(D_{km}^{\nu\gamma})^\dagger = D_{mk}^{\gamma\nu*}$$

Let us now resequence the matrices in (4.24) according to the scheme

$$T_{n'm'} = D_{km}^{\nu\gamma} \text{ and } N_{n'm'}^{(P)} = M_{lm}^{(P)\mu\gamma} \quad (4.25)$$

$$\text{where } n' = (\nu-1)k + k = \nu k$$

$$m' = (\gamma-1)m + m = \gamma m \quad (4.26)$$

Then transformation law (4.24) becomes

$$T_{nm} N_{mj}^{(2)} (T_{j\ell})^\dagger = N_{n\ell}^{(1)} . \quad (4.27)$$

However, (4.27) is strictly the same as Eq. (3.19) of the scalar example in Section 3. In the next section, we consider the solution of (3.19) and (4.27).

5. ON THE SOLUTION OF THE TRANSLATION OPERATOR T.

A. Conservation of Energy

For both the scalar and vector cases in Sections 3 and 4, the condition of conservation of energy requires the translation operator T to satisfy an equation of the form

$$T E^{(2)} T^\dagger = E^{(1)} \quad (5.1)$$

where $E^{(P)}$, $P=1,2$ are Hermetian $N \times N$ matrices which are positive definite and known. Thus (see Gantmacher, p. 337 theorem 21), there exists unitary matrices U and V such that

$$E^{(2)} = E^{(2)\dagger} = U \Lambda U^\dagger, \Lambda_{ij} = \lambda_i \delta_{ij}, \lambda_i > 0 \quad (5.2)$$

$$E^{(1)} = E^{(1)\dagger} = V M V^\dagger, M_{ij} = \mu_i \delta_{ij}, \mu_i > 0$$

Let

$$T = Y V U^\dagger \quad (5.3)$$

then from (5.1), (5.2), and (5.3) it follows that

$$(Y V U^\dagger) (U \Lambda U^\dagger) (Y V U^\dagger)^\dagger = V M V^\dagger$$

This simplifies to the form

$$W \Lambda W^\dagger = M, \text{ where } W = Y^\dagger Y V \quad (5.4)$$

Let

$$W = M^{1/2} P \Lambda^{-1/2}, P P^\dagger = I \quad (5.5)$$

where any particular root of the possible $2N$ diagonal matrix roots to $M^{1/2}$ and $A^{-1/2}$ can be used. By direct substitution of (5.5) into (5.4) it is seen that W is a solution to (5.4) for any unitary matrix P . Combining (5.3), (5.4) and (5.5), we see that the general solution to (5.1) is given by

$$T = CQD, \quad (5.6)$$

where

$$C = C^\dagger = VM^{1/2}V^\dagger, \quad D = UA^{-1/2}U^\dagger \quad (5.7)$$

$$\text{and } Q Q^\dagger = I \quad (5.8)$$

Direct substitution of (5.6) into (5.1) shows that T is a solution to (5.1) for any unitary $N \times N$ matrix Q . The condition of unitarity imposes N^2 conditions on the $2N^2$ parameters in an arbitrary complex $N \times N$ matrix (Hamer mesh p. 290). Thus the conservation of energy Eq. (5.1) fixes N^2 or half of the $2N^2$ parameters in the translation operator T . We now show that the condition of causality fixes the remaining N^2 parameters to yield the unique transformation operator T .

B. Causality

For the transformation T to be causal, the nm th element of T is required to have the form

$$(T)_{nm} = t_{nm} = e^{ikr_{nm}} F_{nm} \quad (5.9)$$

where $\frac{d}{dk} F_{nm} = 0$

Then, for example, Eq. (3.8) takes the form:

$$y_n(\omega) = \sum_{m=1}^N r_{nm}(\omega) x_m(\omega) \quad (5.10)$$

In (5.10) we have chosen to emphasize the frequency dependence ω of the nodal values y_n and x_m . Taking Fourier transforms of both sides of (5.10), e.g.,

$$\mathcal{F}^{-1} \left(y_n(\omega) \right) \stackrel{\text{Def}}{=} \int_{-\infty}^{\infty} y_n(\omega) e^{-i\omega t} \frac{d\omega}{2\pi} = y_n(t) \quad (5.11)$$

yields through use of (5.9),

$$y_n(t) = \sum_{m=1}^N F_{nm} x_m \left(t - r_{nm}/c \right) \quad (5.12)$$

Thus, comparing (5.9) with (5.12) one can see the physical significance of the parameter r_{nm} as the delay time required for the energy to transit the path from the m^{th} nodal point on the surface S_1 to the n^{th} nodal point on the surface S_2 .

As a first step in this process, it is convenient to define a phase retardation matrix R as

$$RC = CQ \quad (5.13)$$

where C and Q are as defined in (5.6) and (5.7). Substitution of (5.13) into (5.1) yields the following equation for R .

$$R E^{(1)} R^\dagger = E^{(1)} \quad (5.14)$$

It can be shown that the general solution R to (5.14) is not unitary. Performing similarity transforms on (5.14) by pre-multiplication by $V^\dagger$ and post multiplication by V yields the equivalent form of (5.14):

$$R' M R'^\dagger = M, \quad R' = V^\dagger R V \quad (5.15)$$

where M is the diagonal positive matrix introduced in (5.2). In component form (5.15) can be written as

$$\sum_{q=1}^N \left(R'_{nq} R'_{qm} - \delta_{nq} \right) \delta_{qm} \mu_q = 0 \quad (5.16)$$

$n, m = 1, 2, 3, \dots, N, \mu_q > 0.$

Thus the N^2 conditions on R' and hence R in (5.16) are equivalent to the N^2 conditions on Q given by (5.8) (unitarity). We now show that the transformation R as defined by Eq. (5.14) defines a Lie group.

Consider the following sets.

$$S = \{R: R \in R^\dagger = E, R \in T(N) \text{ and } E \in HP(N)\}$$

where

$$HP(n) = \{A: A = A^\dagger, A \text{ is positive definite and } A \in T(N)\}$$

$$T(N) = \{B: B \text{ is an } N \times N \text{ complex matrix}\}$$

We show that S is a Lie group.

(1) Let $R_1, R_2 \in S$, then

$$R_2 R_1 \in (R_2 R_1)^\dagger = R_2^\dagger R_1^\dagger = R_2 \in R_2^\dagger = E$$

therefore, $R_2 R_1 \in S$.

(2) $I \in S$ since $I = I^\dagger$ and

$$I \in I^\dagger = E$$

(3) Let $R \in S$ then since $\det R = \pm 1$, R is invertible, and $R \in R^\dagger = E$

$$R^{-1} R \in R(R^\dagger)^{-1} = E$$

$$= R^{-1} \in (R^\dagger)^{-1} = R^{-1} \in (R^{-1})^\dagger = E.$$

since $(R^\dagger)^{-1} R^\dagger = I$

$(R R^{-1})^\dagger = I = (R^{-1})^\dagger R^\dagger$ so that $(R^\dagger)^{-1} = (R^{-1})^\dagger$, and

therefore, $R^{-1} \in S$.

Note that R is not necessarily unitary. For example, one can easily construct 2×2 matrices R which satisfy (5.14) but are not unitary.

As a first attempt at solution, let us, nonetheless, assume R is unitary. This simplifies matters somewhat as evidenced by the following theorem.

Theorem 5.1

If $R E R^\dagger = E^{(1)}$, where E is Hermetian and positive definite, then R is unitary if and only if R commutes with E .

Proof: If R commutes with E then

$$E R R^\dagger = E \quad (5.17)$$

Since E is Hermetian and positive definite, E is invertible and hence $RR^\dagger = I = R^\dagger R$. Thus R is unitary. On the other hand, if R is unitary then multiply $R E R^\dagger = E$ on the right by R to obtain

$$RE = ER \quad (5.18)$$

One can further show that as a consequence of unitarity, and the assumed distinct (nondegenerate) spectrum of E , that if E is diagonalized by the unitary transformation V , i.e.,

$$M = V^\dagger E V, \quad M_{ij} = \mu_i \delta_{ij}, \quad \mu_i \neq \mu_j, \quad i \neq j \quad (5.19)$$

then so is R , i.e.,

$$R' = V^\dagger R V \text{ is also diagonal.}$$

Thus the unitary transform V diagonalizes both $E^{(1)}$ and R in addition to C .

It is consistent with our theory to represent R and hence Q (by theorem 5.1) in terms of the Hermetian matrix r viz

$$R = Q = e^{ikr}, \quad r = r^\dagger \quad (5.20)$$

then for $k > 0$ a real parameter

$$RR^\dagger = R^\dagger R = I \quad (5.21)$$

By the causality principle in the form (5.9) we have shown that it is required that

$$T = R C D = \hat{R} \otimes C D \quad (5.22)$$

where the Schur product $\otimes$ is defined as

$$A = B \otimes C$$

$$A_{nm} = B_{nm} C_{nm}$$

for arbitrary matrices B, C . In the Schur product, no summation is implied. To determine $\hat{R}$, it is convenient to think of T as a function of k . Then for

$$\hat{R}_{nm} = e^{ik\hat{r}_{nm}} \quad (5.23)$$

$$T'(0) = \frac{d}{dk} T(k) \Big|_{k=0} = i \hat{r} \otimes T(0) \quad (5.24a)$$

On the other hand, however, because of (5.22) and the fact that V diagonalizes R we have

$$T(k) = C e^{ikr} D, \quad e^{ikr} = V e^{ik\Lambda_R} V^\dagger \quad (5.24b)$$

where Λ_R is the diagonal matrix

$$\Lambda_R = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_N \end{pmatrix} \quad (5.25)$$

combining (5.24) and (5.25) and solving for $\hat{r}_{nm}$ gives

$$\hat{r}_{nm} = (C F D)_{nm} / (CD)_{nm}. \quad (5.26)$$

$$\text{where } F = V A_R V^\dagger \quad (5.27)$$

In summary then, the electromagnetic translation operator T is given by the Schur product

$$T = \hat{R} \times C D \quad (5.28)$$

where

$$\hat{R}_{nm} = e^{ik\hat{r}_{nm}}, \quad r_{nm} \neq \hat{r}_{nm}.$$

$$\text{Hence } T_{nm}(k) = e^{ik\hat{r}_{nm}} (CD)_{nm} \text{ where } \hat{r}_{nm} \text{ is given by (5.26).}$$

The distinction between the distance matrices $\hat{r}$ and r is that except in the case of diagonal matrices, the position of the element r_{nm} is not preserved under the operation of matrix exponentiation. The remaining unknown is the diagonal matrix A_R . The diagonal elements of the phase retardation matrix R are of the form (see 5.24b)

$$e^{ikd_n}, \quad d_n > 0, \quad n=1,2,\dots,N \quad (5.29)$$

The physical significance of these elements is determined by interpreting the meaning of the diagonal T matrix: In this special case, the nodes on surface S_1 are lifted to the corresponding nodes on S_2 without mixing. Allowing k to be the free space wave number is consistent with letting d_n be the actual physical distance between corresponding (nearest neighbor) nodes on S_1 and S_2 . This interpretation requires $\vec{E}$ and $\vec{H}$ at the n^{th} node on S_1 and S_2 to be

perpendicular to the path along which d_n is measured. Then (5.29) is equivalent to the one dimensional translation operator. To see this, we define an operator t_d as follows

$$t_d f(x) = f(x+d) \quad (5.30)$$

Thus the effect of t_d is to translate the argument of an arbitrary function $f(x)$ to $x+d$. But by Taylor's theorem

$$f(x+d) = \sum_{n=0}^{\infty} f^{(n)}(x) \frac{d^n}{n!} \quad (5.31)$$

where $f^{(n)}(x)$ is the n th derivative of $f(x)$. Let the operator $e^{\frac{d}{dx}}$ be interpreted in terms of its power series. This gives

$$e^{\frac{d}{dx}} f(x) = \sum_{n=0}^{\infty} f^{(n)}(x) \frac{d^n}{n!} \quad (5.32)$$

For one dimensional (or ray solutions) to Maxwell fields that are outward radiating for a time factor $e^{-i\omega t}$ one has the operational relationship

$$\frac{d}{dx} = ik \quad (5.33)$$

and hence $t_d = e^{ikd} \quad (5.34)$

It is noted that the quantities $\hat{F}_{nm}$ as defined by (5.26) are constrained to be positive by the condition of causality. (See Eq. 5.12.) A solution strategy for F as defined by (5.27) and hence R as defined by (5.20), (5.21) and (5.22) is to choose Λ_R to minimize the matrix norm of the difference between the right and left hand side of (5.26) viz:

$$I(d_1, d_2 \dots d_N) = \| \hat{F} \otimes CD - C F D \| \quad (5.35)$$

subject to the constraint

$$d_i > 0, \quad i = 1, 2, 3 \dots N$$

In (5.35), the nm^{th} element of the matrix $\hat{F}$ is taken to be the distance between the n^{th} node on S_1 and the m^{th} node on S_2 . In (5.29), the unknowns are thus $d_1, d_2 \dots$ on which are to be determined by the N equations

$$\frac{\partial I}{\partial d_n} = 0 \quad , \quad n = 1, 2, 3 \dots N \quad (5.36)$$

subject to the constraint $d_n > 0$. This then completes the formal sketch of the solution of the translation operator T for finite dimensional subspaces.

In the next section, we consider an alternative formulation of the translation operator in terms of the more conventional Green's function approach. Certain comparisons and similarities will be evident.

6. THE STRATTON-CHU FORMULATION

Using the vector identity

$$\nabla \cdot (\vec{A} \times \nabla \times \vec{B} - \vec{B} \times \nabla \times \vec{A}) \quad (6.1)$$

$$= \vec{B} \cdot \nabla \times \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \nabla \times \vec{B}$$

and the divergence theorem, one obtains the integral relationship

$$\oint_S (\vec{A} \times \nabla \times \vec{B} - \vec{B} \times \nabla \times \vec{A}) \cdot d\vec{S} = \int_V \{ \vec{B} \cdot \nabla \times \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \nabla \times \vec{B} \} d^3x \quad (6.2)$$

The surface and volume integrals, and normal vectors are defined as usual. The vector fields $\vec{A}$ and $\vec{B}$ must be suitably continuous. Eq. (6.2) can be used to relate vector fields at an arbitrary space-time point external to a closed spatial surface over which the field is known. In our application, the region exterior to the surface S is assumed to be homogeneous, lossless, and source-free. Let $\vec{A}$ and $\vec{B}$ be electromagnetic quantities, i.e.,

$$\begin{aligned} \vec{A} &= \vec{E} \\ \vec{B} &= \hat{a} g, \quad g = \frac{e^{ikR}}{4\pi R}, \quad \hat{a} \text{ an arbitrary constant vector.} \end{aligned} \quad (6.3)$$

In a source free region,

$$\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = 0, \quad \nabla \times \vec{E} = i\omega\mu_0 \vec{H} \quad (6.4)$$

$$\begin{aligned} \text{while} \quad \nabla \times \nabla \times \vec{B} &= -\nabla^2 \vec{A} \vec{g} + \nabla \nabla \cdot \vec{A} \vec{g} \\ &= k^2 \vec{A} \vec{g} + \vec{A} \delta^{(3)}(\vec{r}-\vec{r}') + \nabla \vec{A} \cdot \nabla \vec{g} \end{aligned} \quad (6.5)$$

where $\delta^{(3)}$ is the three dimensional Dirac delta function.

Let the left hand side of (6.2) be I_S and the right hand side be I_V . Substitution of $\vec{A}$ and $\vec{B}$ into I_S yields:

$$I_S = \int_S \{ \vec{E} \times \nabla \times (\vec{A} \vec{g}) - \vec{A} \times \nabla \times (i\omega\mu_0 \vec{H}) \} \cdot \hat{n} dS' \quad (6.6)$$

This can be manipulated into the form

$$I_S = \int_S \{ i\omega\mu_0 \vec{g} \hat{n} \times \vec{E} + (\hat{n} \times \vec{E}) \times \nabla \vec{g} \} \cdot \vec{A} dS' \quad (6.7)$$

Similarly, the right hand side of (6.2) upon substitution of choices (6.3) and (6.4) for $\vec{A}$ and $\vec{B}$ becomes:

$$I_V = \int_V \{ \vec{A} \cdot \vec{E}(\vec{r}') \delta^{(3)}(\vec{r}-\vec{r}') - \vec{E} \cdot \nabla (\vec{A} \cdot \nabla \vec{g}) \} d^3x' \quad (6.8)$$

Let us choose S and V and the observation point $\vec{r}$ is V as shown in Fig. 0.1.

Figure 0.1 Geometry

Thus V is an annular region bounded by S_1 and S_3 which define S . The electric field $\vec{E}$ is required to satisfy a radiation condition which as, Sommerfeld has shown [11], is precisely what is needed to show that as S_3 is removed to infinity in a limit process, the contribution to I_S due to S_3 vanishes. The second term in (6.8) can be integrated by parts such that the divergence theorem can be used to put this term into a surface integral form. At this point, the arbitrary constant vector $\hat{a}$ resides as an exterior dot product in both I_S and E_V . Equating these yields the Stratton Chu vector field formulation.

$$\vec{E}(\vec{x}) = - \oint_{S_1} [i\omega\mu_0 g \hat{n} \times \vec{H} + \hat{n} \times \vec{E} \nabla g + \hat{n} \cdot \vec{E} \nabla g] dS_1 \quad (6.9)$$

The minus sign in (6.9) arises from the choice in a sign of the unit normal vector $\hat{n}$. In (6.9) it is pointing out from S_1 as depicted in Fig. 1.

Equation (6.9) can be transformed into the time domain with Fourier transform theory. Thus

$$\mathcal{E}(\vec{x}, t) = \int_{-\infty}^{\infty} \vec{E}(\vec{x}, \omega) e^{-i\omega t} \frac{d\omega}{2\pi} \equiv \mathcal{F}^{-1} \vec{E}(\vec{x}, \omega) \quad (6.10)$$

Now the convolution theorem for the cross product of two frequency domain functions is also required, i.e.,

$$\begin{aligned} \mathcal{F}^{-1} \left(\vec{A}(\omega) \times \vec{B}(\omega) \right) &= \vec{a}(t) \times \vec{b}(t) \\ &\equiv \int_{-\infty}^{\infty} \vec{a}(\tau) \times \vec{b}(t-\tau) d\tau \end{aligned} \quad (6.11)$$

Since

$$g(t) = \mathcal{F}^{-1} (e^{ikR} 4\pi R) = \delta(t-R/c) 4\pi R \quad (6.12)$$

for $R = |\vec{x} - \vec{x}'|$

the time domain form of (6.9) is

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi} \int_{S_1} \left\{ \mu_0 1/R \hat{n}' \times \frac{\partial \vec{H}}{\partial \hat{t}}(\vec{x}', \hat{t}) - \hat{n}' \times \vec{E}(\vec{x}', \hat{t}) \times \nabla' \left(\frac{1}{R} \right) \right. \\ \left. - \hat{n}' \cdot \vec{E}(\vec{x}', \hat{t}) \nabla' (1/R) \right\} u(\hat{t}) dS' \quad (6.13)$$

where $\hat{t} = t - R/c$ and $u(\hat{t}) = \begin{cases} 1, & \hat{t} \geq 0 \\ 0, & \text{otherwise} \end{cases}$

The unit step function in (6.13) accounts for the fact that the electromagnetic field is assumed to commence at $t = 0$. The time $\hat{t}$ is retarded according to the distance R between the observation point $\vec{x}$ and the secondary source point $\vec{x}'$ on the surface S_1 . Let us define the unit vector $\hat{R}$ to be

$$\hat{R} = \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|} \quad (6.14)$$

Then $\nabla' (1/R) = -\frac{\hat{R}}{R^2}$

and (6.13) takes the form

$$\vec{E}(\vec{x}, t) = 1/4\pi \int_{S_1} \frac{u(\hat{t})}{R^2} \left\{ \mu_0 \hat{n}' \times \frac{\partial \vec{H}}{\partial \hat{t}} + \hat{R} \times (\hat{n}' \times \vec{E}) - \hat{R}(\hat{n}' \cdot \vec{E}) \right\} dS' \quad (6.15)$$

In (6.15), and for the remainder of this derivation, we have reverted to the bold face straight letters for fields, which should cause no confusion. Also, the arguments of the fields under the integral have also been suppressed. From a computational point of view the last two terms in (6.15) can be improved upon. One can show that

$$\begin{aligned} \hat{R} \times (\hat{n}' \times \mathbf{E}) - \hat{R}(\hat{n}' \cdot \mathbf{E}) &= [\hat{n}' \cdot \hat{R}(\hat{n}'\hat{n}' - \bar{\mathbf{I}}) - \hat{R}\hat{n}'] \cdot \mathbf{E} \\ &\equiv \bar{\mathbf{L}} \cdot \mathbf{E} \end{aligned} \quad (6.16)$$

The proof of this identity is given in Appendix 3.

The next step is to obtain an explicit constraint equation which relates nodal field values on surfaces S_1 and S_2 . The resulting matrix equation will be a rigorous boundary condition to terminate finite difference or finite element transient vector field problems. To maintain accuracy of order $(\Delta x)^2$ in the finite difference schemes, it is necessary to use interpolation functions ϕ_i which are C_1 continuous (continuous through first derivatives). Then the surface fields on S_1 are piecewise linear. For N nodal values on S_1 we have

$$\bar{\mathbf{E}}(\bar{\mathbf{x}}', \hat{t}) \approx \sum_{i=1}^N \phi_i(x', y', z') \bar{\mathbf{e}}_i(\hat{t}) \quad (6.17)$$

$$\bar{\mathbf{H}}(\bar{\mathbf{x}}', \hat{t}) \approx \sum_{i=1}^N \phi_i(x', y', z') \bar{\mathbf{h}}_i(\hat{t})$$

$$\bar{\mathbf{x}}' \in S_1$$

The interpolation functions ϕ_i have the property that, when evaluated at the j th nodal value, they are zero or one, i.e.,

$$\phi_i(x_j, y_j, z_j) = \delta_{ij} \quad (6.18)$$

where δ_{ij} is the Kronecker delta. A convenient choice of the interpolation functions when used in conjunction with a rectangular grid are the rectangular based pyramidal functions. We shall discuss then subsequently.

Both time and space are discrete variables in the boundary value constraint required. In the discrete case, we need the rigorous relationship between fields on a surface S_2 which surrounds S_1 as shown in Fig. 6.2. Let $\vec{x}_j^{(p)}$ be a nodal position vector on surface S_p where $p = 1, 2$.

Figure 6.1. Boundary Constraint Geometry.

Let us use the notation

$$\vec{E}_j^n = \vec{E}(\vec{x}_j^{(2)}, t_n) \quad (6.19)$$

Then relationship (6.15) at a specific space time point on S_2 becomes

$$\bar{E}_j^n = \sum_{i=1}^N \int_{S_i} dS_i \left(\bar{A}_{ji}^n \cdot \bar{e}_i(\hat{t}) + \bar{B}_{ji}^n \cdot \frac{\partial \bar{h}_i(\hat{t})}{\partial \hat{t}} \right) \quad (6.20)$$

where

$$\bar{B}_{ji}^n = \frac{\mu_0 u(\hat{t})}{4\pi R} \phi_i(x', y', z') \hat{r}' \times \bar{I} \quad (6.21)$$

$$\bar{A}_{ji}^n = \frac{\mu_0 u(\hat{t})}{4\pi R^2} \phi_i(x', y', z') (\hat{n}' \cdot \hat{R} (\hat{n}' \hat{n}' - \bar{I}) - \hat{R} \hat{n}') \quad (6.22)$$

$$\text{and } R = |\bar{x}_j^{(2)} - \bar{x}_i^{(1)}| = \bar{r}' I, \quad \hat{t} = t_n - R/c$$

where $\bar{r}'$ is the local radius vector on the i^{th} surface patch with origin at $\bar{x}_i^{(1)}$. In (6.20), S_i is the surface region on S_1 over which ϕ_i has support. To complete our formal expression, we must also perform a temporal interpolation over the discrete time values. Hence we let

$$\bar{e}_i(\hat{t}) = \sum_m \bar{E}_i^{(1)m} T_m(\hat{t}) \quad (6.23)$$

$$\bar{h}_i(\hat{t}) = \sum_m \bar{H}_i^{(1)m} T_m(\hat{t})$$

where the functions $T_m(\hat{t})$ are triangular shaped interpolation functions which are C_1 and have the property (compare with Eq. (6.18),

$$T_m(t_n) = \delta_{nm} \quad (6.24)$$

Explicitly

$$T_n(t) = \begin{cases} 1 - \frac{|t - n\Delta t|}{\Delta t} & , \quad |t - n\Delta t| < \Delta t \\ 0 & , \quad \text{otherwise} \end{cases} \quad (6.25)$$

Pictorially, the first three functions are shown in Fig. 6.3.

Figure 6.3. Triangular interpolation functions T_0 , T_1 and T_2 .

Substitution of time series (6.23) into (6.20) yields the desired result

$$\bar{\varepsilon}_j^n = \sum_{i,m} \left\{ \bar{A}_{ji}^{nm} \cdot \bar{\varepsilon}_i^{(1)m} + \bar{B}_{ji}^{nm} \cdot \bar{H}_i^{(1)m} \right\} \quad (6.26)$$

where

$$\bar{B}_{ji}^{nm} = \frac{\mu_0}{4\pi} \int_{S_i} \left\{ \frac{u(\hat{t})}{R} \phi_i(x', y', z') T_m(\hat{t}) \hat{n}' \times \hat{I} \right\} dS_i \quad (6.27)$$

$$\bar{A}_{ji}^{nm} = \frac{1}{4\pi} \int_{S_i} \left\{ \frac{u(\hat{t})}{R^2} \phi_i(x', y', z') \frac{\partial T_m(\hat{t})}{\partial \hat{t}} \left(\hat{n}' \cdot \hat{R} (\hat{n}' \hat{n}' - \hat{I}) - \hat{R} \hat{n}' \right) \right\} dS_i$$

For numerical stability of finite difference schemes, it is known [12] that the computational velocity must be somewhat greater than the speed of light in the medium.

Thus

$$\frac{\Delta x}{\Delta t} = \alpha_0 c \quad (6.28)$$

$$1 < \alpha_0 < 2$$

where α_0 is a numerical factor. For given values of the integers i, j and n it is possible to compute the values of m which will contribute to (6.25).

$$M_{\text{Max}} = \left[n - \frac{\alpha_0 R_{ji}}{\Delta x} + \sqrt{3} \alpha_0 - 1 \right] \quad (6.29)$$

$$M_{\text{Min}} = \left] n - \frac{\alpha_0 R_{ji}}{\Delta x} + \sqrt{3} \alpha_0 - 1 \right[$$

$$\text{Max}(1, M_{\text{Min}}) \leq m \leq M_{\text{Max}} \quad (6.30)$$

In (6.29) and (6.30) we have used the notation

$[x]$ = least integer greater than or equal to x .

$]x[$ = greatest integer less than or equal to x .

Notice m cannot be less than unity since $\hat{t} \geq 0$. The maximum number of terms in the summation over m as determined from (6.29) and (6.30) to be

$$M_{\text{Max}} - M_{\text{Min}} \leq 7. \quad (6.31)$$

Thus at most 7 terms are required in the summation over m . The $\sqrt{3}$ in (6.29) arises from the assumption that the fundamental cube in the finite difference lattice has dimension $2\Delta x$ on a side. Then $\sqrt{3} \Delta x$ is the largest distance from the center of the cube to its surface.

As an example of a spatial interpolation function ϕ_i , as introduced in Eqs. (6.17) and (6.18), let us consider the special case when S_1 is a rectangular parallelepiped. Then the surfaces of S_1 can be chosen to be constant coordinate planes in three-dimensional Cartesian coordinates. Let such an interpolation function be defined on the x, y

plane. In Fig. 6 we have drawn such a function which is appropriate for a cubic nodal point lattice. They are a natural extension of the one-dimensional triangle interpolation functions drawn in Fig. 6.3.

Figure 6.4. The pyramidal surface interpolation function $\phi_{00}(x,y)$

Since three points determine a plane, it is easy to determine from Fig. 6.4 that an analytic expression for $\phi_{00}(x,y)$ is given by

$$\begin{aligned} \phi_{00}(x,y) = & (1 - u(|x| - \Delta x)) (1 - u(|y| - \Delta y)) \left[u\left(|x| - \frac{|\Delta x|}{y\Delta y}\right) \left(1 - \frac{|x|}{\Delta x}\right) \right. \\ & \left. + u\left(|y| - \frac{|\Delta y}{\Delta x} x\right) \left(1 - \frac{|y|}{\Delta y}\right) \right] \end{aligned} \quad (6.32)$$

The interpolation function ϕ_{nm} centered on the position $n\Delta x, m\Delta y$ is simply related to ϕ_{00} .

$$\phi_{nm}(x,y) = \phi_{00}(x-n\Delta x, y-m\Delta y) \quad (6.33)$$

In an application such as in Eq. (6.17), it is convenient to use single rather than double indices as in Eq. (6.33). In a two-dimensional array, let N_x and N_y be the total number of nodal points in the x and y directions, respectively. Similarly, let n_x and n_y be a particular (x,y) coordinate. Then the transformation

$$i = N_x (n_y - 1) + n_x \quad (6.34)$$

where $1 \leq n_x \leq N_x$, $1 \leq n_y \leq N_y$, $1 \leq i \leq N_x N_y$

assigns a unique integer to surface patches in a natural 'book' ordering.

Appendix 1: Nesting property of the ϕ_i .

The pyramidal interpolation functions as shown in Fig. 6.4, and defined by Eq. (6.32), are nested such that in a lattice $2\Delta x$ on a side containing nine nodal points, a total of 9 interpolation functions have simultaneous non-zero values. This arrangement is shown in Fig. 6.5.

Appendix 2:

If both $\vec{E}$ and $\vec{H}$ are required in the transformation law (6.26) can be rather easily modified as follows.

Let

$$\vec{P}(\vec{x}, t) = \vec{E} + \hat{i} \eta_0 \vec{H} \quad (6.35)$$

In (6.35) the symbol $\hat{i}$ is distinguished from $i = \sqrt{-1}$ but serves a similar role, and $\eta_0 = (\mu_0/\epsilon_0)^{1/2}$ is the free space wave impedance ($\sim 377 \Omega$). Since in free space

$$\nabla \times \vec{E} = i\omega\mu_0 \vec{H} \quad (6.35)$$

$$\nabla \times \vec{E} = -i\omega\epsilon \vec{E}$$

$$\text{so } \nabla \times \vec{P} = \hat{i}/c \frac{\partial \vec{P}}{\partial t} = -\hat{i} \frac{\omega}{c} \vec{P} \quad (6.37)$$

Figure 6.5. Nested interpolation around point P.
(Note only second quadrant of point P has been drawn).

Comparing (6.36) and (6.37), it is seen that the systematic replacement

$$\bar{E} \rightarrow \bar{P}$$

$$\mu_0 \rightarrow \hat{i}/c \quad (6.38)$$

$$\bar{H} \rightarrow -\bar{P}$$

in Eq. (6.20) renders the more general transformation law

$$\bar{p}_j^n = \sum_{i,m} \bar{T}_{ji}^{nm} \cdot p_i^{(1)m} \quad (6.39)$$

where

$$\bar{T}_{ji}^{nm} = \bar{A}_{ji}^{nm} - i/\mu_0 /c \bar{B}_{ji}^{nm}$$

In Fig. 6.6, we demonstrate how surface values are interpolated from adjacent nodal planes so that all components of the electromagnetic field can be referenced to a common point such as $\bar{x}_i^{(1)}$ as required in Eq. (6.26). For example, with reference to Fig. 6.6, interpolation at the point P : (0, 0, 0) yields the relations:

$$E_x(0, 0, 0) = \frac{1}{4\Delta x \Delta z} (E_x(-1, 0, -1) + E_x(-1, 0, 1) + E_x(1, 0, -1) + E_x(1, 0, 1))$$

$$E_y(0, 0, 0) = \frac{1}{4\Delta x \Delta z} (E_y(0, -1, -1) + E_y(0, -1, 1) + E_y(0, 1, -1) + E_y(0, 1, 1))$$

$$E_z(0, 0, 0) : (\text{Natural node point})$$

$$H_y(0, 0, 0) = \frac{H_y(-1, 0, 0) + H_y(1, 0, 0)}{2\Delta x}$$

$$H_x(0, 0, 0) = \frac{H_x(0, -1, 0) + H_x(0, 1, 0)}{2\Delta y}$$

H_z : not required.

Fig. 6.6 Interpolation Scheme

7. CONCLUSIONS AND RECOMMENDATIONS

If one postulates at the outset that the Maxwellian fields in a region are finite dimensional, we have shown how an algebra can be constructed to translate fields from one surface to another. It is natural to do this in the frequency domain where the concept of power flux is easily defined and utilized.

We have shown that an N dimensional field requires a $2N^2$ parameter transformation operator. Conservation of energy utilizes N^2 degrees of freedom. The remaining N^2 degrees of freedom are required to enforce causality. In terms of the bookkeeping, conservation of energy defines the translation operator to within an arbitrary $N \times N$ unitary transformation. Such a transformation contains N^2 free parameters. Causality, on the otherhand, requires the specification of the N^2 distances r_{nm} between the N nodal points on S_1 and N nodal points on S_2 . Thus the $2N^2$ degrees of freedom are uniquely determined.

We have also shown that both scalar and vector fields follow a very similar development. The vector case requires, in general, nine times as many elements.

The alternative, and at this time more certain, Stratton-Chu formulation of the problem was derived in Section 6. The general form of the solutions for both the algebraic and Stratton-Chu formulations are seen to be quite similar.

The next step in this progression is yet to be accomplished: It is recommended that a simple scalar problem with a well-known solution be formulated using the algebraic Green's function method to establish the validity of this new approach.

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