INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

University of
Maryland
College Park

DOE/ER/03443-79

Technical Note BN-894

MASTER

ON THE NUMERICAL SOLUTION OF
HELMHOLTZ EQUATION BY THE
FINITE ELEMENT METHOD

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January 1979

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* Research supported in part by the Department of Energy under contract
DOE E(40-1)3443.

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1. Introduction

In this paper we consider the numerical solution of the boundary value problem:

$$(1.1) \quad L\phi \equiv \Delta\phi + \lambda\phi = \gamma \quad \text{in } \Omega$$

$$(1.2) \quad \phi = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, λ is not an eigenvalue of $-\Delta$ and $\gamma \in H^{r-1}(\Omega)$ ($r \geq 2$). In [6] a similar problem has been treated by the method of least squares, namely the boundary value problem.

$$(1.3) \quad \Delta\phi + q\phi = f \quad \text{in } \Omega$$

$$(1.4) \quad \phi = 0 \quad \text{on } \Gamma_D$$

$$(1.5) \quad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \Gamma_N,$$

or equivalently:

$$(1.6) \quad \text{div}(\underline{u}) + q\phi = f \quad \text{in } \Omega,$$

$$(1.7) \quad \nabla\phi - \underline{u} = 0 \quad \text{in } \Omega,$$

$$(1.8) \quad \phi = 0 \quad \text{on } \Gamma_D, \quad \underline{u} \cdot \underline{\nu} = 0 \quad \text{on } \Gamma_N,$$

where Ω is a bounded region in $\mathbb{R}^n$ with boundary $\Gamma_D \cup \Gamma_N$ and $\underline{\nu}$ is the outward normal to the boundary.

In order to obtain optimal estimates in L_2 for ϕ and $\underline{u}$ using linear elements, a very restrictive condition (the so-called "Grid Decomposition Property") is imposed on the elements which excludes the use of some of the usual triangulations (see [5], [6], [7]). Except for the introduction of the "GDP" the proof in [6] proceeds along the lines of the techniques used in [8] where, under the assumption that f is sufficiently smooth, an optimal L_2 error estimate for the least squares method is obtained for the scalar function ϕ , the solution of the Dirichlet problem for Laplace's equation. The results in [6] are incorrect, because the solution $(\phi, \underline{u})$ of (1.6-1.7) does not have adequate regularity.

In the present paper we obtain an optimal L_2 error bound in a general context which contains as a special case an optimal L_2 error estimate for $\nabla\phi$ with linear elements for problem (1.1), (1.2) without requiring the "Grid Decomposition Property" and f needs only to be in $H^1(\Omega)$. The key to our approach lies in reducing the solution of boundary value problem (1.1-1.2) by a two-stage finite element procedure to the solution of a 2x2 first order elliptic system of Petrovski type for which suitable shift theorems are known [10], [11]; for the 3x3 system (1.6-1.7) which is elliptic in the sense of Agmon-Douglis-Nirenberg, there are in general no such shift theorems.

In section 2 we give the finite element formulation of our problem and in section 3 we obtain L_2 error estimates for our method. In section 4, using linear elements with a triangulation which does not satisfy the "GDP" condition [6], we give a numerical example for the case $\lambda = \frac{1}{2}$ in (1.1). The computed L_2 -error estimates agree with the theoretical estimates given in section 3.

2. Finite Element Formulation

In what follows, we use the usual notations and definitions (for Sobolev

spaces, norms, etc.) found in [4], [9]. Let $\Omega \subseteq \mathbb{R}^2$ be a Lipschitzian domain, and let $S^1 = H_0^1(\Omega)$. Given $\gamma \in H^{r-1}(\Omega)$ ($r \geq 2$) and $\lambda \in \mathbb{R}$, not an eigenvalue of $-\Delta$, we wish to find $\phi \in S^1$ such that

$$(2.1) \quad L\phi \equiv \Delta\phi + \lambda\phi = \gamma \quad \text{in } \Omega.$$

In practice, e.g., potential flow problems, one is often more interested in $\nabla\phi$ than in ϕ . Let t be the tangent to $\partial\Omega$, and let

$$(2.2) \quad S^2 = \{v \in [H^1(\Omega)]^2 : v \cdot t = 0 \text{ on } \partial\Omega\}.$$

If $u \in S^2$ satisfies

$$(2.3) \quad Lu \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial u}{\partial x_1} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial u}{\partial x_2} = \begin{bmatrix} \gamma - \lambda\phi \\ 0 \end{bmatrix} \equiv f,$$

then $u = \nabla\phi$.

Note that (2.1) and (2.3) may be expressed in a weak form. The weak form of (2.1) will be to find $\phi \in S^1$ such that

$$(2.4) \quad \int_{\Omega} [\nabla\phi \cdot \nabla\psi - \lambda\phi\psi] = \int_{\Omega} -\gamma\psi \quad \forall \psi \in S^1,$$

while that for (2.3) will be to find $u \in S^2$ such that

$$(2.5) \quad (Lu, Lv) = (f, Lv) \quad \forall v \in S^2.$$

The two-stage finite element method is based on replacing S^1 and S^2 in (2.4) and (2.5) by finite-dimensional subspaces. We let $\{S_h^1\}_{h>0}$ and $\{S_h^2\}_{h>0}$ be families of finite-dimensional subspaces of S^1 and S^2 ,

respectively. The first stage of the method is to find the solution

$\phi_h \in S_h^1$ of

$$(2.6) \quad \int_{\Omega} \nabla \phi_h \cdot \nabla \psi_h - \lambda \phi_h \psi_h = \int_{\Omega} -\gamma \psi_h \quad \forall \psi_h \in S_h^1$$

The second stage consists of setting

$$(2.7) \quad f_h \equiv \begin{pmatrix} \gamma - \lambda \phi_h \\ 0 \end{pmatrix}$$

and then finding $u_h \in S_h^2$ such that

$$(2.8) \quad (Lu_h, Lv_h) = (f_h, Lv_h) \quad \forall v_h \in S_h^2$$

Note that (2.6) is the usual Galerkin method for approximating ϕ , while

(2.8) is a least-squares method for approximating $u = \nabla \phi$.

We assume that S_h^1 and S_h^2 satisfy the usual approximation properties.

That is, there exists a constant $C_A > 0$, depending only on Ω , and a positive integer k , such that the following hold:

- (i) Approximation in S_h^1 : for each $\psi \in S^1 \cap H^{k+1}(\Omega)$, there exists $\tilde{\psi}_h \in S_h^1$ such that

$$||\psi - \tilde{\psi}_h||_{\ell} \leq C_A h^{k+1-\ell} ||\psi||_{k+1} \quad (\ell=0, 1)$$

- (ii) Approximation in S_h^2 : for each $v \in S^2 \cap [H^{k+1}(\Omega)]^2$, there exists $\tilde{v}_h \in S_h^2$ such that

$$||v - \tilde{v}_h||_{\ell} \leq C_A h^{k+1-\ell} ||v||_{k+1} \quad (\ell=0, 1)$$

For instance, S_h^1 and S_h^2 may be taken to be finite element subspaces of S^1 and S^2 (respectively) consisting of piecewise polynomials of degree less than or equal to k .

3. Error Analysis

In order to discuss convergence of the method, we need the following ellipticity estimates (see [4], [9], [10]): there exist positive constants C_1 and C_2 (depending only on Ω) such that for $s \geq 0$,

$$(3.1) \quad C_1 ||\psi||_{s+2} \leq ||L\psi||_s \leq C_2 ||\psi||_{s+2} \quad \forall \psi \in S^1 \cap H^{s+2}(\Omega)$$

and

$$(3.2) \quad C_1 ||v||_{s+1} \leq ||Lv||_s \leq C_2 ||v||_{s+1} \quad \forall v \in S^2 \cap [H^{s+1}(\Omega)]^2.$$

Hence for sufficiently smooth γ , there exists $r \geq 2$ such that

$$(3.3) \quad \phi \in H^{r+1}(\Omega),$$

and thus

$$(3.4) \quad u = \nabla \phi \in H^r(\Omega).$$

We assume that the degree k of the subspaces S_h^1 and S_h^2 satisfies $k \leq r - 1$.

The analysis of the first stage is done in sections 5.3.2 and 6.3.2 of [3]; we have the following

Theorem 3.1.

- (i) There exists a unique $\phi_h \in S_h^1$ satisfying (2.6).

(ii) There is a $C > 0$, depending only on Ω and λ , such that

$$||\phi_h||_1 \leq C ||\gamma||_0.$$

(iii) There is a $C > 0$, depending only on Ω and λ , such that

$$||\phi - \phi_h||_s \leq Ch^{k+1-s} ||\phi||_{k+1} \quad (s=0, 1).$$

We now consider the second stage of the algorithm. First, we prove existence, uniqueness, and stability of the solution to (2.8).

Theorem 3.2. There is a unique $u_h \in S_h^2$ satisfying (2.8). Moreover, there is a $C > 0$, depending only on Ω and λ , such that

$$(3.5) \quad ||u_h||_1 \leq C ||\gamma||_0.$$

Proof: Existence and uniqueness of u_h is shown in the usual manner, i.e., by showing that for any basis of S_h^2 , the coefficient matrix in the linear system for (2.8) is symmetric and positive definite. Using (2.8) with $v_h = u_h$ (3.2), (2.7), and (ii) of Theorem 3.1, we find

$$C_1 ||u_h||_1^2 \leq (Lu_h, Lu_h) = (f_h, Lu_h)$$

$$\leq C_2 ||f_h||_0 ||u_h||_1$$

$$\leq C_2 [||\gamma||_0 + |\lambda| ||\phi_h||_1] ||u_h||_1$$

$$\leq C_2 [1 + C|\lambda|] ||\gamma||_0 ||u_h||_1,$$

which implies (3.5).

In order to analyze the convergence of the second stage, we need a bound on the error in $H^1(\Omega)$.

Theorem 3.3. There is a constant $C > 0$, depending only on Ω and λ , such that

$$||u - u_h||_1 \leq Ch^k ||u||_{k+1}.$$

Proof: Using (2.5) with v replaced by v_h and (2.8), we find

$$(3.6) \quad (L(u - u_h), Lv_h) = (f - f_h, Lv_h) \quad \forall v_h \in S_h^2.$$

Hence (3.2) and (3.6) imply

$$\begin{aligned} c_1 ||u_h - \tilde{u}_h||_1^2 &\leq (L(u_h - \tilde{u}_h), L(u_h - \tilde{u}_h)) \\ &= (L(u - \tilde{u}_h), L(u_h - \tilde{u}_h)) - (L(u - u_h), L(u_h - \tilde{u}_h)) \\ &= (L(u - \tilde{u}_h), L(u_h - \tilde{u}_h)) - (f - f_h, L(u_h - \tilde{u}_h)) \\ &\leq c_2 [||u - \tilde{u}_h||_1 + ||f - f_h||_0] ||u_h - \tilde{u}_h||_1, \end{aligned}$$

where $\tilde{u}_h$ is chosen as an approximation property (ii). Hence,

$$||u_h - \tilde{u}_h||_1 \leq c_1^{-1} c_2 [||u - \tilde{u}_h||_1 + ||f - f_h||_0],$$

so that

$$\begin{aligned} (3.7) \quad ||u - \tilde{u}_h||_1 &\leq ||u - \tilde{u}_h||_1 + ||u_h - \tilde{u}_h||_1 \\ &\leq C [||u - u_h||_1 + ||f - f_h||_0]. \end{aligned}$$

Now using approximation property (ii) with $\ell = 1$ and

$$(3.8) \quad \|f - f_h\|_0 = |\lambda| \|\phi - \phi_h\|_0 \leq C|\lambda|h^{k+1}\|\phi\|_{k+1} \leq Ch^{k+1}\|u\|_{k+1}$$

in (3.7), the result follows immediately.

Finally, we use a duality argument to prove our main result:

Theorem 3.4. There is a constant $C > 0$, depending only on Ω and λ , such that

$$\|u - u_h\|_0 \leq Ch^{k+1}\|u\|_{k+1}.$$

Proof: Let $e_h = u - u_h$, and let $w \in L(S^2)$ satisfy

$$(3.9) \quad (Lv, w) = (e_h, v) \quad \forall v \in S^2.$$

Integration by parts, along with $v \in S^2$, yields

$$\begin{aligned} (Lv, w) &= (v, -Lw) + \int_{\partial\Omega} w_1(v \cdot n) + w_2(v \cdot t) \\ &= (v, -Lw), \end{aligned}$$

provided $w_1 = 0$ on $\partial\Omega$. Thus $w \in L(S^2)$ is the weak solution to

$$(3.10) \quad -Lw = e_h \quad \text{in } \Omega,$$

$$(3.11) \quad w_1 = 0 \quad \text{on } \partial\Omega.$$

The boundary condition (3.11) is complementary to (3.10); thus (3.10) is elliptic in $\bar{\Omega}$. Hence the results of Roitberg and Šeftel [10] imply that there is a constant $C > 0$, depending only on Ω , such that

$$(3.12) \quad ||w||_1 \leq C ||e_h||_0$$

Writing $w = Lr$ and using the analogue of (3.2) for problem (3.10), (3.11), we have

$$C_1 ||r||_2 \leq ||w||_1 \leq C_2 ||e_h||_0 ;$$

thus there exists $r_h \in S_h^2$ such that for $w_h = Lr_h$, we have

$$(3.13) \quad ||w - w_h||_0 \leq C ||r - r_h||_1 \leq Ch ||r||_2 \leq Ch ||e_h||_0$$

Now (2.5), (2.7), (2.8), (3.9), (3.12), and (3.13) yield

$$\begin{aligned} ||e_h||_0^2 &= (Le_h, w) = (Le_h, w - w_h) - (f - f_h, w - w_h) + (f - f_h, w) \\ &\leq C[||e_h||_1 + ||f - f_h||_0] ||w - w_h||_0 + ||f - f_h||_0 ||w||_1 \\ &\leq Ch[||e_h||_1 + ||f - f_h||_0] ||e_h||_0 + C ||f - f_h||_0 ||e_h||_0, \end{aligned}$$

i.e.,

$$(3.14) \quad ||e_h||_0 \leq C[h ||e_h||_1 + ||f - f_h||_0]$$

The use of (3.8) and Theorem 3.3 in (3.14) completes the proof.

4. Numerical Results

In this section, we report on the results of a numerical experiment which demonstrates the effectiveness of the method. We chose $\Omega = [0,1] \times [0,1]$. The right-hand side γ was chosen so that the exact solution to (2.1) was

$$\phi(x,y) = (\sin \pi x) (\sin \pi y),$$

and an equidistant directional right-triangular grid was used, the triangles having legs of length h .

The L_2 error and order of convergence of the approximation to $\nabla\phi$ are given in Table 4.1. In Table 4.1, we see the results for the case $\lambda = \frac{1}{2}$, which supports the assertion of Theorem 3.4, i.e., it is possible to achieve $O(h^2)$ accuracy in the computed gradient when using linear elements. A numerical example for the case $\lambda = 0$ is given in [2], where the first stage is not needed.

h^{-1}	L_2 -error	L_2 -order
3	.1724531300	1.60713
4	.1086120100	1.72586
5	.0738966160	1.80067
6	.0532163890	1.84973
7	.0400139980	1.88318
8	.0311173530	1.90682
9	.0248578610	1.92408
10	.0202965820	1.93701
11	.0168750420	1.94694
12	.0142453410	1.95472
13	.0121821190	1.96092
14	.0105344380	1.96593
15	.0091982600	

Table 4.1

RESULTS FOR $\lambda = \frac{1}{2}$

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