MASTER**STABILITY OF HELICALLY SYMMETRIC
EQUILIBRIUM TO IDEAL MHD
PERTURBATIONS**

BY

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It is well known that the stability of a plasma configuration is determined by the eigenvalues of the linearized perturbation equations. In the case of a helically symmetric equilibrium, the helically symmetric eigenvalues which arise in two-dimensional analysis are not sufficient. In cylindrical geometry, the helically symmetric eigenvalues which arise in two-dimensional analysis are not sufficient. In cylindrical geometry, the helically symmetric eigenvalues which arise in two-dimensional analysis are not sufficient. In cylindrical geometry, the helically symmetric eigenvalues which arise in two-dimensional analysis are not sufficient.

I. INTRODUCTION

There is currently substantial experimental evidence that disruptions in tokamaks are strongly correlated with tearing mode activity at low order rational surfaces.¹⁻⁴ These modes are driven by the free energy associated with the nonuniform current density profile.⁵ The modes do not exist in the ideal MHD description because they result in changes in magnetic field topology (reconnection); such changes are not allowed in ideal MHD. The linear growth rates are intermediate between the resistive diffusion rate $1/\tau_R = c\eta/a^2$ and the Alfvén transit rate $1/\tau_A = B_0/\sqrt{\rho}a$. Here a is the column radius, η the resistivity, ρ the mass density, and B_0 the poloidal magnetic field strength.

The nonlinear state of the collisional tearing mode has been the subject of extensive numerical study. Until recently, these studies have been limited to two dimensional computations; viz., the time development of helically symmetric island states on a cylindrically symmetric equilibrium⁶⁻⁹ (the cylinder, with its ends identified, being treated as a model for a small inverse aspect ratio torus). These two-dimensional nonlinear numerical studies (of low β cylindrical columns) have shown the creation of magnetic islands at the mode rational surface where $m - nq = 0$. Here m and n are the azimuthal and axial mode numbers respectively and $q(r) \equiv RB_0(r)/rB_z$ is the safety factor. It is just at such a surface that the ideal MHD constraint $\mathbf{E} \cdot \mathbf{B} = 0$ is violated. Since these nonlinear states have a

(helical) symmetry direction, magnetic surfaces are present at all times. This is fundamental to the transport properties of the column. If field lines are confined to the neighborhood of surfaces, transport losses will be determined by the relatively small fluxes perpendicular to these surfaces. Thus the creation of magnetic islands of a single helicity can explain the experimentally observed quenching of the discharge, only for the case where the island width itself becomes comparable to the discharge radius.¹⁰

If, however, these magnetic surfaces are destroyed, then some field lines ergodically fill a volume and the much larger fluxes parallel to the field line direction can lead to enhanced radial losses¹¹⁻¹³ and possible disruption of the discharge. Magnetic surface destruction can result from three dimensional effects associated with toroidal coupling terms¹⁴⁻¹⁶ or excitation of modes with more than one helicity.¹⁶⁻¹⁸ Recent three-dimensional simulations of tearing mode development in a zero β cylindrical column have demonstrated that modes of several helicities can be driven to large amplitudes.¹⁸ This could lead to a destruction of magnetic surfaces over a substantial fraction of the column radius. These simulations have been interpreted via a heuristic theory in which linear tearing eigenmodes localized at different mode rational surfaces are nonlinearly coupled.¹⁹ The mode-coupling equations are normalized to the simulation results in order to obtain the unknown matrix elements

and then the results for the nonlinear growth rates are scaled to realistic tokamak parameters.

The generation of such symmetry-breaking modes could be due to a fast time-scale ($\sim \tau_\lambda$) instability of the helically symmetric state attained in the two-dimensional simulations.¹⁷ Such an instability could play a role during the rapid quenching of the discharge. It has been previously shown that two-dimensional equilibria are unstable to symmetry preserving instabilities which tend to reduce the number of islands; i.e., lead to coalescence.²⁰⁻²¹ This coalescence instability seems to require a threshold island thickness for onset.²⁰ It is therefore possible that, during the evolution of the tearing modes, which may be considered a succession of MHD equilibria because of the disparity in time scales, the island width may pass through a critical value at which a rapidly growing symmetry-breaking instability onsets. The nonlinear evolution of such an instability could lead to additional resistive reconnection and destruction of magnetic surfaces, again on an intermediate time scale.

Our contribution is the calculation of the stability of single helicity nonlinear tearing mode states to MHD perturbations with allowance for variation of the perturbation in the helical symmetry direction of the island "equilibrium." The stability is analyzed by numerically solving the linearized equations as an initial value problem. This technique allows us to study the growth rate and mode structure of the most

unstable mode. This mode will dominate the perturbation amplitude after a sufficiently long time if the initial conditions are not orthogonal to it. In solving the initial value problem rather than the corresponding eigenvalue problem, we have, however, given up information about the less rapidly growing and stable modes.

We have limited our explicit calculation to the stability of $m = 2$, $n = 1$ islands, which emanate from the $q = 2$ surface, to perturbations of different helicity. These islands can achieve large widths for realistic current profiles; furthermore, once the island width exceeds the tearing layer width, the evolution occurs on the resistive time scale.²² The poloidal flux function ψ then obeys, at each instant, the equilibrium equation

$$\nabla^2 \psi = J(\psi) - 2 ,$$

with J the axial current density. The concept of MHD stability of a sequence of such states is then a systematically valid one.

If we define n' as the wavevector of the perturbation in the ignorable direction [see Eq. (16)], we find that, for $n' = 0$, the coalescence instability is recovered. As n' is increased, the growth rate decreases because the stabilizing energy required to bend the strong axial magnetic field is increased. We find that the mode becomes stable for $m = 2$, $n = 1$ islands if n' is made large enough ($n' = 1/2$) to satisfy the axial periodicity

requirements as applied to the cylindrical model of the small inverse aspect ratio torus. We conclude that, for an $m = 2$, $n = 1$ island equilibrium, excitation of other helicities in the three-dimensional numerical calculations¹⁸ may be related to nonideal or nonlinear effects.

In Section II the equations and ordering scheme are presented and the linearized equations are derived. The numerical results and discussion are contained in Section III.

II. EQUATIONS

The results in this section have been previously derived by Strauss.²³ We present them here for completeness and clarity. We begin by applying the small inverse aspect ratio tokamak ordering

$$\epsilon \sim \frac{L_1}{L_2} \sim \frac{B_1}{B_2} \sim \frac{p^{1/2}}{B_2} \ll 1 \quad (1)$$

to the MHD equilibrium equations

$$0 = \frac{1}{c} \nabla \times \mathbf{B} - \nabla p \quad (2)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} \quad (3)$$

Here $L_\perp$ and L_z are characteristic dimensions for variation perpendicular to and along the axial direction, respectively, $B_\perp$ and B_z are the corresponding magnetic field components and p is the pressure. From Eq. (2) we find that the variation $\Delta B_z \sim \epsilon^2 B_z$. It then follows from Eq. (3) that $J_z \sim \epsilon$ and $J_\perp \sim \epsilon^2$. Taking the curl of Eq. (2) and using these results we find that, to lowest order,

$$\mathbf{B} \cdot \nabla J_z = 0. \quad (4)$$

We now assume a helical symmetry direction and introduce the independent variables

$$\begin{aligned} r &= (x^2 + y^2)^{1/2} \\ \tau &= \theta - \frac{n}{m} \frac{z}{R} \end{aligned} \quad (5)$$

z

where r , θ , z are the usual cylindrical coordinates. Here $R \sim L_z$ is the "major radius" of the cylindrical tokamak and the ratio n/m determines the helical pitch. For equilibria which are independent of z , we may introduce the magnetic flux function ψ through the definition⁶

$$\underline{B} = \underline{\hat{z}} \times \nabla \psi + \frac{rn}{Rn} B_z \underline{\hat{\theta}} + B_z \underline{\hat{z}} \quad (6)$$

so that

$$\underline{B} \cdot \nabla \psi = 0 . \quad (7)$$

The carets denote unit vectors. Equations (4) and (7) taken together imply that

$$J_z = J_z(\psi) \quad (8)$$

and substitution of (6) into Ampere's law (3) yields the equilibrium equation

$$\nabla_{\perp}^2 \psi = \frac{4\pi}{c} J(\psi) - \frac{2n}{mR} B_z . \quad (9)$$

Small perturbations with velocity $\underline{v}$ about solutions of Eq. (9) satisfy the linearized equations

$$n_o M \frac{\partial \underline{v}}{\partial t} = -\nabla p + \frac{1}{c} (\underline{J} \times \underline{B}_o + \underline{J}_o \times \underline{B}) \quad (10a)$$

$$= -\nabla \left(p + \frac{\underline{B}_o \cdot \underline{B}}{4\pi} \right) + \frac{1}{4\pi} (\underline{B}_o \cdot \nabla \underline{B} + \underline{B} \cdot \nabla \underline{B}_o) ,$$

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}_0) \quad (10b)$$

$$= (\underline{B}_0 \cdot \nabla) \underline{v} - \underline{B}_0 \nabla \cdot \underline{v} - (\underline{v} \cdot \nabla) \underline{B}_0,$$

$$\frac{\partial p}{\partial t} + \alpha p_0 \nabla \cdot \underline{v} + \underline{v} \cdot \nabla p_0 = 0. \quad (10c)$$

Here equilibrium quantities have a zero subscript, M is the ion mass, and α is the adiabatic index. In order for the large restoring pressure force $\nabla \underline{B}_0 \cdot \underline{B}/4\pi$ to be balanced by the potentially destabilizing terms $\underline{B}_0 \cdot \nabla \underline{B} + \underline{B} \cdot \nabla \underline{B}_0$, the perturbation must satisfy $\nabla \cdot \underline{v} \sim \epsilon^2 v_1/a$. Comparison of perpendicular and z components of the momentum equation ($\partial/\partial z \sim \epsilon/a$) yield the additional estimate $v_z \sim \epsilon v_1$. These two estimates allow the introduction of a stream function

$$\underline{v} = \hat{z} \times \nabla \phi \quad (11)$$

to the required order. Introduction of (11) into (10b) shows that we may express the perturbed magnetic field in terms of a flux function ψ through

$$\underline{B} = \hat{z} \times \nabla \psi \quad (12)$$

which satisfies

$$\frac{\partial \psi}{\partial t} = \hat{z} \times \nabla \psi_0 \cdot \nabla \phi + B_z \hat{z} \cdot \nabla \phi \quad (13)$$

Here, as elsewhere, the operation $\hat{z} \cdot \nabla$ is understood to be performed with τ fixed.

Performing the operation $\hat{z} \cdot \nabla \times$ on Eq. (10a) and assuming constant density n_c , we obtain the equation

$$\frac{\partial}{\partial t} \nabla_{\perp}^2 \phi = \frac{1}{4\pi n_0 M} [\hat{z} \times \nabla \psi_0 \cdot \nabla \nabla_{\perp}^2 \psi_0 + \hat{z} \times \nabla \psi_0 \cdot \nabla \nabla_{\perp}^2 \psi + B_z (\hat{z} \cdot \nabla) \nabla_{\perp}^2 \psi] \quad (14)$$

for the evolution of the stream function. Numerical solutions of Eqs. (13) and (14) for particular solutions ψ_0 of Eq. (9) are presented in the next section.

III. NUMERICAL RESULTS AND DISCUSSION

The equilibrium functions $\psi_0(r, \tau)$ were obtained from the nonlinear tearing mode two-dimensional mass code.⁷ The $m = 2$ islands grew from initial perturbations of "flat" current profiles which corresponded to initial safety factor profiles of the form

$$q(r) = c[1 + (r/r_0)^8]^{1/4} \quad (15)$$

Flat current profiles were chosen to insure large island widths since a threshold width for instability was expected. The initial $q = 2$ surface was located at $r \approx 0.7 r_{\max}$ and the values of c and r_0 were adjusted to yield $q(0)$ of 1.4 and 1.1. Since the results are similar for the two values of $q(0)$, we present here only those for $q(0) = 1.1$. Contours of the flux function ψ_0 for an island width $W \sim 0.4 r_{\max}$ are shown in Fig. 1.

Since $\partial\psi_0/\partial z = 0$, solutions of Eqs. (13) and (14) were sought of the form

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} (r, t) \sim \exp(in'z/R) \begin{pmatrix} \phi \\ \psi \end{pmatrix} (r, \tau, t) \quad (16)$$

and the time behavior was studied for several values of n' .

The linearized equations were advanced via a leapfrog scheme. Fourier analysis in τ was used to invert the Laplace operator which appears on the left-hand side of Eq. (14). Equation (13) for ψ was solved directly in (r, τ) space. Typically 50 evenly spaced radial grid points and 32 τ grid points were employed.

An attempt was made to choose an initial perturbation which would efficiently excite the coalescence instability. This instability acts to reduce the number of islands by convecting flux within the islands towards one x point and away from the other. The return flow occurs exterior to the islands. For $m = 2$ islands, the principal τ dependence of the

perturbation is of the form $\exp(\pm i\tau)$. The τ dependence of ψ_0 couples all odd τ modes $\exp(im'\tau)$, $m' = \pm 1, \pm 3, \pm 5, \dots$. Because of the finite resolution of the numerical difference scheme, we can hope to accurately represent only those modes $m' < N_\tau/2$ where N_τ is the number of grid points used to represent the τ variable. Fortunately, previous results for the coalescence instability show that the amplitude of harmonics decreases rapidly as $|m'|$ increases,²⁰ enabling us to accurately represent the most unstable mode with an achievable value for N_τ . We choose the initial conditions $\psi = 0$,

$$\phi_{\pm m'} \sim \pm i r^{|m'|} (r - r_x)(r - r_{\max}) \quad (17)$$

with r_x the radius of the island x points which were at $\tau = 0, \pi$. When n' was small enough for instability to occur, excitation with any odd value of m' resulted in growth of the mode.

After a few growth times, the amplitude of the most unstable mode overwhelms that of the other modes and the spatial structures of ϕ and ψ become time independent. Figures (2a) and (2b) show, respectively, the level contours of ϕ and ψ for $n' = 0$ at time $t = 18 t_0$, where $t_0 \equiv \rho^{1/2} R_m / (B_z n)$. For these parameters, the level contours show no perceptible change after time $t = 12 t_0^{-1}$, whereupon the mode amplitudes take on exponential time behavior, with growth rate $\gamma = 0.31 t_0^{-1}$. The growth rate monotonically

decreases with increasing n' as shown in Fig. 3. For an $m = 2$ equilibrium configuration, and for m' odd, n' must be a half-integer in order to fulfill the periodicity requirements in a large aspect ratio torus. Neither of the equilibria studied [$q(0) = 1.1, 1.4$] showed any evidence of instability when $n' = 0.5$, even for large island widths $w \approx 0.4 r_{\max}$.

In conclusion, we have demonstrated that for an $m = 2$, $n = 1$ island equilibrium, excitation of other helicities which satisfy the periodicity requirements in a small inverse aspect ratio torus is not related to an ideal MHD instability. The observed excitation of these helicities in the three-dimensional numerical calculations may thus be related to effects ignored in our calculations such as: nonideal terms in the Ohm's law, nonlinear effects, or time dependence of the $m = 2$ island. It is noteworthy, however, that the minimum n' needed to satisfy the periodicity conditions in a torus is given by $n' = 1/m$, m being the azimuthal mode number of the equilibrium island. Therefore, our results do not preclude the possibility of an ideal MHD instability of differing helicity in the case of a higher m island equilibrium.

ACKNOWLEDGMENTS

We gratefully acknowledge useful discussions regarding numerical questions with Drs. S. C. Jardin and J. Manickam.

This work was supported by United States Department of Energy Contract No. EY-76-C-02-3073, and also by United States Air Force Office of Scientific Research Contract No. F 44620-75-C-0037.

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Fig. 1. Helically symmetric equilibrium flux surfaces ψ_0 ⁷⁸²²⁶² taken from MASS code for a run with a "flat" initial current profile and an on-axis safety factor $q(0) = 1.1$.

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Fig. 2. Contours of constant amplitude of the perturbation velocity potential ϕ (a) and vector potential ψ (b) at time $t/t_0 = 18$, for $n' = 0$.

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Fig. 3. Growth rate vs n' .