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PERTURBATION THEORY AT LARGE ORDER:
THE CHOICE OF INTEGRATION CONTOUR*

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ABSTRACT

I discuss two technical points in the method of Lipatov for extracting the large order behavior of field theories. The first is an adjustment of the integration contour required to make the action bounded below. The second is a method for treating fluctuations in the coupling constant exactly.

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I. INTRODUCTION

Investigations in quantum field theory are based, for the most part, on the expansion of the Green's functions in powers of the coupling constant $G(g) \sim \sum G_N g^N$. It has long been of interest to determine how the coefficients G_N behave in the large N limit. Great progress has been made on this problem recently, following the work of Lipatov [1].

The work of Lipatov in field theory was to some extent anticipated by that of Langer [2] in statistical mechanics and of Bender and Wu [3] in quantum mechanics. Considerable advances have been made since Lipatov's paper [4-9]. In the body of this paper I will refer to the work of Brezin, LeGuillou, and Zinn-Justin [4], who discuss the general method and elucidate what happens when the perturbation theory vacuum becomes unstable, and to the work of Collins and me [5], who relate the large N behavior of G_N for $g\phi^4$ theory in $0 + 1$ dimensions to the behavior of the discontinuity of $G(g)$ near the tip of its cut on the negative g axis [3] and give the rules for calculating to any order in $1/N$. (See also Ref. 7).

The method of Lipatov can be quickly summarized for, say, a $g\phi^4$ theory. Lipatov writes G_N as an integral in g space with an effective action $S_{eff} = S_0[\phi] + g S_1[\phi] + N \ln g$. He then argues that the large N behavior of this integral is controlled by the integration near a certain stationary point $\phi_c^{(n)}, g_c^{(n)}$ of the effective action.

In this paper I address two questions that arise in this procedure. The first is a question of principle. As noted by Brezin, LeGuillou, and Zinn-Justin [4], if, as seems natural, the g -integral in Lipatov's formulation runs over a fixed circle $g = re^{ie}$, then the g -integral must be done first, before the ϕ -integral. This is because the action becomes infinitely large

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and negative as $\phi \rightarrow \infty$ when $g \approx \pm iR$. But one is then hard pressed to justify the expansion of the iterated integral about a joint saddle point $\phi_c^{(N)}, g_c^{(N)}$, especially when the action is unbounded from below. I will argue that the g -integral becomes a properly convergent joint integral, suitable for a joint saddle point expansion, if one lets the radius R of the g -integration contour depend on ϕ in a certain way.

The second question addressed relates to practical calculations. If one wishes to calculate the $1/N$, $1/N^2$, etc., corrections to the leading behavior of G_N using Lipatov's method, one must calculate Feynman diagrams that describe the fluctuations $\delta\phi, \delta g$ away from the stationary point. I show how to treat the g fluctuations exactly, so that only the ϕ fluctuations need to be accounted for by perturbation theory. (The resulting perturbation theory is essentially ordinary small coupling perturbation theory about a non-trivial classical solution of the field equations.)

II. THE INTEGRATION CONTOUR

The simplest model that illustrates the points that I wish to discuss is that of a quantum system with one canonical coordinate $\phi(t)$ and the hamiltonian

$$H(g) = \frac{1}{2} \pi^2 + \frac{1}{2} \phi^2 - \gamma g \phi^3 + \frac{1}{2} g^2 \phi^4 \quad (2.1)$$

where $\pi(t)$ is the canonical momentum, $[\pi, \phi] = -i$. The large order behavior of perturbation theory in this model was first discussed by Brezin, LeGuillou, and Zinn-Justin^[4], who used it to illustrate the change in the behavior of perturbation theory when the $\phi=0$ ground state becomes unstable.

Consider the large order behavior of the partition function

$$Z(g) = \text{Tr} [\exp(-H(g) T)] \quad (2.2)$$

in this model. The large order analysis is based on the path integral

representation

$$Z(g) = \int d[\phi] \exp(-S[\phi, g]) \quad (2.3)$$

where

$$S[\phi, g] = \int_{-T/2}^{+T/2} dt \left(\frac{1}{2} \dot{\phi}(t)^2 + \frac{1}{2} \phi(t)^2 - \gamma g \phi(t)^3 + \frac{1}{2} g^2 \phi(t)^4 \right) \quad (2.4)$$

The partition function has a perturbation expansion for small g :

$$Z(g) \sim \sum_N Z_N g^N \quad (2.5)$$

The coefficient Z_N is, by definition, the sum of the N th order vacuum bubble Feynman diagrams of the theory. It has the path integral representation

$$Z_N = \int d[\phi] \frac{1}{2\pi i} \oint \frac{dg}{g} \exp(-S_{\text{eff}}[\phi, g]) \quad (2.6)$$

where the g contour circles the origin and

$$\begin{aligned} S_{\text{eff}} &= S_2[\phi] - \gamma g S_3[\phi] + g^2 S_4[\phi] + N \ln g \\ S_2 &= \int dt \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi^2 \right) \\ S_3 &= \int dt \phi^3 \\ S_4 &= \int dt \frac{1}{2} \phi^4 \end{aligned} \quad (2.7)$$

We will follow the original saddle point expansion method of Lipatov^[1] in order to find the large N behavior of Z_N . We expect the region of integration in which $\text{Re } S_{\text{eff}}$ is smallest to dominate the integral for large N . Thus we seek a point $\phi_c^{(N)}, g_c^{(N)}$ at which $\text{Re } S_{\text{eff}}$ takes its minimum value. However we must first face a question that was ignored in Lipatov's pioneering work: Over what contour does the g -integration in Eq. (2.6) run? The most natural choice is a circle with a fixed radius R , but $\text{Re } S_{\text{eff}}$ does not have a global minimum on this contour. In fact, $\text{Re } S_{\text{eff}}$ becomes arbitrarily large and negative at $g \approx \pm iR, \phi(t) \rightarrow \infty$.

Another aspect of the same difficulty is that, with this contour, the g -integral in Eq. (2.6) must be done first, before the ϕ -integral. This problem was noted by Brezin, LeGuillou and Zinn-Justin and was dealt with in the case $\gamma=0$ by actually doing the g -integral first. This gives

a factor $(-S_4[\phi])^{N/2}$ or, equivalently, a term $-\frac{1}{2}N \ln(-S_4[\phi])$ in the effective action. (See also the recent work of Ref. 8, which uses this method.) Another approach^[5], inspired by the work of Bender and Wu^[3], is to effectively do the g -integral last as an integral along the negative g -axis of the discontinuity of $Z(g)$. Neither of these approaches is very convenient for the case $\gamma \neq 0$.

A better device for avoiding divergences at large ϕ is to let the radius R of the contour for the g -integral be a function $R_N[\phi]$ of ϕ . One must choose $R[\phi]$ so that $\text{Re } S_{\text{eff}}$ does not become negative and large for large ϕ . An optimum choice can be made as follows: For a given ϕ and a given $R = |g|$, find the "worst" value of $\text{Re } S_{\text{eff}}$ as $\Theta = \arg g$ ranges over 0 to 2π , that is, the smallest (most negative) value of $\text{Re } S_{\text{eff}}$. Now choose $R[\phi]$ so that this worst value of $\text{Re } S_{\text{eff}}$ is as "good" as possible, that is as large as possible. The result is

$$R_N[\phi] = \left(\frac{N}{2S_4[\phi]} \right)^{\frac{1}{2}} \quad \text{if } \gamma^2 S_3[\phi]^2 < 8NS_4[\phi] \quad (2.8)$$

$$R_N[\phi] = \frac{|\gamma S_3[\phi]|}{4S_4[\phi]} \left\{ 1 - \left(1 - \frac{8NS_4[\phi]}{\gamma^2 S_3[\phi]^2} \right)^{1/2} \right\}$$

$$\quad \text{if } \gamma^2 S_3[\phi]^2 > 8NS_4[\phi].$$

With this choice for $R[\phi]$, the divergences at large ϕ for fixed angle Θ are absent. In fact, it is not difficult to show[†] that

$$|e^{-S_{\text{eff}}} < C_N e^{-S_2[\phi]} (S_4[\phi] + \gamma^2 S_3[\phi]^2)^{N/2} \quad (2.9)$$

so that the integral over ϕ and g is now absolutely convergent. In addition it is easy to see that $\text{Re } S_{\text{eff}}[\phi, g = R e^{i\Theta}]$ is bounded from below[†]. It is therefore sensible to expand about the (global) minimum of $\text{Re } S_{\text{eff}}$. We now turn to this task.

III. THE STATIONARY POINTS

The next step in Lipatov's procedure is to locate the stationary points $\phi_c^{(N)}$, $g_c^{(N)}$ of the effective action. We briefly review this step here, then discuss the relation of the stationary points to the contour introduced in the previous section.

It is useful to define new variables

[†]At the "worst" angle, $g_c = R \exp(i\Theta_c)$ satisfies $\partial S_{\text{eff}} / \partial g = 0$, so $S_{\text{eff}} = S_2 - g^2 S_4 - N + N \ln g$. The solution (2.8) for R satisfies $R^2 \leq N/2S_4$. Thus $\text{Re } S_{\text{eff}} \geq S_2 - \frac{1}{2}N + N \ln R$. Then separate lower bounds for R for the two cases in Eq (2.8) lead to the inequality (2.9). The further bound $\text{Re } S_{\text{eff}} > \text{const.}$ follows from (2.9) and the inequalities^[10] $S_4[\phi] < \text{const. } S_2[\phi]^2$, $S_3[\phi]^2 < \text{const. } S_2[\phi]^6$.

$$\begin{aligned}\tilde{\phi}(t) &= N^{-1/2} \phi(t) \\ \tilde{g} &= N^{1/2} g\end{aligned}\quad (3.1)$$

The effective action as a function of $\tilde{\phi}$ and $\tilde{g}$ is

$$S_{\text{eff}}^{(N)} = N \left(S[\tilde{\phi}, \tilde{g}] + \ln \tilde{g} \right) + \frac{1}{2} N \ln N \quad (3.2)$$

where $S[\tilde{\phi}, \tilde{g}] = S_2[\tilde{\phi}] - \lambda \tilde{g} S_3[\tilde{\phi}] + \tilde{g}^2 S_4[\tilde{\phi}]$ as in Eq. (2.4).

Once one finds the (N-independent) stationary points, $\tilde{\phi}_c, \tilde{g}_c$ of $S[\tilde{\phi}, \tilde{g}] + \ln \tilde{g}$, the stationary points of $S_{\text{eff}}^{(N)}$ are known:

$$\begin{aligned}\phi_c^{(N)} &= N^{1/2} \tilde{\phi}_c \\ g_c^{(N)} &= N^{-1/2} \tilde{g}_c\end{aligned}\quad (3.3)$$

In Ref 4, one can find a calculation of the action $S_{\text{eff}}^{(N)}[\phi_c^{(N)}, g_c^{(N)}]$ at the leading stationary point. That paper gives S_{eff} as a function of the coefficient λ of the $g\phi^3$ term and discusses how the leading large order behavior of perturbation theory changes as λ increases and the original perturbation theory ground state becomes unstable. The interesting special case $\lambda = 1$ is further discussed in Ref 9.

Let us call the stationary points $\phi_c^{(N)}, g_c^{(N)}$ with the smallest value of $\text{Re } S_{\text{eff}}$ the leading stationary points. (There are more than one leading stationary points, with equal values of $\text{Re } S_{\text{eff}}$).

In the case $\lambda = 0$, the solutions $\phi_c^{(N)}(t)$ are real and are thus included in the integration $d[\phi]$. In addition, the integration contour in g -space passes through $g_c^{(N)}$: the construction of the "optimum" choice of $R[\phi]$ in Section II amounts to finding $g_c^{(N)} = R e^{i\theta_c}$ that satisfies $\partial S_{\text{eff}}[\phi, g]/\partial g = 0$. Thus the stationary points $\phi_c^{(N)}, g_c^{(N)}$ lie on the combined ϕ, g -integration contour.

The solutions $\phi_c^{(N)}(t)$ are again real for $1 < |\lambda|$. In this case, the contour of Section II passes through one of the two solutions $g_{\pm}$ of $\partial S_{\text{eff}}[\phi, g]/\partial g = 0$, call it g_- . One of g_- and g_+ must be $g_c^{(N)}$. A detailed calculation shows that $g_c^{(N)} = g_-$ (i.e. the contour passes through $\phi_c^{(N)}, g_c^{(N)}$) for $1 + \epsilon < \lambda$, where $1 + \epsilon \approx 1.103$ is the solution of $\frac{2}{3} - 7\lambda^2 + \frac{1}{2}\lambda(7\lambda^2 - 5) \ln[(\lambda+1)/(\lambda-1)] = 0$.

In the case $0 < |\lambda| < 1$, the solutions $\phi_c^{(N)}(t)$ are complex valued and in the case $1 < |\lambda| < 1 + \epsilon$ one finds $g_c^{(N)} = g_+$. Thus for $0 < |\lambda| < 1 + \epsilon$ the stationary points $\phi_c^{(N)}, g_c^{(N)}$ do not lie on the ϕ, g -contour constructed in Section II. In this case, the ϕ, g -contour should be deformed so that it passes through the leading stationary points.

One can easily show (following Ref. 5) in the case $\lambda = 0$ that $\text{Re } S_{\text{eff}}^{(N)}[\phi, g]$ is larger everywhere else on the integration contour of Section II than it is at the leading stationary points $\phi_c^{(N)}, g_c^{(N)}$. It is almost certain that this property holds in the case $1 + \epsilon < |\lambda|$ also. In the remaining case, $0 < |\lambda| < 1 + \epsilon$, I leave it as a conjecture that the integration contour can be deformed in such a way that $\text{Re } S_{\text{eff}}[\phi, g] > \text{Re } S_{\text{eff}}[\phi_c^{(N)}, g_c^{(N)}]$ when ϕ, g is on the contour and $(\phi, g) \neq (\phi_c^{(N)}, g_c^{(N)})$.

IV. THE EXPANSION ABOUT A STATIONARY POINT

Consider the integration near one of the leading stationary points $\phi_c^{(N)}, g_c^{(N)}$. Following the discussion of the previous section, we assume that the contour passes through $\phi_c^{(N)}, g_c^{(N)}$ and that $\text{Re } S_{\text{eff}}$ takes its minimum value there. One sees from Eq (3.2) that, as a function of $\tilde{\phi} = N^{-1/2} \phi$ and $\tilde{g} = N^{-1/2} g$, $\exp(-S_{\text{eff}})$ becomes more and more sharply peaked at $\tilde{\phi}_c, \tilde{g}_c$ as $N \rightarrow \infty$. Thus one expects that the large N behavior of Z_N can be extracted by making a saddle point expansion about $\phi_c^{(N)}, g_c^{(N)}$. [1]

It is, however, useful to employ a slight modification of the most straightforward expansion procedure. The modified procedure, to which this section is devoted, results in simpler rules for generating the expansion in powers of $1/N$.

Since only the part of the contour near the stationary point contributes in the large N limit, one can replace the original contour by any contour that passes through $\phi_c^{(N)}, g_c^{(N)}$ in a direction of increasing $\text{Re } S_{\text{eff}}[\phi, g]$. A convenient choice of contour follows from the observation that all of the points $g_c^{(N)} = N^{-1/2} \tilde{g}_c$ lie on a radial line

$$g = \lambda \tilde{g}_c \quad (4.1)$$

in the complex g -plane. We choose a contour parameterized by a positive real number λ and a function $\eta(t)$, with g given by (4.1) and $\phi(t)$ by

$$\phi(t) = \lambda^{-1} \tilde{\phi}_c(t) + \eta(t) \quad (4.2)$$

The function $\eta(t)$ is to lie in the plane in complex function space determined by the condition that

$$\langle \eta | \mathcal{H} | \eta \rangle = \int dt dt' \left[\frac{\delta^2 S[\phi, \tilde{g}_c]}{\delta \phi(t) \delta \phi(t')} \right]_{\phi=\tilde{\phi}_c} \eta(t) \eta(t') \quad (4.3)$$

be real and positive. (The operator $\mathcal{H}$ will turn out to have one negative eigenvalue [1,4,5]. The projection of η onto the corresponding eigenvector should therefore be purely imaginary. There is also a zero eigenvalue, to be discussed shortly).

Evidently this contour passes through the stationary point $\phi_c^{(N)} = N^{1/2} \tilde{\phi}_c, g_c^{(N)} = N^{-1/2} \tilde{g}_c$ at $\eta = 0, \lambda = N^{-1/2}$. The second variation of S_{eff} at the stationary point is easily calculated by writing S_{eff} in the form

$$S_{\text{eff}} = \lambda^{-2} S[\tilde{\phi}_c + \lambda \eta, \tilde{g}_c] + N \ln(\lambda \tilde{g}_c)$$

and temporarily treating $\lambda \eta$ and λ as independent variables. One finds

$$\delta^2 S_{\text{eff}} = \frac{1}{2} \langle \eta | \mathcal{H} | \eta \rangle + N^2 (\delta \lambda)^2 \quad (4.4)$$

Thus $\text{Re } S_{\text{eff}}$ increases as one moves away from the stationary point. (In addition, the contour is tangent to the "steepest descent" plane at the stationary point, so that $\delta^2 S_{\text{eff}}$ is real.)

It is now possible to simplify the saddle point expansion procedure: Instead of expanding about the joint saddle point $\eta = 0, \lambda = N^{-1/2}$, first expand the η -integral about its saddle point at $\eta = 0$, then do the λ -integral exactly. The η -integral takes the form

$$g^{-N} \int d[\eta] e^{-S[\phi + \eta, g]}$$

where $g = \lambda \tilde{g}_c$. The function $\phi_c = \lambda^{-1} \tilde{\phi}_c$ is the stationary point of $S[\phi, g]$. The integral has an expansion for small coupling^[5]

$$g^{-N} (\det \hat{H})^{-1/2} e^{-S[\tilde{\phi}_c, \tilde{g}_c]/\lambda^2} A g^\alpha [1 + a_1 g + a_2 g^2 + \dots]$$

(4.5)

Here $\det \hat{H}$ is the product of all the non-zero eigenvalues of the second variation operator $\hat{H}$. The invariance group of the action (in the present model; time translation invariance) that leads to zero eigenvalues of $\hat{H}$ is integrated over exactly, giving the factor $A g^\alpha$ as discussed in Refs. 1, 4, and 5. For each negative eigenvalue of $\hat{H}$ there is also a factor ± 1 included in $(\det \hat{H})^{-1/2}$. This factor arises from rotating the integration contour for the corresponding direction in η -space

to the imaginary axis; the $\pm$ sign is determined by finding the direction of rotation needed to keep $\text{Re } S_{\text{eff}}^2 > 0$ as one moves continuously from the old contour to the new. The coefficients $a_1, a_2, \dots$ are to be calculated perturbatively using the Feynman rules for the expansion of a field theory about a (non-trivial) solution of the field equations. This expansion is discussed in detail, and a_1 is calculated, for the case $\gamma = 0$ in Ref. 5. Finally, it is useful to note that $S[\tilde{\phi}_c, \tilde{g}_c]$ is real and positive; in fact the variational equation for $\tilde{g}_c$ at fixed $\tilde{\phi}$ is $S[\tilde{\phi}_c, \tilde{g}_c] = \frac{1}{2}$.

The remaining λ -integral of the J^{th} term in the series (4.5) is

$$Z_N^{(J)} = \frac{1}{2\pi i} (\det \hat{H})^{-1/2} A a_J \tilde{g}_c^{J+\alpha-N} \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{J+\alpha-N} e^{-S[\tilde{\phi}_c, \tilde{g}_c]/\lambda^2}$$

It would, of course, be possible to evaluate this integral in the large N limit by a saddle point expansion; that is essentially what one would be doing if one made the most straightforward saddle point expansion of the original ϕ, g -integral. However, it is easier to simply evaluate the integral exactly. One finds $Z_N \sim \sum_J Z_N^{(J)}$ where

$$Z_N^{(J)} = \frac{1}{2\pi i} (\det \hat{H})^{-1/2} A a_J (\tilde{g}_c^2 S[\tilde{\phi}_c, \tilde{g}_c])^{1/2(J+\alpha-N)} \frac{1}{2} \Gamma\left(\frac{N-J-\alpha}{2}\right).$$

(4.6)

Successive terms in the series are smaller by a factor

$$(\tilde{g}_c^2 S)^{1/2} \Gamma\left(\frac{1}{2}[N-J-\alpha-1]\right) / \Gamma\left(\frac{1}{2}[N-J-\alpha]\right) \sim (\tilde{g}_c^2 S/N)^{1/2}$$

at large N .

V. COMPARISON WITH THE DISCONTINUITY METHOD

In the case $\gamma = 0$ there is another method available to find the large N behavior of Z_N . This method is based on the method of Bender and Wu and is discussed in detail in Ref. 5. One calculates the discontinuity of $Z(g^2)$ across its cut on the negative g^2 -axis in the limit of small g . Then one reconstructs Z_{2N} as an integral of $|g^2|^{-N}$ disc Z along the cut. The actual calculations to be performed in this discontinuity method are exactly the same as in the method discussed in this paper, with the role of the integration along the cut being played by the λ -integral that is saved to be done at the end.

The discontinuity method has a technical feature that is useful for constructing the complete $1/N$ expansion of G_N for a Green's function

$G(t_1, \dots, t_N; g^2)$. The Green's function is calculated by writing $G(t_1, \dots, t_N; g^2) = Z(g^2)^{-1} I(t_1, \dots, t_N; g^2)$, where I has a simple path integral expression and the factor Z^{-1} removes the vacuum bubble contributions from I . In the discontinuity method with $\gamma = 0$ one calculates $\text{disc } I(g^2)$, $I(g^2)_+$, $\text{disc } Z(g^2)$ and $Z(g^2)_+$ in perturbation theory near the tip of the cut along the negative g^2 axis. From this information it is straightforward to calculate $\text{disc } G(g^2)$ near the tip of the cut and thus to determine the Feynman rules for the complete large N expansion of G_N . Thus, for instance, one can verify the cancellation in G_N of certain infrared divergences that occur in both I_N and Z_N . The determination of this expansion directly from the large N expansions of I_N and Z_N would be more difficult (although it is not hard to find the leading terms.) Evidently this same construction can be adapted to the method developed in this paper, which, in the end, is calculationaly the same as the discontinuity method.

The distinguishing feature of the discontinuity method is that one must assume analyticity in a neighborhood of the origin in the g^2 plane except for a cut along the negative g^2 axis. The required analyticity is known to hold in the case of ϕ^4 field theory in $0 + 1$ dimensions discussed here. However, serious doubts have been raised about whether this same analyticity holds in $3 + 1$ dimensions^[11]. At first it would seem that the Lipatov method as discussed here escapes from these doubts because, since the g -integral is performed inside the functional integral, no analyticity for $Z(g^2)$ is assumed. However, in more than $0 + 1$ dimensions counter terms are required in the action. In $1 + 1$ and $2 + 1$ dimensions these counterterms are polynomials in the renormalized coupling constant g^2 and no obvious difficulty occurs. In $3 + 1$ dimensions, however, the counterterms contain all powers of g^2 and may not be analytic even in a cut off theory. It is therefore not clear that one can treat the counter terms as if they were finite analytic functions

of g^2 and thus extract the coefficient of g^N in the integrand of the functional integral by integrating $g^{-(N+1)}$ times this integrand along a small circle about the origin.

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