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Monte Carlo Calculations in Lattice Gauge Theory*

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ABSTRACT

I review recent progress in numerical studies of the Feynman path integral for non-Abelian gauge theories. In the quarkless theory, numerical relations between the glueball mass, the confinement potential, and the scale of asymptotic freedom have been obtained. Renormalization group analysis has confirmed that the Gell-Mann Low function for the lattice SU(2) theory of Wilson has no non-trivial zeros. New phase transitions have been found in SU(4) and SU(5) gauge theories. These may be an artifact of the Wilson form for the action.

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MONTE CARLO CALCULATIONS IN LATTICE GAUGE THEORY⁺

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ABSTRACT

I review recent progress in numerical studies of the Feynman path integral for non-Abelian gauge theories. In the quarkless theory, numerical relations between the glueball mass, the confinement potential, and the scale of asymptotic freedom have been obtained. Renormalization group analysis has confirmed that the Gell-Mann Low function for the lattice SU(2) theory of Wilson has no non-trivial zeros. New phase transitions have been found in SU(4) and SU(5) gauge theories. These may be an artifact of the Wilson form for the action.

Although this is a conference on perturbative studies of the strong interaction gauge theory, I will emphasize non-perturbative results obtained by Monte Carlo studies of the Feynman path integral. I will first summarize the few physical numbers that have been calculated with this technique. Then I will discuss some recent renormalization group studies verifying asymptotic freedom in lattice gauge theory. Finally I will present evidence for some new phase transition for non-Abelian gauge groups. These may be a spurious consequence of the form of the lattice action.

In elementary particle physics, Monte Carlo simulation provides a technique for the numerical evaluation of path integrals. We can borrow this tool from statistical mechanics because the Feynman integral is mathematically equivalent to a partition function. As numerical work requires a well defined theory, a non-perturbative ultraviolet cutoff is necessary. For gauge theories, Wilson's lattice formulation elegantly serves this purpose.¹ The lattice is merely a regulator, and physical results require a continuum limit at the end of any calculation.

The method is truly non-perturbative. Convergence is excellent in both strong and weak coupling domains and is slow only near singularities or "phase transitions" in the corresponding statistical system. Nevertheless, inherent limitations exist. Statistical errors fall with the square root of the computing time; consequently certain parameters displaying large "thermal" fluctuations may be impractical to compute. Also, in four dimensional space time the linear dimensions of the system are necessarily strongly limited in lattice units. For the gauge groups SU(2) and SU(3), lattices of up to 10^4 sites have been used.² Recently Bhanot and Rebbi have used a discrete approximation to SU(2) on a 16^4 site lattice.³

A Monte Carlo algorithm generates a sequence of field configuration where ultimately the probability density for any given state is proportional to the Boltzman factor

$$P \sim e^{-\beta S}$$

(1)

⁺Talk at Conference on Perturbative Quantum Chromodynamics, Florida State University, Tallahassee, 25-28 March 1981.

where S is the action of the fields and β is proportional to the inverse of the bare coupling constant squared. We thus do "experiments" on small "crystals" in thermal equilibrium. As the entire field configuration is stored in the computer memory, any desired correlation function can be extracted. In a pure gauge theory the natural order parameters are Wilson loops, the path ordered exponential of the gauge fields integrated around a closed space-time contour.

The first physical parameter arising from this analysis relates the long distance linear potential responsible for quark confinement to the scale of asymptotic freedom. As a Wilson loop is made larger, its numerical expectation should fall exponentially with the area enclosed by the loop. The coefficient of the area in the exponent gives the strength of the linear long-range interquark potential $E \rightarrow Kr$. With Monte Carlo this coefficient is measured as a function of the bare coupling. Asymptotic freedom relates the bare charge to the lattice spacing

$$g_0^{-2}(a) = \beta_0 \ln(1/\Lambda_0^2 a^2) + (\beta_1/\beta_0) \ln[\ln(1/\Lambda_0^2 a^2)] + O(g_0^2) \quad (2)$$

Here β_0 and β_1 are the first two coefficients in the perturbative expansion of the Gell-Mann-Low function

$$a \frac{dg_0}{da} = \beta_0 g_0^3 + \beta_1 g_0^5 + O(g_0^7) \quad (3)$$

and Λ_0 is an integration constant parametrizing the scale of the asymptotic freedom phenomenon. For $SU(N)$ gauge theory β_0 and β_1 are

$$\beta_0 = \frac{11}{3} \frac{N}{16\pi^2} \quad (4)$$

$$\beta_1 = \frac{34}{3} \left(\frac{N}{16\pi^2} \right)^2$$

By studying the linear potential as a function of the lattice spacing we extrapolate to the continuum limit and obtain for $SU(3)$

$$\frac{\Lambda_0}{\sqrt{K}} = (5.0 \pm 1.5) \times 10^{-3} \quad (5)$$

A recent perturbative calculation by Hasenfratz and Hasenbratz⁴ relates Λ_0 to Λ^{MOM} , the scale of asymptotic freedom as defined from the three point vertex of the continuum theory in Feynman gauge. They find

$$\frac{\Lambda^{\text{MOM}}}{\Lambda_0} = 83.5 \quad (6)$$

If we accept the string model connection between K and the Regge slope, we are led to conclude for the pure gauge theory without including effects of light quarks²

$$\Lambda^{\text{MOM}} = 170 \pm 50 \text{ MeV} \quad (7)$$

A second non-perturbative number to come out of Monte Carlo analysis is the mass gap of the theory. Here statistical errors have been severely limiting accuracy. So far only SU(2) has been studied with the result for the pure gauge theory^{3,5}

$$\frac{m_g}{\Lambda_0} \sim 200 \quad (\text{SU}(2)) \quad (8)$$

where the error is probably of order 30 percent. This number has been estimated independently from correlations between separated gauge invariant operators and from finite lattice effects.

A third and less observable number which has been studied via Monte Carlo is the physical temperature at which confinement is lost and the vacuum becomes a "hot gluon soup." Using periodicity in time to represent physical temperature, several authors have found for the critical temperature of SU(2) gauge theory⁶

$$T_c \sim \frac{1}{3} \sqrt{K} \quad (9)$$

Naively extrapolating to SU(3) this gives 160 MeV. This number is a factor of 5-10 below the estimated glueball mass. Thus we have the surprising conclusion of a phase transition at a temperature considerably below the energy of the first excitation above the vacuum state.

I now turn to some recent renormalization group studies of SU(2) gauge theory.⁷ By measuring a physical observable on two length scales, one can relate the bare coupling with two values of the cut-off. Let $P(r/a, g_0(a))$ be a dimensionless observable defined on a physical scale r , with lattice cutoff a , and at bare coupling $g_0(a)$. If P is finite in the continuum limit, we expect

$$P\left(\frac{r}{a}, g_0(a)\right) = P\left(\frac{2r}{a}, g_0\left(\frac{a}{2}\right)\right) + O(a^2) \quad (10)$$

The order a^2 corrections depend on the renormalization scheme chosen. As lattices in practice are rather small, we must be optimistic in dropping these corrections. Iterating Eq. (10) with r replaced by $2^{-n}r$ gives

$$P\left(\frac{r}{a}, g_0\left(\frac{a}{2^n}\right)\right) = P\left(\frac{2r}{a}, g_0\left(\frac{a}{2^{n+1}}\right)\right) \quad (11)$$

Measuring an observable on two scales determines the dependence of the bare coupling on cutoff as the continuum limit is approached.

To carry out this procedure numerically, we need an observable which is easily measured in Monte Carlo simulation. Unfortunately, the Wilson loop by itself cannot be used because of ultraviolet divergences in the continuum limit. A simple perturbative calculation

shows a linear divergence proportional to the loop perimeter and a logarithmic infinity from sharp corners, inevitable on a lattice. These divergences will all cancel in ratios of loops with the same perimeter and number of corners but different overall shapes. With this motivation, consider the quantity

$$P\left(\frac{r}{a}, g_0\right) = 1 - \frac{W\left(\frac{r}{a}, \frac{r}{a}\right) W\left(\frac{r}{2a}, \frac{r}{2a}\right)}{W\left(\frac{r}{a}, \frac{r}{2a}\right)^2} \quad (12)$$

Here $W(I, J)$ denotes the expectation value of a rectangular Wilson loop of dimensions I by J in lattice units. To implement Eq. (1), we consider $r = 2a$ and $r = 4a$ in defining the function of the bare coupling

$$F(g_0) = 1 - \frac{W(2, 2) W(1, 1)}{(W(2, 1))^2} \quad (13)$$

$$G(g_0) = 1 - \frac{W(4, 4) W(2, 2)}{(W(4, 2))^2} \quad (14)$$

In Fig. 1 Monte Carlo measurements of these quantities are plotted versus g_0^2 . Note that $F(g_0)$ always lies below $G(g_0)$, indicating by Eq. (11) that the bare coupling decreases as the cutoff is reduced. Except at $g_0 = 0$, F and G do not cross each other, as would be expected at a conventional second order phase transition.

Asymptotic freedom predicts precisely how $g_0(a)$ vanishes in the continuum limit. Eq. (2) implies that for a factor of two change in cutoff

$$\frac{1}{g_0^2\left(\frac{a}{2}\right)} = \frac{1}{g_0^2(a)} + 2\beta_0 \ln 2 + O(g_0^2) \quad (15)$$

In Fig. 2 I show these functions F and G but with the argument of G shifted by the amount indicated in Eq. (15). Note the good agreement of F and G for weak coupling. This is a test of the neglected finite cutoff correction from Eq. (10).

For small distances the Wilson loop has a perturbative expansion. To lowest order this gives for the ratios considered here

$$1 - \frac{W\left(\frac{r}{a}, \frac{r}{a}\right) W\left(\frac{r}{2a}, \frac{r}{2a}\right)}{\left(W\left(\frac{r}{a}, \frac{r}{2a}\right)\right)^2} = p_1 g^2(r) + O(g_0^4, g_0^2 \frac{a^2}{r^2}) \quad (16)$$

where

$$\begin{aligned} p_1 &= \frac{3}{16\pi^2} (8 \arctan 2 + 2 \arctan 1/2 - 2\pi - 4\ln(5/4)) \\ &= 0.049559841. \dots \end{aligned} \quad (17)$$

Fig. 1. The functions F and G for the SU(2) theory.

Fig. 2. Testing asymptotic freedom.

The expansion parameter $g^2(r)$ represents an effective coupling at scale r . To this lowest order the renormalization prescription is arbitrary; indeed I can define the coupling such that higher order perturbation corrections vanish. Thus the function F is a measure of an effective coupling at $r = 2a$

$$g^2(r = 2a) = F(g_0(a))/p_1 \quad (18)$$

when I have neglected the finite cutoff correction. In Fig. 3 I plot $g^{-2}(2a)$ versus $g_0^{-2}(a)$. Asymptotic freedom predicts a linear relation between these quantities as g becomes small

$$g^{-2}(2a) - g_0^{-2}(a) = \beta_0 \ln \frac{2\Lambda}{\Lambda_0} + O(g_0^2) \quad (19)$$

Here Λ is the scale parametrizing the logarithmic behavior of $g^2(r)$ in analogy to Eq. (2)

$$g^{-2}(r) = \beta_0 \ln(1/\Lambda^2 a^2) + (\beta_1/\beta_0) \ln[\ln(1/\Lambda^2 a^2)] + O(g^2(r)). \quad (20)$$

The observed unit slope in Fig. 3 represents an additional strong check on the smallness of the neglected finite cutoff corrections. From the intercept we find

$$\beta_0 \ln(2\Lambda/\Lambda_0) = 0.35 \quad (21)$$

or

$$\Lambda = 22\Lambda_0. \quad (22)$$

Fig. 3. The inverse renormalized charge squared at $r = 2a$ versus the inverse bare charge squared.

Fig. 4. Four Monte Carlo runs for SU(5) gauge theory at $g_0^{-2} = 1.67$.

A perturbative calculation of the next order term in Eq. (16) can check this result. This need not be done with lattice perturbation theory because of the existing value for $\Lambda_{\text{MOM}}/\Lambda_0$ in Eq. (6). This verification of the number in Eq. (21) would provide a final check on the cutoff corrections and would make nontrivial contact between Monte Carlo methods and higher order perturbative gluon dynamics.

I now turn to some new unexpected phase transitions in pure lattice gauge theory. Recently Halliday and Schwimmer and Lautrup and Greensite have studied the gauge group SO(3) with the standard Wilson action.⁸ In the continuum limit this model should exhibit the same physics as SU(2) because the Lie algebras are identical. However, the lattice theory undergoes a first order phase transition. From an analysis of my own which confirms their result, I find $g_c^2 = .60 \pm .01$.

In Fig. 4 I show the results of four Monte Carlo runs with the gauge group SU(5) on a 3^4 site lattice.⁹ All runs are at the same coupling, $g_0^{-2} = 1.67$. The lower and upper sets of points represent the evolution from ordered and disordered initial states, respectively. The intermediate runs are from superheated and supercooled states. Note the appearance of two distinct final equilibrium values for the internal energy. This represents a first order phase transition. A similar transition appears for SU(4) at $g_0^{-2} = 1.28$.

As the above groups are non-Abelian, presumably the mentioned transitions are not deconfining. I conjecture that they are artifacts of the Wilson form for the action. To see that phase transitions can be induced by a poor choice of action, consider the gauge group SU(2) and generatize the action for a plaquette to the two parameter form

$$S_{\square} = \frac{1}{2} \beta (1 - \frac{1}{2} \text{Tr } u_{\square}) + \frac{1}{3} \beta_A (1 - \frac{1}{3} \text{Tr}_A u_{\square}) . \quad (23)$$

Here the first term is the usual Wilson form and in the second term Tr_A represents the trace of the matrix u considered in the adjoint (spin 3) representation of $\text{SU}(2)$. This model has three simple limits: (1) $\beta_A = 0$ is the usual $\text{SU}(2)$ model, (2) $\beta = 0$ is the $\text{SO}(3)$ model, and (3) when $\beta_A \rightarrow \infty$ the model reduces to the Z_2 lattice gauge theory. In both of the latter cases the model is known to have a non-trivial phase structure. Gyan Bhanot and I have been using Monte Carlo simulation to follow these transitions into the phase diagram. Both the Z_2 and $\text{SO}(3)$ transitions remain and join at a triple point located at $\beta = .55 \pm .03$, $\beta_A = 2.34 \pm .03$. From this point a single first order line continues into the diagram until it ends at a critical point with coordinates $\beta = 1.57 \pm .05$, $\beta_A = .78 \pm .05$. This line points at but does not reach the region of the β axis where Lautrup and Nauenberg found a peak in the specific heat of the usual $\text{SU}(2)$ model.¹⁰ This peak is thus the reflection of a nearby singularity in a larger coupling constant space.

In the continuum limit the bare coupling is related to β and β_A by the expression

$$\frac{4}{g_0^2} = \beta + \frac{8}{3} \beta_A \quad (24)$$

Consequently, a continuum limit can be taken in many directions in the β, β_A plane. If there is a unique continuum limit, physics should not depend on the direction chosen. However, before the continuum limit, lattice artifacts can appear; indeed one can encounter the phase transitions mentioned above. These do not signal deconfinement, however, as one can continue around them. My conjecture for the $\text{SU}(N)$ models at large N is that this nontrivial phase structure accidentally overlaps the β axis.

To conclude, lattice gauge theory is a rapidly evolving field. The phase structure is turning out to be fascinatingly complicated and yet there is presumably only one physical continuum limit. We are all working on finding ways to calculate the details of that theory of quark-gluon dynamics.

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