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November 1978

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Available from
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U. S. Department of Commerce
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Springfield, VA 22161

Price: Printed Copy, \$ 4.00 Domestic; \$ 8.00 Foreign
Microfiche, \$ 3.00 Domestic; \$ 4.50 Foreign

A Minicourse in Lie Perturbative Methods*

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ABSTRACT

We give a skeletal presentation of the mechanics of Lie perturbative methods. The discussion is intended to enable the reader to begin to use the methods himself. The technique is illustrated by a specific example, in which we derive the ponderomotive Hamiltonian for a particle moving in a curl-free electric field.

* This research was supported by the U. S. Department of Energy

Given any function G of the phase variables $\underline{z} \equiv (\underline{q}, \underline{p})$, we define the operator $\check{G}$ by

$$\check{G} \equiv \{G, \} \equiv \partial G / \partial \underline{q} \cdot \partial / \partial \underline{p} - \partial G / \partial \underline{p} \cdot \partial / \partial \underline{q}. \quad (1)$$

One can show¹ that the operator $T_G \equiv e^{\check{G}} \equiv \sum_{n=0}^{\infty} (1/n!) \check{G}^n$ induces a canonical transformation (C.T.).

Now if we have a Hamiltonian of the form

$$H^0(\underline{z}) \equiv H(\underline{z}) \equiv \sum_{m=0}^{\infty} \epsilon^m H_m(\underline{z}) \quad (2)$$

where $H_0 = H_0(\underline{p})$ (so that the zero-order problem is exactly soluble). we can induce C.T.s of the type T_G , to remove the $\underline{q}$ -dependent part of H , to successively higher orders in the perturbation parameter ϵ , as follows. We write

$$\begin{aligned} H^1 &\equiv e^{\epsilon \check{G}_1} H_0 = \left(1 + \epsilon \check{G}_1 + \frac{1}{2} \epsilon^2 \check{G}_1^2 + \dots\right) (H_0 + \epsilon H_1 + \epsilon^2 H_2 + \dots) \\ &= H_0 + \epsilon (\check{G}_1 H_0 + H_1) + \epsilon^2 \left(\frac{1}{2} \check{G}_1^2 H_0 + \check{G}_1 H_1 + H_2\right) + O(\epsilon^3). \end{aligned} \quad (3)$$

Writing $H^1 = \sum_{m=0}^{\infty} \epsilon^m H_m^1$, we read off the H_m^1 from Eq.(3), to as high order as desired. Up to $O(\epsilon^2)$, one has

$$H_0^1 \equiv H_0, \quad H_1^1 \equiv \check{G}_1 H_0 + H_1 = -\check{H}_0 G_1 + H_1, \quad (4)$$

$$H_2^1 \equiv \frac{1}{2} \check{G}_1^2 H_0 + \check{G}_1 H_1 + H_2.$$

Here we have used the property $\check{G}F = -\check{F}G$, which follows from (1), in obtaining the second form for H_1^1 .

We now remove the q dependence to $O(\epsilon)$, and thereby determine the generator G_1 , by stipulating that

$$H_1^1 \equiv -\check{H}_0 G_1 + H_1 = \bar{H}_1, \quad (5)$$

where $\bar{H}_1 = \bar{H}_1(\underline{p})$ means an average over the zero-order orbit of $H_1(q, \underline{p})$. With this differential equation defining G_1 , we have subtracted all the secularity out of G_1 and the C.T. it induces (i.e. G_1 oscillates about zero), in addition to removing the q -dependence in H up to (ϵ^2) . To see the former property, we note from Eq. (1) that $-\check{H}_0 = (d/dt)_0$, the time derivative along the unperturbed trajectory, and so

$$G_1 = \int_0^t (0\text{-traj.}) dt' (\bar{H}_1 - H_1), \quad (6)$$

where by definition of $\bar{H}_1$, the right-hand side is nonsecular.

Now we proceed analogously to second order.

$$\begin{aligned} H^2 &\equiv e^{\epsilon^2 \check{G}_2} H^1 = (1 + \epsilon^2 \check{G}_2 + \dots) (H_0^1 + \epsilon H_1^1 + \epsilon^2 H_2^1 + \dots) \\ &\equiv K^1(\underline{p}) + \epsilon^2 H_2^2 + O(\epsilon^3), \end{aligned} \quad (7)$$

where $K^1(\underline{p}) \equiv H_0^1 + \epsilon H_1^1$ (the q -independent part of H^1), and

$$H_2^2 \equiv -\check{H}_0 G_2 + H_2^1 = -\check{H}_0 G_2 + \frac{1}{2} \check{G}_1 (\bar{H}_1 + H_1) + \check{H}_2. \quad (8)$$

The second form for H_2^2 here comes from the third of Eqs.(4) and Eq. (5). Analogous to (5), we determine G_2 , and remove the $\underline{q}$ -dependence of H^2 to $O(\epsilon^2)$, by requiring

$$H_2^2 \equiv -\check{H}_0 G_2 + H_2^1 = \overline{H_2^1}. \quad (9)$$

Thus

$$G_2 = \int_{(0-\text{traj})}^t (\overline{H_2^1} - H_2^1) \quad (10)$$

The extension of this process to arbitrary order in ϵ should be clear.

Time Dependent H

In the event that $H = H(\underline{z}, t)$, we reduce the problem to the situation already treated by extending our phase space to have one additional degree of freedom, with new coordinate t , conjugate momentum E , and Hamiltonian

$$H(\underline{q}, t; \underline{p}, E) \equiv H(\underline{q}, \underline{p}, t) + E. \quad (11)$$

The system develops in a new time variable θ (so $\dot{f} \equiv df/d\theta$). The equations of motion are then

$$\dot{t} = \partial H / \partial E = 1 \quad (\text{hence } t = \theta), \quad (12a)$$

$$\dot{E} = -\partial H / \partial t = -\partial H / \partial t, \quad \text{and} \quad (12b)$$

$$\dot{\underline{q}} = \partial H / \partial \underline{p}, \quad \dot{\underline{p}} = -\partial H / \partial \underline{q}, \quad (12c)$$

as before. [We may think of the term E in H as a reservoir, feeding energy in and out of the term H representing our original system, at just the rate needed to give H its time dependence. Thus Eq. (12b) is to be expected.] We now treat this θ -independent Hamiltonian, with exactly soluble zero order part $H_0(\underline{p}, E) \equiv H_0(\underline{p}) + E$, just as described above.

Example

We consider the motion of a particle, moving freely except for the perturbing influence of any number of electrostatic plane waves, each having wave vector $\underline{k}_\ell$ and frequency ω_ℓ . The Hamiltonian is then

$$H(\underline{z}, t) = H_0(\underline{p}) + \epsilon H_1(\underline{q}, t), \quad (13)$$

where $H_0(\underline{p}) \equiv p^2/(2m)$, $H_1(\underline{q}, t) \equiv \sum_{\ell} V_{\ell} \exp i(\underline{k}_{\ell} \cdot \underline{q} - \omega_{\ell} t)$, and $V_{-\ell} = V_{\ell}^*$, and $(\underline{k}_{-\ell}, \omega_{-\ell}) = -(\underline{k}_{\ell}, \omega_{\ell})$. The unperturbed trajectories of $H = H + E$ are given by

$$\underline{p}(\theta) = \underline{p} = \text{constant}, \quad E = \text{constant}, \quad (14)$$

$$\underline{q}(\theta) = \underline{q}(0) + \theta \underline{v}, \quad t = \theta,$$

where $\underline{v} \equiv \partial H / \partial \underline{p} = \underline{p}/m$. We thus have $\bar{H}_1 = \bar{H}_1 = 0$, and so Eq. (5) reads

$$\begin{aligned}
0 &= H_1^1 \equiv H_1 + \{G_1, H_0\} \\
&= H_1 + \partial G_1 / \partial t + (\partial G_1 / \partial \underline{q}) \cdot (\partial G_1 / \partial \underline{p}) - (\partial G_1 / \partial \underline{p}) \cdot (\partial G_1 / \partial \underline{q}) \\
&\equiv H_1 + dG_1 / d\theta)_0.
\end{aligned} \tag{15}$$

Thus

$$\begin{aligned}
G_1(\underline{z}, t) &= - \int_{(0 \text{ traj.})}^{\theta} d\theta' \sum_{\ell} V_{\ell} \exp i[\underline{k}_{\ell} \cdot (\underline{q}(0) + \theta' \underline{v}) - \omega_{\ell} \theta'] \\
&= -i \sum_{\ell} [V_{\ell} \exp i(\underline{k}_{\ell} \cdot \underline{q} - \omega_{\ell} t)] (\omega_{\ell} - \underline{k}_{\ell} \cdot \underline{v})^{-1}
\end{aligned} \tag{16}$$

From Eq. (8), we have, finally

$$H_2^2 = \overline{H_2^1} = \frac{1}{2} \overline{\tilde{G}_1 H_1} \equiv \frac{1}{2} \overline{\{G_1, H_1\}}. \tag{17}$$

Eq. (17) is the standard expression for the ponderomotive Hamiltonian.

Using (16) in (17), one readily obtains the more explicit form

$$H_2^2 = (2m)^{-1} \sum_{\ell} |V_{\ell} \underline{k}_{\ell}|^2 (\omega_{\ell} - \underline{k}_{\ell} \cdot \underline{v})^{-2}. \tag{18}$$

We now see at a calculational level how to obtain this result using Lie methods, as well as how to proceed in other problems. The interested reader is referred to the bibliography for a fuller development of the mathematical theory,^{1,2} as well as for applications to problems in plasma physics, such as magnetic field self-generation³, expressions for the

ponderomotive force in magnetized plasmas,³ mode coupling in inhomogeneous magnetized plasmas,⁴ and particle motion and guiding center theory.^{5,6}

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This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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