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TITLE: Beam Beam Deflection and Signature Curves for Elliptic Beams.

Abstract

In this note we will present closed expressions for the beam-beam deflection angle for arbitrary elliptic beams including tilt. From these expressions signature curves, i.e. systematic deviations from the round beam deflection curve due to ellipticity or tilt are derived. In the course of the presentation we will prove that it is generally impossible to infer individual beamsizes from beam-beam deflection scans.

1 Introduction

Bassetti and Erskine [1] first calculated the electric field of an upright elliptic beam. Their work is presented in a coherent fashion in [2] by R. Talman. Here we will generalize their result to the case where the "active" beam, i.e. the beam that produces the electric field is tilted in the x-y plane. Since the electric field is proportional to the kick a sample positron experiences, we have to average the electric field over the positron distribution in order to calculate the kick of the centroid of the positron beam. We will show that the centroids' kick only depends on the convolution of the distribution functions of the active and passive beams. For gaussian beams this implies that it is impossible to disentangle individual beam sizes from beam-beam scans. Finally we derive "signature curves" by expanding the full expression for the centroid kick for tilted elliptic beams to first order in the ellipticity and the tilt around the round beam case.

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2 The single particle kick

In the coordinate system $\bar{S}$, where the active beam is upright the kick a positron experiences is given by [2]

$$\Delta \bar{y}' + i \Delta \bar{x}' = \frac{NK\sqrt{\pi}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \left\{ w \left[\frac{\bar{x} + i\bar{y}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right] - e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}} w \left[\frac{\bar{x} \frac{\sigma_y}{\sigma_x} + i\bar{y} \frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right] \right\} \quad (1)$$

where we have $K = -2r_e/\gamma$ and N is the number of electrons in the active beam. σ_x and σ_y are the major axes of the active beam. $w(z)$ is the complex error function [4]. The transformation back into the laboratory frame is defined by $U = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$ or

$$x + iy = e^{i\alpha} (\bar{x} + i\bar{y}) \quad (2)$$

and

$$\Delta y' + i \Delta x' = e^{-i\alpha} (\Delta \bar{y}' + i \Delta \bar{x}') .$$

Now it remains to reexpress the right hand side of eq. 1 in terms of laboratory system quantities, namely

$$\begin{pmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{21} & \tilde{\sigma}_{22} \end{pmatrix} = U \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} U^T . \quad (3)$$

Noting that the expressions appearing as arguments in the complex error functions $w(z)$ transform according to

$$\begin{aligned} \sigma_x^2 - \sigma_y^2 &= e^{-2i\alpha} (\tilde{\sigma}_{11} - \tilde{\sigma}_{22} + 2i\tilde{\sigma}_{12}) \\ \bar{x} \frac{\sigma_y}{\sigma_x} + i\bar{y} \frac{\sigma_x}{\sigma_y} &= e^{-i\alpha} \left[\frac{\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}}{\sqrt{\tilde{\sigma}_{11}\tilde{\sigma}_{22} - \tilde{\sigma}_{12}^2}} x + i \frac{\tilde{\sigma}_{11} + i\tilde{\sigma}_{12}}{\sqrt{\tilde{\sigma}_{11}\tilde{\sigma}_{22} - \tilde{\sigma}_{12}^2}} y \right] \end{aligned} \quad (4)$$

we obtain for the single particle kick

$$\begin{aligned} \Delta y' + i \Delta x' &= \frac{NK\sqrt{\pi}}{\sqrt{2(\tilde{\sigma}_{11} - \tilde{\sigma}_{22} + 2i\tilde{\sigma}_{12})}} \left\{ w \left[\frac{x_1 + ix_2}{\sqrt{2(\tilde{\sigma}_{11} - \tilde{\sigma}_{22} + 2i\tilde{\sigma}_{12})}} \right] - e^{-\frac{1}{2}(\tilde{\sigma}^{-1})_{ij}x_jx_j} w \left[\frac{(\tilde{\sigma}_{22} - i\tilde{\sigma}_{12})x_1 + i(\tilde{\sigma}_{11} + i\tilde{\sigma}_{12})x_2}{\sqrt{\tilde{\sigma}_{11}\tilde{\sigma}_{22} - \tilde{\sigma}_{12}^2} \sqrt{2(\tilde{\sigma}_{11} - \tilde{\sigma}_{22} + 2i\tilde{\sigma}_{12})}} \right] \right\} \end{aligned} \quad (5)$$

where we have used $x = x_1$ and $y = x_2$ for convenience.

3 The kick of the centroid of the passive beam

In order to calculate the centroids kick we will first prove that the kick angle only depends on the convolution of the distribution functions of the active and the passive beam.

To this end we note that the single particle kick is proportional to the electric field the active beam produces which in turn is given by its charge distribution convoluted with a Greens function G . We can write

$$\Delta \vec{x}'(\vec{y}) = \int_{\mathbf{R}^2} d^2x G(\vec{y} - \vec{x}) \psi_{active}(\vec{x}) . \quad (6)$$

Here we rewrite $\Delta \vec{x}'(\vec{y}) = (\Delta y' + i\Delta x')(x_1, x_2)$. The centroids' kick is then given by the convolution of $\Delta \vec{x}'(\vec{y})$ with the distribution function of the passive beam which amounts to averaging the kick each individual positron experiences over the transverse extension of the positron (passive) beam

$$\langle \Delta \vec{x}'(\vec{X}) \rangle = \int_{\mathbf{R}^2} d^2y \psi_{passive}(\vec{X} - \vec{y}) \Delta \vec{x}'(\vec{y}) . \quad (7)$$

$\vec{X}$ denotes the offset of the passive beam with respect to the center of the active beam. Inserting eq. 6, exchanging the order of integration and substituting $\vec{x} = \vec{y}$ and $\vec{y} = \vec{X} - \vec{x} + \vec{y}$ we obtain

$$\langle \Delta \vec{x}'(\vec{X}) \rangle = \int_{\mathbf{R}^2} d^2\hat{x} G(\vec{X} - \vec{\hat{x}}) \int_{\mathbf{R}^2} d^2\hat{y} \psi_{active}(\vec{\hat{y}}) \psi_{passive}(\vec{\hat{x}} - \vec{\hat{y}}) \quad (8)$$

where we have used that the Jacobian of the transformation is unity. This equation has the same structure as eq. 6 except that the distribution function of the active beam is substituted by the convolution of both distribution functions.

We have thus proven that the centroids kick only depends on the convolution of the active and the passive beam. This makes it impossible to extract information about individual beams alone. It is remarkable that this is true for any type of interaction, characterized by the Greens function $G(\vec{x})$. It is particularly true for the Coulomb interaction that governs the beam-beam deflection.

We are now in a position to write down the centroids' kick for the interaction of two gaussian bunches directly without doing extra algebra. We can do that because the convolution of two gaussian distributions yields a gaussian distribution with a correlation matrix that equals the sum of the individual correlation matrices. Furthermore, comparing eq. 6 and eq. 8 we note that $\vec{X}$ has taken the place of $\vec{y}$. Consequently we can use the substitutions in eq. 5

$$\begin{aligned}\tilde{\sigma}_{ij} &\longleftrightarrow \Sigma_{ij} = \tilde{\sigma}_{ij} + \sigma_{ij} \\ x_i &\longleftrightarrow X_i\end{aligned}\tag{9}$$

to finally obtain the centroids' kick

$$\begin{aligned}\langle \Delta y' + i\Delta x' \rangle &= \frac{NK\sqrt{\pi}}{\sqrt{2(\Sigma_{11}-\Sigma_{22}+2i\Sigma_{12})}} \left\{ w \left[\frac{X_1+iX_2}{\sqrt{2(\Sigma_{11}-\Sigma_{22}+2i\Sigma_{12})}} \right] \right. \\ &\quad \left. - e^{-\frac{1}{2}(\Sigma^{-1})_{ij}X_iX_j} w \left[\frac{(\Sigma_{22}-i\Sigma_{12})X_1+i(\Sigma_{11}+i\Sigma_{12})X_2}{\sqrt{\Sigma_{11}\Sigma_{22}-\Sigma_{12}^2}\sqrt{2(\Sigma_{11}-\Sigma_{22}+2i\Sigma_{12})}} \right] \right\}.\end{aligned}\tag{10}$$

The final expression depends only on the sum of the correlation matrices of both beams. This reflects the fact that it is impossible to extract individual beam sizes from beam-beam deflection scans.

However, this expression can be used to analyze the effect of the beam sizes on the deflection curves. Since it gives the horizontal and vertical deflection for arbitrary beam separations (X_1, X_2) , we can generalize the notion of a beam-beam scan to the scan along a path. By path we mean the map $(X_1(t), X_2(t))$ where t is the path length that varies in a suitable interval. The scans are then defined as $\Delta x'(X_1(t), X_2(t))$ and $\Delta y'(X_1(t), X_2(t))$. Simulations are easily performed by implementing eq. 7 on a computer and plotting the deflections $\Delta x'(t)$ and $\Delta y'(t)$ for various path's.

4 Applications and Signature curves

We will now pursue a different approach and expand eq. 10 about the round beam case. By this we mean that we expand the r.h.s. of eq. 10 in the quantities

$$\epsilon_1 = \frac{2\Sigma_{12}}{\Sigma_{11} + \Sigma_{22}}, \quad \epsilon_2 = \frac{\Sigma_{11} - \Sigma_{22}}{\Sigma_{11} + \Sigma_{22}}.\tag{11}$$

VERTICAL SCAN: X= 0.0 , -10.0 < Y < 10.0

Fig. 1: The in-plane scan, depicting the well-known beam-beam deflection curve (solid= $\text{Re}(F_0)$) for a vertical scan over ± 10 in units of Σ . The dot-dashed curve shows the signature curve for ellipticity ($\text{Re}(F_2)$). The signature curve for tilt (dashed= $\text{Re}(F_1)$) is flat, i.e. the signature of a tilted beam does not show up in the in-plane-scan.

Here we introduce the round beam approximation $\Sigma^2 = \frac{1}{2}(\Sigma_{11} + \Sigma_{22})$. From the definitions of the ϵ_i it is clear that ϵ_1 describes the total tilt of the beams and ϵ_2 the ellipticity. The expansion is done by using the asymptotic expansion for large arguments z of the complex error function [3]

$$w(z) = \frac{i}{\sqrt{\pi}} \left\{ \frac{1}{z} + \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdots (2m-1)}{2^m z^{2m+1}} \right\} . \quad (12)$$

Retaining only terms to first order we obtain

VERTICAL SCAN: X= 0.0 , -10.0 < Y < 10.0

Fig. 2: The out-of-plane curves for a vertical scan. Both the regular deflection curve (solid= $\text{Im}(F_0)$) and the signature curve for ellipticity (dot-dashed= $\text{Im}(F_2)$) are flat here. Only the signature curve for tilt (dashed= $\text{Im}(F_1)$) shows a characteristic asymmetry.

$$\begin{aligned}
 \langle \Delta y' + i\Delta x' \rangle &= NK \{ F_0(X_1, X_2; \Sigma) + \epsilon_1 F_1(X_1, X_2; \Sigma) + \epsilon_2 F_2(X_1, X_2; \Sigma) \} \\
 F_0(X_1, X_2; \Sigma) &= i \frac{1 - e^{-(X_1^2 + X_2^2)/2\Sigma^2}}{X_1 + iX_2} \\
 F_1(X_1, X_2; \Sigma) &= \frac{-2\Sigma^2 \left(1 - e^{-(X_1^2 + X_2^2)/2\Sigma^2} \right)}{(X_1 + iX_2)^3} \\
 &\quad - i e^{-(X_1^2 + X_2^2)/2\Sigma^2} \frac{2X_1 X_2 (X_1^2 + X_2^2 + 2\Sigma^2) + 2i\Sigma^2 (X_1^2 - X_2^2)}{2\Sigma^2 (X_1^2 + X_2^2)} \\
 F_2(X_1, X_2; \Sigma) &= \frac{2i\Sigma^2 \left(1 - e^{-(X_1^2 + X_2^2)/2\Sigma^2} \right)}{(X_1 + iX_2)^3} \\
 &\quad - i e^{-(X_1^2 + X_2^2)/2\Sigma^2} \frac{(X_1^2 - X_2^2)(X_1^2 + X_2^2 + 2\Sigma^2) - 4i\Sigma^2 X_1 X_2}{2\Sigma^2 (X_1^2 + X_2^2)}
 \end{aligned} \tag{13}$$

F_0 indeed describes the deflection curve of two round beams and F_1 and F_2 describe the deviations thereof due to tilted or elliptic beams, respectively. For

a particular scan along a path $(X_1(t), X_2(t))$ the functions $F_1(X_1(t), X_2(t); \Sigma)$ and $F_2(X_1(t), X_2(t); \Sigma)$ are the signature of tilt or ellipticity. Therefore, if we look at the differences between the round beam fit to a set of data and the real data we see whether the beams are tilted or elliptic.

Fig. 1 and Fig. 2 show the round beam deflection curve (solid), F_1 (dashed) and F_2 (dotdashed) for a vertical scan ($X_1 = 0, X_2 = t$). The horizontal axis is normalized to Σ and the vertical axes to NK . The round beam curve exhibits the well known form in the y-direction (in-plane) and remains flat in the x-direction (out-of-plane). Clearly the signature of tilt ($\Sigma_{12} \neq 0$) is an asymmetry in the out-of-plane curve and the signature of the ellipticity is a distortion of the maxima of the in-plane curve.

These signatures, however will be hard to detect. For an elliptic beam with an aspect ratio of 2:1 the deviation of the full beam-beam deflection curve (in-plane) to the round beam approximation is only on the 10 % level of the maximum deflection angle. The signature of tilt in the out-of-plane curve is hampered by an offset of the beam, which results in a bell-shaped out-of-plane curve superimposed on the signature of the tilt. Here, however, one can discriminate the effects by their symmetry. The offset shows up as a symmetric deflection scan and the tilt as an asymmetry in this scan.

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6 References

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