ASYMPTOTIC STABILITY BOUNDARIES OF
BALLOONING MODES IN CIRCULAR TOKAMAKS

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ABSTRACT

The model ballooning mode equation of Connor, Hastie, and Taylor for large-aspect-ratio circular tokamaks is analyzed in the limit of large pressure gradient, and corresponding expressions for stability boundaries are derived. In particular, it is found that for a fixed radial wave number, there exists an infinite sequence of unstable bands, and that minimizing over the radial wave numbers leads to asymptotic merging between the neighboring bands.

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It is generally accepted that ideal magnetohydrodynamic (MHD) ballooning instabilities place important limitations on the achievable β values in tokamaks [1,2]. Here, $\beta \equiv$ plasma pressure/magnetic pressure. The discovery of the so-called high- β second stability region [3-5] thus has had a significant impact on tokamak research and makes it desirable to obtain analytical insights to the stability properties of high- β ballooning modes. Previous analytical studies [4,6] on this subject are confined to small or moderate magnetic shear in order to employ the two-scale-length expansion and, hence, tend to emphasize the average (over the surface) effects. On the other hand, in an insightful but basically numerical study, Greene and Chance [7] have pointed out the importance of local properties (e.g., the local magnetic shear) in determining the second stability boundary. The present work attempts to understand analytically the effects of such local properties. To facilitate the analyses, we employ the ballooning-mode equation proposed by Connor, Hastie, and Taylor (CHT) for modeling large-aspect-ratio high- β tokamaks with circular magnetic surfaces [2]. Furthermore, we take the limit of large (negative) pressure gradients so that, as will be shown later, the local structures become more pronounced. As a result, we are able to derive analytic expressions for the asymptotic stability boundaries. Our finding, in addition to demonstrating specifically the ideas suggested by Greene and Chance, also show that for fixed radial wave numbers, there exist infinite sequences of unstable bands, which significantly modify the high- β tokamak stability properties.

The CHT ballooning mode equation is given as

$$\left[\frac{d}{d\eta} f(\eta) \frac{d}{d\eta} + k(\eta) \right] \phi(\eta) = 0; \quad -\infty < \eta < \infty, \quad (1)$$

where

$$f(\eta) = 1 + I^2(\eta) \quad , \quad (2)$$

$$I(\eta) = \alpha \sin \eta - s(\eta - \theta_k) \quad , \quad (3)$$

$$k(\eta) = \alpha \cos \eta - I(\eta)\alpha \sin \eta \quad , \quad (4)$$

$s = rq'/q$ is the (average) magnetic shear, $\alpha = -2Rq^2p'/B^2$ measures the pressure gradient, and the finite radial wave number ($|\theta_k| \leq \pi$) is also included [8,9]. Letting $\hat{\phi} = f^{1/2}\phi$, Eq. (1) can be rewritten as

$$[d^2/d\eta^2 - V(\eta)]\hat{\phi} = 0 \quad , \quad (5)$$

where

$$V(\eta) = (I'/f)^2 - \alpha \cos \eta / f \quad , \quad (6)$$

and $I' = \alpha \cos \eta - s$. Note, in Eq. (6), I' , $\alpha \cos \eta$, and f correspond to, respectively, effects of local magnetic shear, normal curvature, and field line bending. Note also that at marginal stability, the boundary condition is $|d\hat{\phi}/d\eta| \rightarrow 0$ as $|\eta| \rightarrow \infty$. The stability boundaries can then be calculated variationally, i.e.,

$$D[\hat{\phi}_t] = \int_{-\infty}^{\infty} d\eta [V(\eta)\hat{\phi}_t^2 + (d\hat{\phi}_t/d\eta)^2] = 0 \quad . \quad (7)$$

Here, $\hat{\phi}_t$ is an appropriately constructed trial function satisfying the above-mentioned boundary conditions.

Before proceeding with the analyses, let us consider some ordering arguments. In the limit $\alpha \gg 1$, we have formally $|I|, |I'| \sim O(\alpha) \gg 1$, and $|f| = |1 + I^2| \sim O(\alpha^2) \gg 1$. $V(\eta)$ is thus negligible except around $\eta = \eta_*$ where $I(\eta_*) = 0$, i.e.,

$$\sin \eta_* = (s/\alpha)(\eta_* - \theta_k) \quad (8)$$

On the other hand, for $\eta \approx \eta_*$, $|f| \sim O(1)$ and the stabilizing local magnetic shear, $|I'|^2$, is formally $O(\alpha)$ bigger than the destabilizing normal curvature, $|\alpha \cos \eta_*|$. Marginal stability thus suggests vanishing local magnetic shear at η_* , i.e., $I'(\eta_*) = 0$ or

$$\cos \eta_* = s/\alpha \quad (9)$$

Combining Eqs. (8) and (9) then gives

$$\alpha_{*l} = s(\eta_{*l} - \theta_k) / \sin \eta_{*l} \quad (10)$$

where $\eta_{*l} = 2l\pi + \eta_{*0}$, $l = \text{integers}$, and $|\eta_{*0} = \cos^{-1}(s/\alpha_{*0})| \leq \pi$. Equation (10) thus suggests the existence of an infinite sequence of asymptotic marginal-stability lines or, in practice, unstable bands, and this observation has strongly motivated the present analyses. Associated with each l th band, there are then the $(2l + 1)$ th and $(2l + 2)$ th stability boundaries denoted as $\hat{\alpha}_{l,1}$ and $\hat{\alpha}_{l,2}$.

To correspond with previous theoretical results [2,4], we first analyze the zeroth band, i.e., the first and second stability boundaries for the $\theta_k = 0$ case. For a fixed s value, as we increase α from zero to $\alpha < s$, I only vanishes at $\eta = 0$. Around $\eta = 0$, we have

$$V(\eta) = V_0(\eta) = \left[\frac{I'_0}{1 + (I'_0 \eta)^2} \right]^2 - \frac{\alpha}{1 + (I'_0 \eta)^2} \quad , \quad (11)$$

where $I'_0 = \alpha - s$. Taking $\hat{\phi}_{t0} = 1$, Eq. (7) gives

$$(s - \alpha)^2 = 2\alpha \quad ,$$

i.e.,

$$\hat{\alpha}_{01} = s + 1 - (2s + 1)^{1/2} \quad , \quad (12)$$

which corresponds to the first stability boundary. As to the second stability boundary, we have $\alpha > s$ and I vanishes at $\eta = \hat{\eta}_0$ in addition to $\eta = 0$. Here, $\hat{\eta}_0^2 = 6(\alpha - s)/\alpha \ll 1$. Thus,

$$V(\eta) = V_0 + \hat{V}_0 \quad ,$$

with $\hat{V}_0$ having the same form as V_0 , except α and η are replaced by $\hat{\alpha} = \alpha \cos \hat{\eta}_0$ and $\eta - \hat{\eta}_0$, respectively. For $\hat{\phi}_{t0} = 1$, we have

$$\hat{\alpha}_{0,2} = s + 0.5 + (4s + 1)^{1/2}/2 \quad . \quad (13)$$

In Fig. 1 are plotted $\hat{\alpha}_{0,1}$ and $\hat{\alpha}_{0,2}$ along with direct numerical results, and good asymptotic agreement is evident. Note also that $\hat{\eta}_0 \rightarrow 0$ and $\hat{\alpha}_{0,1}, \hat{\alpha}_{0,2} \rightarrow s$ asymptotically, which is consistent with $I'(0) = 0$.

We now discuss qualitatively finite- θ_k effects on the stability of the zeroth band. With $\theta_k \neq 0$, $I(\eta) = 0$ at $\eta = \eta_{*0} \neq 0$. The asymptotic marginal stability condition, $I'(\eta_{*0}) = 0$, then suggests $\alpha_{*0} > s$. That is, $\theta_k \neq 0$ tends to tilt the zeroth unstable band toward the α -axis in Fig. 1 or, in other words, stabilize (destabilize) the first (second) stability boundary. In the following, we shall present general analyses for the l th band.

Noting the up-down symmetry ($\eta \rightarrow -\eta$, $\theta_k \rightarrow -\theta_k$), we shall confine the analyses to the $\eta > 0$ region and let $-\pi \leq \theta_k \leq \pi$. Consider the l th region, $2l\pi \leq \eta \leq (l+1)2\pi$. It is easy to see that, as s/α is reduced toward the value at which $I = I' = 0$, we first encounter $I' = 0$ and, yet, $I \neq 0$. Let $I'(\eta_{*l}) = 0$, i.e., $\cos \eta_{*l} = s/\alpha_l$. Also, we let $I(\hat{\eta}_{l-1}) = 0$ such that $\hat{\eta}_{l-1} < \eta_{*l}$ and $I'(\hat{\eta}_{l-1}) \neq 0$. There, thus, exists a large stabilizing potential barrier at $\hat{\eta}_{l-1}$. To minimize this stabilization, we employ the following trial function

$$\hat{\phi}_{tl} = \begin{cases} c_l(\eta - \hat{\eta}_{l-1} - \Delta_1) & , \quad \hat{\eta}_{l-1} + \Delta_1 \leq \eta \leq \eta_{*l} - \Delta_2 \\ 1 & , \quad \eta \geq \eta_{*l} - \Delta_2 \\ 0 & , \quad \text{otherwise} \end{cases} \quad (14)$$

Here, $1 \gg \Delta_1, \Delta_2 \gg$ corresponding scale lengths of the potential structures, and $c_l = (\eta_{*l} - \hat{\eta}_{l-1} - \Delta_1 - \Delta_2)^{-1}$. Note that contributions due to Δ_1 and Δ_2 are asymptotically negligible. We then obtain, from Eq. (7),

$$2(s/I'_{*l})_l = (2I_{*l})_l^{-1} + [c/\pi(I'_{*l})_l^{1/2}]_l (2I_{*l})_l^{3/2} \quad (15)$$

Here, I_{*l} and I'_{*l} are values at $\eta = \eta_{*l}$. The stabilizing terms in the right-hand side of Eq. (15) have a minimum at

$$I_{*lm} = \frac{1}{2} \left(\frac{2\pi}{3c_l} \right)^{2/5} (I'_{*lm})^{1/5} \sim 0(\alpha_l^{1/5}) \quad , \quad (16)$$

where we have noted that $c_l = 2/3\pi$ and $I'_{*lm} = \alpha_l$. Expanding Eq. (15) around I_{*lm} yields

$$s/\alpha_l = \frac{5}{6} \pi^{-4/5} \alpha_l^{-1/5} + \frac{5}{2} \pi^{-12/5} \alpha_l^{-3/5} (I_{*l} - I_{*lm})^2 \quad . \quad (17)$$

From Eq. (17), we can readily obtain the asymptotic stability boundaries of the l th unstable band. Note that $\eta_{*l} \approx (4l + 1)\pi/2$, we then have $\alpha_l = \alpha_{*l} + \delta\alpha_l$ with $\alpha_{*l} = s(\eta_{*l} - \theta_k)$ and $\delta\alpha_l = \pm (2/5)\pi^{12/5}(\eta_{*l} - \theta_k)^{-1}\alpha_{*l}^{3/10}$.

Noting that $\eta_{*l}(\theta_k = -\pi) = \eta_{*l+1}(\theta_k = \pi)$, minimizing the stability boundaries over θ_k thus leads to merging between neighboring asymptotic unstable bands. The effective second stability boundary, therefore, is given by Eq. (17) with $I_{*l} = I_{*lm}$, i.e.,

$$\alpha = 4.0 \times s^{1.25} \quad . \quad (18)$$

Recently, a similar result has been obtained independently by Llobet et al. [10]. Figure 2 summarizes both numerical and analytical results. It clearly demonstrates that minimization over θ_k modifies significantly, for moderate and large α values, the high- β second stability boundary from that of the $\theta_k = 0$ case. On the other hand, for weak and moderate $\alpha(s)$ values, finite- θ_k effects are negligible since the relevant scale length becomes longer than the connection length.

In summary, the present work has demonstrated both analytically and numerically that finite (global) radial wave number (θ_k) affects qualitatively the high- β second stability boundaries of circular tokamaks. Physically, this is due to the dependence of stability on local properties as first suggested by Greene and Chance. Analytical expressions for the marginal stability boundaries are also derived and agree sufficiently well with numerical solutions. While our analyses have been limited to the model CHT equation, many features obtained here appear to be quite universal and it would be interesting to investigate realistic equilibria along similar lines.

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FIGURE CAPTIONS

Figure 1. First and second stability boundaries in the (s, α) plane for the $\theta_k = 0$ case. The region denoted as "stable" is further destabilized by finite- θ_k values.

Figure 2. Stability boundaries in the (s, α) plane after minimizing over θ_k .

Fig. 1

Fig. 2

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