MASTER**Is Yang-Mills Equation a Totally Integrable System?***

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I. Introduction

Certainly I shall not be able to answer the question raised in the title of this talk. What I hope to convey to you is why even we can ask such a bold question.

The path ordered phase factor $\phi_{ab} \equiv P \exp \int_a^b dx_\mu A_\mu(x)$ along a loop (or part of a loop) has many desirable features. It is the minimal necessary object¹ describing a gauge theory, has simple gauge transformation properties² and is the spin-like quantity for lattice gauge theory.³ Thus it is natural, may even be essential, that we formulate gauge theory in the loop space. It was realized in 1979 that Yang-Mills equations give chiral-like equations in the loop space.^{4,5} This beautiful realization immediately raised the hope that the loop-space chiral equations may also be a total integrable system, just like the ordinary-space chiral equations,⁶ and thus lead to the full solutions of the Yang-Mills equations. However, here we want to discuss some intricate properties of the integrability conditions of the loop space chiral equations, which do not have their correspondence in the ordinary chiral equations.

In Section I I shall derive the loop-space chiral equation and demonstrate that they give the Bianchi identity and the Yang-Mills equation. In Section II the conservation laws, linear system and the integrability conditions for the two-dimensional chiral equation are discussed in order to provide some motivation and guidance. In Section III parallel development, similar to that in Section II, is attempted for the loop-space chiral equation in two dimensions. Here the subtlety involved in the integrability is elaborated. It is pointed out that similar complications arise also for the three dimensional case. In the four dimensional case little progress is made beyond the chiral equations.

II. Loop-space Formulation of Gauge Theory

Here we shall derive the loop space chiral equation and its implication on gauge theories. Consider a loop specified by $\ell = 02s''s'l0$, as shown in Fig. (1) and an infinitesimally distorted new loop $\ell' = 02s''s''s'l0$, also shown in Fig. (1)

Fig. (1)

The phase factor of path ordered product of gauge potential

$$\begin{aligned}\psi(\ell) &\equiv \phi_{02s''s'l0} = P \text{Exp} \left(i \oint_{\ell} A_{\mu} dx_{\mu} \right) \\ &= \prod_{(i)} [1 + A_{\mu}(x^{(i)}) dx_{\mu}^{(i)}] \end{aligned} \quad (2.1a)$$

where the $\prod_{(i)}$ denotes the product of infinitesimal segments along the closed path ℓ . Similarly along the path ℓ' ,

$$\psi(\ell') \equiv \phi_{02s''s''s'l0} \quad (2.1b)$$

Then the difference of the two phase factors is

$$\psi(\ell') - \psi(\ell) = \phi_{02s} \phi_{ss''s''s's} \phi_{s'l0} \quad (2.2a)$$

Here $\phi_{ss''s''s's}$ is just the phase factor along the infinitesimal closed path $ss''s''s's$. Careful analysis shows that⁷ is just the curvature flux going through the area, i.e.,

$$\phi_{ss''s''s's} = f_{\mu\nu}[x(s)] \delta x^{\mu} dx^{\nu} \quad (2.2b)$$

$$\text{where } f_{\mu\nu}(x(s)) = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} - [A_{\mu}, A_{\nu}] \quad (2.3)$$

Now we take the limit $s \rightarrow s''$, and define the functional differentiation to be

$$\frac{\delta\psi(l)}{\delta x^\mu(s)} \equiv \frac{\psi(l') - \psi(l)}{\delta x^\mu(s) ds} = \phi_{02s} f_{\mu\nu}[x(s)] \dot{x}^\nu(s) \phi_{s10} , \quad (2.4)$$

and obviously

$$\psi^{-1}(l) \frac{\delta\psi(l)}{\delta x^\mu(s)} = \phi_{s10}^{-1} f_{\mu\nu}[x(s)] \dot{x}^\nu(s) \phi_{s10} \equiv \mathcal{F}_\mu(s, l) . \quad (2.5)$$

Now we want to functionally differentiate Eq. (2.5) at another point s' . Besides the differentiation on the phase factor ϕ_{s10} , which is non-zero only if $s' < s$, we have differentiations at the end points of the phase factor and on the ordinary function. So in addition to Eq. (2.4), we need the following equations, to obtain the functional differentiation of Eq. (1.5),

$$\frac{\delta f[x(s)]}{\delta x_\mu(s')} = \frac{\partial f[x(s)]}{\partial x_\mu(s)} \delta(s' - s) , \quad (2.6a)$$

$$\frac{\delta\phi_{01s}}{\delta x_\mu(s')} = \phi_{01s} A_\mu[x(s)] \delta(s' - s) , \quad (2.6b)$$

$$\frac{\delta \dot{x}_\mu(s)}{\delta x_\nu(s')} = \delta_{\mu\nu} \frac{d}{ds'} \delta(s' - s) , \quad (2.6c)$$

to obtain the functional differentiation of Eq. (2.5)

$$\begin{aligned} \frac{\delta \mathcal{F}_\nu(s', l)}{\delta x_\mu(s)} &= [\mathcal{F}_\mu(s, l), \mathcal{F}_\nu(s', l)] \theta(s' - s) \\ &+ \delta(s - s') \phi_{s'10}^{-1} \mathcal{D}_\mu f_{\nu\lambda}[x(s')] \dot{x}_\lambda(s') \phi_{s'10} \\ &+ \frac{d\delta(s' - s)}{ds'} \phi_{s'10}^{-1} f_{\mu\nu}[x(s')] \phi_{s'10} , \end{aligned} \quad (2.7)$$

where $\mathcal{D}_\mu$ denotes the covariant derivative, $\mathcal{D}_\mu = \partial_\mu + [A_\mu(x), \cdot]$. In the limit

$$\lim_{s' \rightarrow s} \frac{\delta \mathcal{F}_\mu(s', l)}{\delta x_\mu(s)} \equiv \frac{\delta \mathcal{F}_\mu(s, l)}{\delta x_\mu(s)} = 0 , \quad (2.8a)$$

from the right hand side of Eq. (2.7), we obtain

$$\mathcal{D}_\mu f_{\mu\lambda}[x(s)] \dot{x}_\lambda(s) = 0, \quad (2.8b)$$

which is just the projected Yang-Mills equation.

Now we shall derive that the loop-space curvatureless definition of Eq. (2.5) will give the projected Bianchi identity. From Eq. (2.7),

$$\begin{aligned} \frac{\delta \mathcal{F}_\mu(s, l)}{\delta x_\nu(s')} &= \frac{\delta \mathcal{F}_\nu(s', l)}{\delta x_\mu(s)} \\ &= [\mathcal{F}_\nu(s', l), \mathcal{F}_\mu(s, l)] \theta(s - s') - [\mathcal{F}_\mu(s, l), \mathcal{F}_\nu(s', l)] \theta(s' - s), \\ &\quad + \delta(s - s') \phi_{s10} \{ \mathcal{D}_\nu f_{\mu\lambda}[x(s)] \dot{x}_\lambda(s) - \mathcal{D}_\mu f_{\nu\lambda}[x(s)] \dot{x}_\lambda(s) \} \phi_{s01} \\ &\quad + \frac{d\delta(s - s')}{ds} \phi_{01s} f_{\mu\nu}[x(s)] \phi_{s01} - \frac{d\delta(s' - s)}{ds'} \phi_{01s'} f_{\nu\mu}[x(s')] \phi_{s01} \end{aligned} \quad (2.9)$$

Using the relation,

$$[\phi(s) - \phi(s')] \frac{d\delta(s - s')}{ds} = \frac{d\phi}{ds} \delta(s - s'), \quad (2.10)$$

the $d\delta(s - s')/ds$ term in Eq. (2.9) becomes

$$\begin{aligned} \frac{d\delta(s - s')}{ds} \{ \phi_{01s} f_{\mu\nu}[x(s)] \phi_{s01} \} &= -\delta(s - s') \frac{dx_\rho(s)}{ds} \frac{\partial}{\partial x_\rho(s)} \{ \phi_{01s} f_{\mu\nu}[x(s)] \phi_{s01} \} \\ &= -\delta(s - s') \dot{x}_\rho(s) \phi_{01s} \mathcal{D}_\rho f_{\mu\nu}[x(s)] \phi_{s01}. \end{aligned} \quad (2.11)$$

Substitute Eq. (2.11) into Eq. (2.9), we find that the loop-space curvatureless equation

$$\frac{\delta \mathcal{F}_\mu(s, l)}{\delta x_\nu(s')} - \frac{\delta \mathcal{F}_\nu(s', l)}{\delta x_\mu(s)} = [\mathcal{F}_\nu(s', l), \mathcal{F}_\mu(s, l)] \theta(s - s') - [\mathcal{F}_\mu(s, l), \mathcal{F}_\nu(s', l)] \theta(s' - s) \quad (2.12)$$

gives the projected Bianchi identity

$$\mathcal{D}_\nu f_{\mu\lambda}[x(s)] \dot{x}_\lambda(s) + \mathcal{D}_\mu f_{\lambda\nu}[x(s)] \dot{x}_\lambda(s) + \mathcal{D}_\lambda f_{\nu\mu}[x(s)] \dot{x}_\lambda(s) = 0 \quad (2.13)$$

The geometric meaning of the loop space equations (2.12) is that the loop phase factor arrived from an initial loop to a given final loop is independent of the different volumes swapped out by the intermediate loops if the Bianchi identity is satisfied.

III. Two Dimensional Chiral Equation

To motivate the further development of the loop-space chiral equation, we shall discuss some of the well known properties that make the two dimensional chiral a totally integrable system.

The chiral fields are simply described by the following equations,

$$A_\mu(x) \equiv g^{-1}(x) \partial_\mu g(x) \quad , \quad \text{curvatureless,} \quad (3.1)$$

$$\partial_\mu A_\mu(x) = 0 \quad , \quad \text{continuity-like,} \quad (3.2)$$

which follows from a Lagrangian density $\mathcal{L} = \text{Tr} \partial_\mu g^{-1}(x) \partial_\mu g(x)$. Following the procedure of Brezin, et al.,⁸ we consider $A_\mu(x)$ being the first conserved current.

$$V_\mu^{(1)}(x) \equiv A_\mu(x) = \epsilon_{\mu\alpha} \partial_\alpha \chi^{(1)}(x). \quad (3.3)$$

We shall see that $\chi^{(1)}(x)$ is needed to construct higher current. So we re-express Eq. (3.3) as

$$\partial_\mu \chi^{(1)}(x) = -\epsilon_{\mu\beta} A_\beta(x). \quad (3.3')$$

Then the integrability of $\chi^{(1)}(x)$ is

$$\partial_\nu \partial_\mu \chi^{(1)}(x) - \partial_\mu \partial_\nu \chi^{(1)}(x) = 0. \quad (3.4)$$

From Eq. (3.3), we have

$$-\epsilon_{\mu\beta} \partial_\nu A_\beta(x) + \epsilon_{\nu\alpha} \partial_\mu A_\alpha(x) = 0, \quad (3.5)$$

$$\text{for } \mu = \nu = 1, \quad -\partial_1 A_2(x) + \partial_1 A_2(x) = 0, \quad (3.5a)$$

$$\text{for } \mu = 1, \nu = 2, \quad -\partial_2 A_2 - \partial_1 A_1 = 0. \quad (3.5b)$$

Equation (3.5a) is automatically true and Eq. (3.5b) is just Eq. (3.2).

Therefore, $\partial_\mu A_\mu = 0$ is the necessary and sufficient condition for the integrability of $\chi^{(1)}(x)$.

Now we construct the second current,

$$v_\mu^{(2)} \equiv \mathcal{D}_\mu \chi^{(1)}, \quad \text{where } \mathcal{D}_\mu \equiv \partial_\mu + A_\mu(x). \quad (3.6)$$

Using Eqs. (3.1) and (3.2), we can easily show $\partial_\mu v_\mu^{(2)} = 0$. Similarly we can construct higher currents

$$v_\mu^{(n)} = \epsilon_{\mu\nu} \partial_\nu \chi^{(n)} = \mathcal{D}_\mu \chi^{(n-1)}. \quad (3.7)$$

Again Eqs. (3.1) and (3.2) give $\partial_\mu v_\mu^{(n)} = 0$. We shall call these $\chi^{(n)}$'s the "generating" function.

Next we shall demonstrate the connection of these infinite conserved currents with the inverse-scattering (or the linear) equations.⁶ Multiplying Eq. (3.7) by L^n and sum over all n ,

$$\sum_{n=1}^{\infty} \epsilon_{\mu\nu} \partial_\nu L^n \chi^{(n)} = L \sum_{n=1}^{\infty} \mathcal{D}_\mu L^{n-1} \chi^{(n-1)}, \quad (3.8)$$

where L is an arbitrary constant, Eq. (3.8) can be rewritten as, due to $\chi^{(0)} = 1$,

$$\epsilon_{\mu\nu} \partial_\nu \sum_{n=0}^{\infty} L^n \chi^{(n)}(x) = L \mathcal{D}_\mu \sum_{n=0}^{\infty} L^n \chi^{(n)}(x).$$

$$\text{Defining } \phi(x, L) \equiv \sum_{n=0}^{\infty} L^n \chi^{(n)}(x) \quad (3.9)$$

$$\text{we obtain } \epsilon_{\mu\nu} \partial_\nu \phi = L \mathcal{D}_\mu \phi, \text{ or } [\partial_\mu - A_\mu - L^{-1} \epsilon_{\mu\nu} \partial_\nu] \phi(x, L) = 0, \quad (3.10)$$

which is the inverse-scattering equations⁹ for the chiral equations.

Now we discuss the integrability conditions of ϕ . From Eq. (3.10) it is easy to obtain

$$\partial_\mu \phi(x, L) = -\frac{1}{1+L^2} [A_\mu(x) + L \epsilon_{\mu\alpha} A_\alpha] \phi(x, L). \quad (3.11)$$

Differentiating Eq. (3.11) once more, and using Eq. (3.11), it follows

$$\begin{aligned} \partial_\nu \partial_\mu \phi(x, L) = & -\frac{1}{1+L^2} [\partial_\nu A_\mu(x) + L \epsilon_{\mu\alpha} \partial_\nu A_\alpha(x)] \phi(x, L) + \\ & \frac{1}{(1+L^2)^2} [A_\mu(x) + L \epsilon_{\mu\alpha} A_\alpha(x)] [A_\nu(x) + L \epsilon_{\nu\beta} A_\beta(x)] \phi(x, L). \end{aligned} \quad (3.12)$$

The integrability conditions of $\phi(x, L)$, $\partial_\mu \partial_\nu \phi(x, L) - \partial_\nu \partial_\mu \phi(x, L) = 0$, for arbitrary L , gives the L^0 term, $\partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu] = 0$; the L^1 term, $-\epsilon_{\mu\alpha} \partial_\nu A_\alpha + \epsilon_{\nu\beta} \partial_\mu A_\beta + [\epsilon_{\mu\alpha} A_\alpha, A_\nu] + [A_\mu, \epsilon_{\nu\beta} A_\beta] = 0$; the L^2 term, $-\partial_\nu A_\mu + \partial_\mu A_\nu + [\epsilon_{\mu\alpha} A_\alpha, \epsilon_{\nu\beta} A_\beta] = 0$; the L^3 term, $\epsilon_{\mu\alpha} \partial_\nu A_\alpha - \epsilon_{\nu\beta} \partial_\mu A_\beta = 0$. They imply simply $\partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu] = 0$, $\partial_\mu A_\mu = 0$. Therefore we see that the original equations of motion provide the integrability of $\phi(x, L)$ and also provide the conditions for constructing infinite number of conserved currents.

These properties, existence of infinite conservation law and the existence of the corresponding linear systems, are the characteristics that make the system totally integrable and eventually lead to the construction of s matrix.¹⁰

IV. Parallel Development for the Loop-Space Chiral Equations - Subtlety

We see from the previous discussion that the loop-space equations Eqs. (2.5), (2.12), (2.8a) are very similar to the chiral equations Eqs. (3.1) and (3.2). We now like to ask the question if parallel development¹¹ can be made for the loop-space chiral equation.

Non-local Conservation Laws: Equation (2.8a) is like a continuity equation so we try to follow the standard procedure and identify the first current as follows, (here we specify in two dimension though the conclusion is general),

$$v_\mu^{(1)}(l, s) \equiv \mathcal{F}_\mu(l, s) = \epsilon_{\mu\nu} \frac{\delta \chi^{(1)}}{\delta x^\nu(s)}. \quad (4.1a)$$

This satisfies Eq. (2.8a), but the question is whether Eq. (2.8a) provides the sufficient conditions for the integration of $\chi^{(1)}$ from Eq. (4.1a). We shall discuss separately the following two possible cases.

Case (1): $\chi^{(1)}$ is a function of the loop above, i.e., Eq. (4.1a) reads

$$\mathcal{F}_\mu(\ell, s) = \epsilon_{\mu\nu} \frac{\delta\chi^{(1)}(\ell)}{\delta x^\nu(s)} \quad (4.1b)$$

Just as the finite dimensional case, the integrability conditions of $\chi^{(1)}(\ell)$ is

$$\frac{\delta\chi^{(1)}(\ell)}{\delta x^\nu(s')\delta x^\mu(s)} - \frac{\delta\chi^{(1)}(\ell)}{\delta x^\mu(s)\delta x^\nu(s')} = 0. \quad (4.2)$$

From $\frac{\delta\chi^{(1)}(\ell)}{\delta x^\nu(s)} = -\epsilon_{\nu\mu} \mathcal{F}_\mu(\ell)$, Eq. (4.2) gives

$$\epsilon_{\mu\alpha} \frac{\delta\mathcal{F}_\alpha(\ell, s)}{\delta x^\nu(s')} = \epsilon_{\nu\alpha} \frac{\delta\mathcal{F}_\alpha(\ell, s')}{\delta x^\mu(s)}, \quad (4.3)$$

$$\text{for } \mu = \nu = 1, \quad \frac{\delta\mathcal{F}_2(\ell, s)}{\delta x^1(s')} = \frac{\delta\mathcal{F}_2(\ell, s')}{\delta x^1(s)}, \quad (4.4a)$$

which is false, unless $s' \rightarrow s$; for $\mu = 1, \nu = 2$,

$$\frac{\delta\mathcal{F}_2(\ell, s)}{\delta x^2(s')} = -\frac{\delta\mathcal{F}_1(\ell, s')}{\delta x^1(s)}, \quad (4.4b)$$

which become Eq. (2.8a) only in the limit $s' \rightarrow s$. Therefore, we see that higher conserved currents cannot be constructed by this procedure.

Case (2): $\chi^{(1)}$ is not only a function of the loop but also a parameter s .

Now Eq. (4.1a) becomes

$$\mathcal{F}_\mu(\ell, s) = \lim_{s' \rightarrow s} \epsilon_{\mu\nu} \frac{\delta\chi^{(1)}(\ell, s)}{\delta x^\nu(s')} = \epsilon_{\mu\nu} \frac{\delta\chi^{(1)}(\ell, s)}{\delta x^\nu(s)}. \quad (4.1')$$

Thus the integrability condition of Eq. (4.1a) becomes

$$\frac{\delta\chi^{(1)}(\ell, s)}{\delta x^\nu(s)\delta x^\mu(s)} - \frac{\delta\chi^{(1)}(\ell, s)}{\delta x^\mu(s)\delta x^\nu(s)} = 0. \quad (4.2')$$

Notice that all parameters coincide at a point s . Then from $\frac{\delta\chi^{(1)}(\ell, s)}{\delta x^\nu(s)} = \epsilon_{\nu\mu} \mathcal{F}_\mu(\ell, s)$ the integrability condition becomes Eq. (3) with $s' \rightarrow s$. Thus the equations of motion Eq. (2.8a) does provide integrability of $\chi^{(1)}(\ell, s)$ from (4.1b). However, the peculiar situation here is that Eq. (4.1b) constraints $\chi(\ell, s)$, only when the parameter s' of $\delta x_\mu(s')$ coincide with s of $\chi(\ell, s)$, thus does not constraint $\chi(\ell, s)$ enough and there are infinite many $\chi(\ell, s)$'s that can satisfy Eq. (4.1').

This is a manifestation that additional informations are needed in order to integrate the system from one point of the loop to the other uniquely.

Since $\chi(l, s)$ can be constructed, now we can follow the standard iterative procedure to construct the n -th current from the $(n-1)$ th current,

$$v_{\mu}^{(n)}(l, s) = \lim_{s' \rightarrow s} \left[\frac{\delta}{\delta x_{\mu}(s')} + \mathcal{F}_{\mu}(l, s) \right] \chi^{(n-1)}(l, s), \quad (4.5)$$

where $\chi^{(n-1)}$ satisfies $v_{\mu}^{(n-1)}(l, s) = e_{\mu\nu} \delta \chi^{(n-1)}(l, s) / \delta x^{\nu}(s)$.

Using the original equations of motion Eqs. (2.8a) and (2.12) one can easily show $v_{\mu}^{(n)}(l, s)$'s are conserved, i.e.,

$$\delta v_{\mu}^{(n)}(l, s) / \delta x_{\mu}(s) = 0. \quad (4.6)$$

Notice here the arbitrariness in $\chi^{(n-1)}(l, s)$ reflects directly in the next current $v_{\mu}^{(n)}(l, s)$.

An explicit example. Actually both cases can be demonstrated explicitly from the solution of the Yang-Mills equation in 2-dimensions. In R^2 we have the following solution for the gauge potentials of the Yang-Mills equations

$$b_1 = -x_2 c, \quad b_2 = 0, \quad \text{and then } f_{12} = c, \quad (4.7)$$

where c is an element of g . Of course we assume $c \neq 0$. It is easily seen that Eq. (4.7) is the general solution of the Yang-Mills equations in R^2 .

From Eq. (2.5) we have

$$\mathcal{F}_1(l, s) = c \frac{dx_2(s)}{ds}, \quad \mathcal{F}_2(l, s) = -c \frac{dx_1(s)}{ds}. \quad (4.8)$$

The analogy of Eq. (4.1a) is

$$\frac{\delta \chi}{\delta x^1(s)} = \mathcal{F}_2(l, s) = -c \frac{dx_1(s)}{ds}, \quad \text{and} \quad \frac{\delta \chi}{\delta x^2(s)} = -\mathcal{F}_1(l, s) = -c \frac{dx_2(s)}{ds}. \quad (4.9)$$

In case (1), $\chi = \chi(l)$ a functional of loop alone, clearly from Eq. (4.9),

$$\frac{\delta \chi(l)}{\delta x^i(s') \delta x^i(s)} = -c \delta(s'-s) \neq \frac{\delta \chi(l)}{\delta x^i(s) \delta x^i(s')} = -c \delta(s-s'), \quad \text{where } i = 1, \text{ or } 2, \quad (4.10)$$

since $\delta(s' - s)$ is odd under interchange of s and s' . But in case (2), if we allow $\chi' = \chi(l, s)$ a functional of both the loop and a parameter, then Eq. (4.9) have infinite number of solutions. One of them has been constructed by Polyakov,¹²

$$\chi(l, s) = -\frac{c}{\pi} \int dt \, du \frac{(s-t)(s-u)}{[(s-t)^2 + (s-u)^2]^2} x_\mu(t) x_\mu(u) , \quad (4.11)$$

and

$$\frac{\delta \chi(l, s)}{\delta x_\mu(t)} = -\frac{2c}{\pi} \int du \frac{(s-t)(s-u)}{[(s-t)^2 + (s-u)^2]^2} x_\mu(u) . \quad (4.12)$$

Using the fact $\delta(x) = \lim_{\epsilon \rightarrow 0} [-2\epsilon x / (x^2 + \epsilon^2)^2]$, so $\lim_{s \rightarrow t} (s-t)(s-u) / [(s-t)^2 + (s-u)^2]^2 = -(\pi/2) \delta(s-u)$, we obtain

$$\lim_{t \rightarrow s} \frac{\delta \chi(l, s)}{\delta x_\mu(t)} = \frac{\delta \chi(l, s)}{\delta x_\mu(s)} = -c \frac{dx_\mu(s)}{ds} ,$$

which is precisely Eq. (4.9). However, this is only one of the infinite ways of constructing the solution. We can replace $(s-t)(s-u) / [(s-t)^2 + (s-u)^2]^2$ by any function, e.g., $\{[(s-t)/(s-u)^3] \text{Exp}[-(s-t)^2/(s-u)^2] + (t \leftrightarrow u)\}$, which is $\delta(s-u)$ in the limit $t \rightarrow s$. This is a consequence of the fact that Eqs. (4.1), do not constraint $\chi(l, s)$ when s' of $\delta x^\mu(s')$ and s of $\chi(l, s)$ are different.

Linear System: Now we discuss the corresponding development of a linear system for the loop-space chiral equations (2.5), (2.12) and (2.8a).

Case (1): In this case the higher order conservation laws do not exist, so the inverse scattering equations cannot be constructed from them using the standard method.⁸ However, to demonstrate a point, following the spirit of Ref. 5, we construct one in analogue to Eq. (3.10),

$$\left(\frac{\delta}{\delta x^\mu(s)} + \mathcal{F}_\mu(l, s) - \gamma \epsilon_{\mu\nu} \frac{\delta}{\delta x^\nu(s)} \right) \phi(l, \gamma) = 0 . \quad (4.13)$$

Note that $\phi(l, \gamma)$ is a functional of loop and the parameter γ . What are the conditions for the integration of $\phi(l, \gamma)$? Eq. (4.13) can be rewritten as

$$\frac{\delta}{\delta x_\mu(s)} \phi(l, \gamma) = -\frac{1}{1+\gamma^2} [\mathcal{F}_\mu(l, s) + \gamma \epsilon_{\mu\alpha} \mathcal{F}_\alpha(l, s)] \phi(l, \gamma) \quad (4.14)$$

Requiring, for arbitrary γ ,

$$\frac{\delta^2 \phi(l, \gamma)}{\delta x^\mu(s) \delta x^\nu(s')} - \frac{\delta^2 \phi(l, \gamma)}{\delta x^\nu(s') \delta x^\mu(s)} = 0, \quad (4.15)$$

one obtains the following conditions for arbitrary s , and s'

$$\frac{1}{\gamma^3} \text{ term: } \epsilon_{\mu\alpha} \frac{\delta \mathcal{F}_\alpha(l, s)}{\delta x^\nu(s')} - \epsilon_{\nu\beta} \frac{\delta \mathcal{F}_\beta(l, s')}{\delta x^\mu(s)} = 0 \quad (4.16)$$

$$\mu=1, \nu=1, \frac{\delta \mathcal{F}_2(l, s)}{\delta x^1(s')} - \frac{\delta \mathcal{F}_1(l, s')}{\delta x^1(s)} = 0, \quad (4.17a)$$

$$\mu=1, \nu=2, \frac{\delta \mathcal{F}_2(l, s)}{\delta x^2(s')} + \frac{\delta \mathcal{F}_1(l, s')}{\delta x^1(s)} = 0; \quad (4.17b)$$

$$\frac{1}{\gamma^2} \text{ term: } -\frac{\delta \mathcal{F}_\mu(l, s')}{\delta x^\nu(s)} + \frac{\delta \mathcal{F}_\nu(l, s)}{\delta x^\mu(s')} + [\epsilon_{\mu\alpha} \mathcal{F}_\alpha(l, s'), \epsilon_{\nu\beta} \mathcal{F}_\beta(l, s)] = 0, \quad (4.18)$$

$$\begin{aligned} \frac{1}{\gamma} \text{ term: } & -\epsilon_{\mu\alpha} \frac{\delta \mathcal{F}_\alpha(l, s')}{\delta x^\nu(s)} + \epsilon_{\nu\beta} \frac{\delta \mathcal{F}_\beta(l, s)}{\delta x^\mu(s')} + \\ & [\epsilon_{\mu\alpha} \mathcal{F}_\alpha(l, s'), \mathcal{F}_\nu(l, s)] + [\mathcal{F}_\mu(l, s'), \epsilon_{\nu\beta} \mathcal{F}_\beta(l, s)] = 0, \end{aligned} \quad (4.19)$$

$$\gamma^0 \text{ term: } \frac{\delta \mathcal{F}_\mu(l, s')}{\delta x^\nu(s)} - \frac{\delta \mathcal{F}_\nu(l, s)}{\delta x^\mu(s')} + [\mathcal{F}_\mu(l, s'), \mathcal{F}_\nu(l, s)] = 0. \quad (4.20)$$

We see that they require much more than the loop space chiral Eqs. (2.5), (2.12) and (2.8a) for integrability.

Case (2): In this case infinite number of conserved currents satisfying Eq. (2.8a) can be constructed, so we have

$$\nabla_\mu^{(n)}(l, s) = \epsilon_{\mu\nu} \frac{\delta}{\delta x_\nu(s)} \chi^{(n)}(l, s) = \left[\frac{\delta}{\delta x_\mu(s)} + \mathcal{F}_\mu(l, s) \right] \chi^{(n-1)}(l, s). \quad (4.21)$$

Multiply Eq. (4.21) by L^n and sum over all n , and then define $\phi(l, s, L)$

$\equiv \sum_{n=0} L^n \chi^{(n)}(l, s)$, we obtain the inverse-scattering equation for $\phi(l, s, L)$, which is in the form of Eq. (4.13) with $\gamma = L^{-1}$,

$$\lim_{s' \rightarrow s} \left[\frac{\delta}{\delta x^\mu(s')} + \mathcal{F}_\mu(l, s) - L^{-1} \epsilon_{\mu\nu} \frac{\delta}{\delta x_\nu(s')} \right] \phi(l, s, L) = 0. \quad (4.13')$$

The integrability conditions in this limit of $s' \rightarrow s$ are just Eqs. (4.17) to (4.20) with $s' \rightarrow s$, which imply Eqs. (2.5), (2.12) and (2.8a), the equations of motion.

V. Summary and Outlook

So far we have restricted our discussions about further development only in two dimensions for the loop-space chiral equations. Similar subtleties exist for the three dimensional case, in which the first conservation law and linear system have been proposed.⁵ In the four dimensional case, little progress has been made beyond the formulation of loop-space chiral equations. We see that much work is still needed. The loop-space chiral equations are not a totally integrable system in the ordinary sense. The loop-space chiral equations do not provide enough information for the integration of loop-space currents from one point of the loop to another in a unique way. However, in spite of such difficulties, the observation that the Yang-Mills equations give the loop-space chiral equations is such a beautiful one, with further insight it is bound to lead to new understanding of the gauge theories.

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