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TITLE: The Response of the SLC Beamstrahlung Monitor

**Abstract**

In this note the number of Cerenkov photons generated per unit time by synchrotron radiation characterized by its characteristic energy $\varepsilon_c$ is calculated as a function of the ratio of $\varepsilon_c$ to the cutoff energy of the Cerenkov monitor. These results are used to determine approximate scaling relations for the beamstrahlung flux.

## 1 Introduction

At the SLC a Cerenkov monitor [1] is used to detect the light emitted by particles in the (strong) beam-beam interaction at the interaction point (IP). The monitor is also exposed to the synchrotron light emitted by the same particles in a strong bend magnet needed to deflect the radiating particles away from the detector. The Cerenkov monitor is needed to discriminate the abundantly generated photons in the bend magnet, having a critical energy of about 2 MeV from the fewer photons generated in the beam-beam interaction, having a critical energies of up to a factor 10 higher.

The Cerenkov monitor consists of a converter plate that converts the incident photons into $e^+e^-$ pairs which travel through a gas volume (Ethylene at 1/3 atmosphere) with refractive index $n = 1 + 2\,10^{-4}$. Here the high energy pairs can emit Cerenkov radiation which is subsequently passed by mirrors through a light channel and detected by photo multiplier tubes (PMT).

Clearly, only pairs with energies above the Cerenkov threshold, here expressed in units of $m_e c^2$

$$\frac{E_{Cutoff}}{m_e c^2} = \gamma_0 = \frac{n}{\sqrt{n^2 - 1}} \tag{1}$$

can emit Cerenkov photons. At the present detector this threshold is approximately $E_{Cutoff} = 25$ MeV. This discriminates the radiation from the bending magnet.

In order to quantify the above argument assume that radiation with characteristic energy $\varepsilon_c$ is incident on the monitor. To calculate the number of Cerenkov photons $dN_C/dt$ generated per unit time we have to take the following effects into account [2].

1. Pair production probability in the converter plate.
2. The probability of finding a pair produced particle at energy E.
3. The number of Cerenkov photons produced per particle.

These three points constitute the monitor response for one incident photon of a certain energy. The effect of a spectrum characterized by $\varepsilon_c$ is the given by the convolution of the detector response and the spectrum. This calculation will be the task of the following sections.

# 2 The Monitor Response

The pair production probability is determined by the crossection of the process $\gamma \rightarrow e^+e^-$. Since we are only interested in energies above the Cerenkov threshold we may use the Heitler–Sauter crossection for an unscreened point nucleus for ultrarelativistic energies [4]

$$\sigma_{\gamma \rightarrow e^+e^-} = \alpha Z^2 r_e^2 \left[\frac{28}{9} \ln(2\gamma_B) - \frac{218}{27}\right] . \tag{2}$$

Here $\alpha = 1/137$ is the fine structure constant, $Z$ is the atomic number of the nucleus ($Z = 26$ for iron), $r_e = 2.8179\,10^{-15}$ m is the classical electron radius and $\gamma_B$ is the photon energiy in units of $m_e c^2$. At the Cerenkov threshold we have $\sigma_{\gamma \rightarrow e^+e^-} = 2\,10^{-28}$ m$^{-2}$.

The probability for the pair production process is the given by

$$P_{\gamma\to e^+e^-} = 1 - e^{Nd\sigma_{\gamma\to e^+e^-}} \approx Nd\,\sigma_{\gamma\to e^+e^-}\ . \tag{3}$$

Here $N$ is the number density of iron nuclei in the converter plate and $d$ is the thickness (0.8 mm). The approximation made in eq. 3 is valid for the present monitor, because at the Cerenkov threshold we have $Nd\sigma_{\gamma\to e^+e^-} \approx 0.013$ and the logarithm containing the photon energies varies only weakly. We conclude that only 1 % to 2 % of the incident photons actually pair-produce.

Following the argument presented in ref. 3 we approximate the probability for finding a pair produced particle at energy $E_e$ by $P(E_e, E_B) = 2/E_B$ for $0 \le E_e \le E_B$. Here $E_B$ is the energy of the incident photon. Physically this means that all particle energies are assumed equally probable in the kinematically allowed range. They only depend on the energy of the incident photon.

Given one particle of energy $E_e = \gamma_e mc^2$ the number of Cerenkov photons it emits is given by

$$N_{c1}(\gamma_e) = \eta\frac{2\pi\alpha l\Delta\nu}{c}\left[1 - \frac{1}{n^2}\left(1 + \frac{1}{\gamma_e^2}\right)\right]\Theta(\gamma_e - \gamma_0) \tag{4}$$

with $\Theta(x) = 1$ for $x > 0$ and 0 elsewhere. where $l$ is the length of the gas volume (21 cm), $\Delta\nu$ is the frequency range accepted by the PMT and $\eta$ is the conversion efficiency of Cerenkov photons to PMT counts.

In order to calculate the number of Cerenkov photons as a function of the incident photon energy we have to integrate out the intermediate electron energies and obtain

$$\begin{aligned}
N_c(\gamma_B) &= \eta N d\,\sigma_{\gamma_B\to e^+e^-}\int_{E_{Cutoff}}^{E_B} dE_e\, P(E_e, E_B)\, N_{c1}(E_e) \\
&= \eta\,\frac{4\pi\alpha l\Delta\nu}{cE_B} N d\,\sigma_{\gamma_B\to e^+e^-}\Theta(\gamma_B - \gamma_0) \\
&\quad \times\Big[(1 - \frac{1}{n^2})(E_B - E_{Cutoff}) + \frac{m^2c^4}{n^2}(\frac{1}{E_B} - \frac{1}{E_{Cutoff}})\Big] \\
&\approx \eta N d\,\sigma_{\gamma_B\to e^+e^-}\frac{4\pi\alpha l\Delta\nu}{c}\Big(\frac{1}{\gamma_0} - \frac{1}{\gamma_B}\Big)^2\Theta(\gamma_B - \gamma_0)
\end{aligned} \tag{5}$$

where we have used eq. 1 to substitute $\gamma_0$ for $n$ and have neglected terms of order $1/\gamma_0^2$ against those of order 1.

This expression can be viewed as a transfer map that converts photons of energy $E_B = \gamma_B mc^2$ into the number of PMT counts.

# 3 The Influence of the Spectral Distribution of the Incident Photons

The spectral characteristics of synchrotron radiation emanating from a charged particle that traverses a section of curved path with bending radius $\rho$ can conveniently be described by the characteristic energy of the emitted radiation. This is given by

$$\varepsilon_c = \frac{3}{2}\frac{\hbar c}{\rho}\gamma^3 \tag{6}$$

where $\gamma$ is the energy of the radiating particle in units of its rest mass.

The emitted power as a function of photon energy $\varepsilon$ is usually parametrized in terms of the quantity

$$y = \frac{\varepsilon}{\varepsilon_c} \tag{7}$$

and can be written as [6]

$$\frac{dW}{dy} = \frac{2}{3}\frac{e^2 c\gamma^4}{4\pi\varepsilon_0\rho^2}\frac{9\sqrt{3}}{8\pi} y \int_y^\infty dx\, \mathrm{K}_{5/3}(x)\ . \tag{8}$$

Here $\mathrm{K}_{5/3}(x)$ is a Bessel function of fractional order [5]. For our purpose, however, the number spectrum is important. It can be expressed similarly by

$$\frac{dN}{dy\, dt} = \frac{1}{E}\frac{dW}{dy} = \frac{1}{\varepsilon_c}\frac{1}{y}\frac{dW}{dy}\ . \tag{9}$$

Expressing the number spectrum with respect to the photon energy we can conveniently write

$$\frac{dN}{dE\, dt} = \frac{1}{\varepsilon_c^2}\frac{1}{y}\frac{dW}{dy} = \frac{1}{\sqrt{3}\pi}\frac{\alpha^2}{r_e mc}\frac{1}{\gamma^2}\int_y^\infty dx\, \mathrm{K}_{5/3}(x) \tag{10}$$

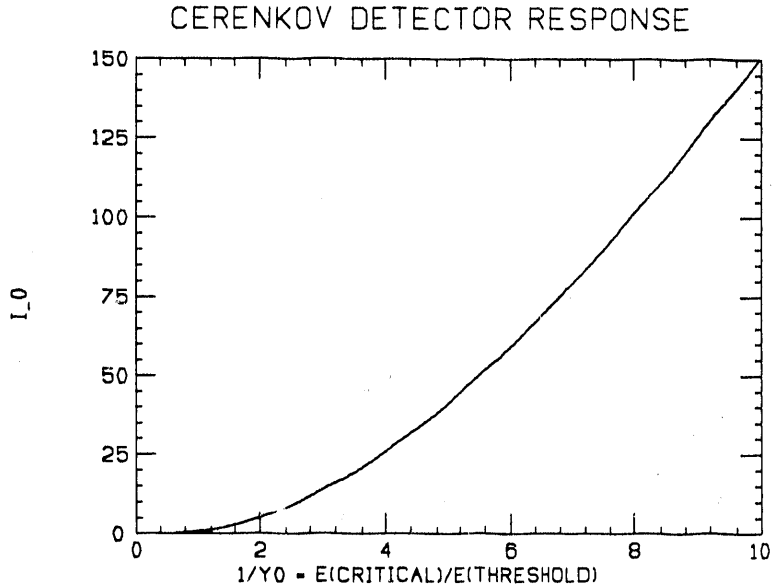

Fig. 1: $I_0(\gamma_0, y_0)$ as a function of $1/y_0$ for $\gamma_0 = 50$

with $\frac{1}{\sqrt{3}\pi}\frac{\alpha^2}{r_e mc} = 1.27\,10^{31}$ 1/Js. In order to calculate the number of Cerenkov monitor counts we have to convolute the response of the detector, eq. 5 and the spectrum, eq. 10. After some algebra we arrive at

$$\begin{aligned}\frac{dN_c}{dt} &= \frac{4}{\sqrt{3}}\alpha^4 Z^2 \eta \frac{N dl r_e}{\gamma_0 \gamma^2}\Delta\nu \\ &\times \frac{1}{y_0}\int_{y_0}^{\infty} dy \left[\frac{28}{9}\ln(2\gamma_0) - \frac{218}{27} - \frac{28}{9}\ln(y_0) + \frac{28}{9}\ln(y)\right] \\ &\times (1 - \frac{y_0}{y})^2 \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x)\end{aligned} \tag{11}$$

where $y_0 = E_{Cutoff}/\varepsilon_c$ denotes the ratio of the Cerenkov monitor threshold and the critical energy of the synchrotron radiation. The double integral appearing in the previous expression can be reduced to a single integral using partial integration. This is illustrated in the appendix. Here we only note that the entire expression can be written as

$$\frac{dN_c}{dt} = \frac{4}{\sqrt{3}}\alpha^4 Z^2 \eta \frac{N dl r_e}{\gamma_0 \gamma^2}\Delta\nu\, I_0(\gamma_0, y_0)\ . \tag{12}$$

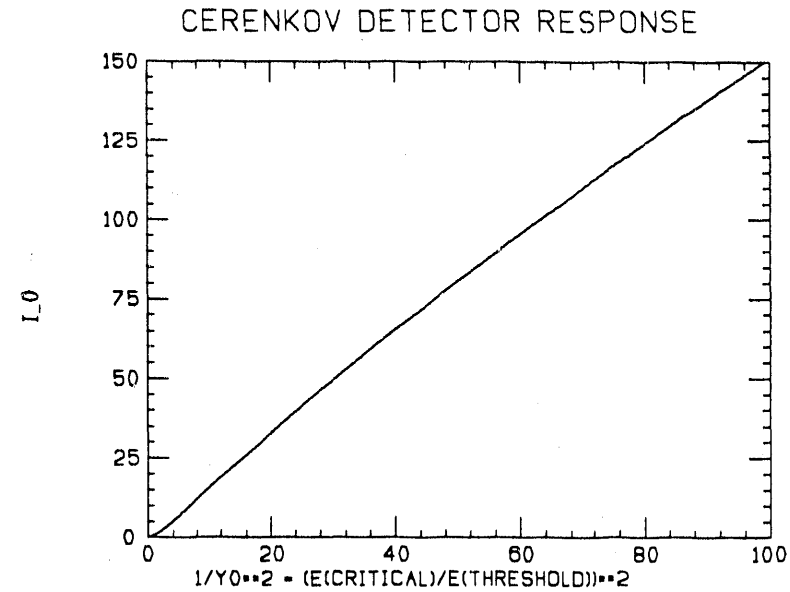

Fig. 2: $I_0(\gamma_0, y_0)$ as a function of $1/y_0^2$ for $\gamma_0 = 50$

with

$$
\begin{aligned}
I_0(\gamma_0, y_0) &= \left[\frac{28}{9}\ln(2\gamma_0) - \frac{218}{27} - \frac{28}{9}\ln(y_0)\right] \\
&\quad \times\left[2\ln(y_0)G_0(y_0) + \frac{1}{y_0}F_1(y_0) - 2F_0(y_0) - y_0F_{-1}(y_0)\right] \\
&\quad +\frac{28}{9}\Big\{(2 + [\ln(y_0)]^2)G_0(y_0) \\
&\qquad +\frac{1}{y_0}H_1(y_0) - H_0(y_0) - y_0H_{-1}(y_0)\Big\}
\end{aligned}
\tag{13}
$$

The functions $F_n, G_0$ and $H_n$ are defined in the appendix. The entire process of synchrotron emission, pair production, Cerenkov effect and photomultiplier response is simply parametrized by the function $I_0(\gamma_0, y_0)$. Since usually the Cerenkov threshold is fixed by the experimental conditions a plot of $I_0(\gamma_0, y_0)$ versus $1/y_0$ as shown in Fig. 1 is proportional to the Cerenkov monitor count per unit time as a function of the characteristic energy of the synchrotron radiation.

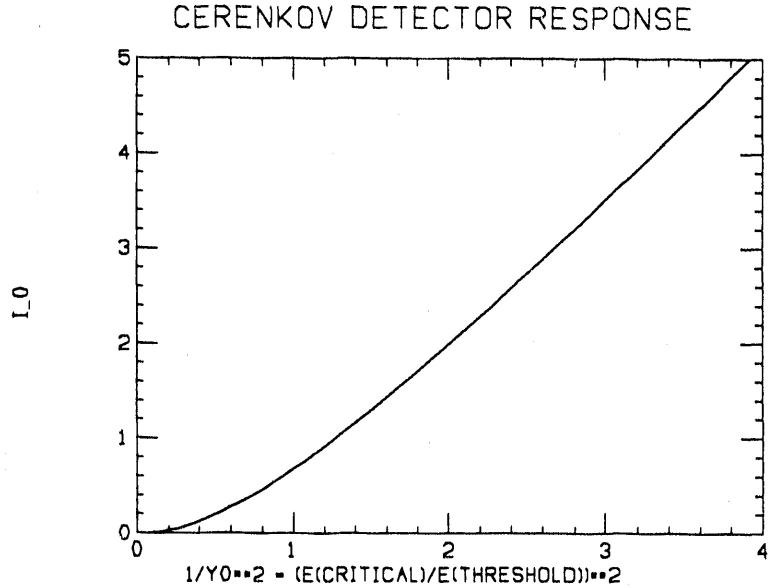


Fig. 3: $I_0(\gamma_0, y_0)$ as a function of $1/y_0^2$ for $\gamma_0 = 50$

Since the critical energy is inversely proportional to the bending radius by virtue of eq. 6, which in turn is inversely proportional to the electric or magnetic field experienced by the radiating particles Fig. 1 effectively shows the response of the Cerenkov monitor to the fields. These fields can either be the magnetic fields of bending magnets or electric fields that cause the electrons to be deflected in the beam-beam interaction. In the first case the magnetic field is constant while the radiating particles traverse the magnet whereas in the latter case the electric field varies as the radiating particle traverses the oppositely running bunch. The total Cerenkov counts have then to be calculated by integrating over the interaction time of the two bunches and averaging over their transverse extension.

Plotting $I_0$ versus $1/y_0^2$ as shown in Fig. 2 one obtains almost a straight line. Since $1/y_0$ is proportional to $\varepsilon_c$ this indicates that in a regime where $\varepsilon_c \gg E_{Cutoff}$ the monitor response is proportional to the total emitted power, which is proportional to $1/\rho^2$ and $\varepsilon_c$ is proportional to $1/\rho$. This means that the Cerenkov monitor response of a beam-beam scan is essentially determined by the total emitted energy.

For small ratios $1/y_0 = \varepsilon_c/E_{Cutoff}$ $I_0$ falls to zero much quicker than $1/y_0^2$ as shown in Fig. 3. This means that for critical energies much smaller than the Cerenkov threshold the response of the monitor is strongly suppressed, as desired. However, it is not zero, since even spectra with small critical energies have high energy tails, even though these tails are suppressed exponentially.

Now we want to compare the counts from the strong bending magnet ($\varepsilon_c$ = 2 MeV, Length = 7.6 m) with those stemming from the beam-beam interaction with two bunches containing $2.5\,10^{10}$ particles and having spatial extensions of 1 mm longitudinally and 2 $\mu$m transversely. In the latter case the average critical energy is about 20 MeV. Taking the different interaction times into account we conclude that the counts from the beam-beam interaction are about 300 times higher.

A second effect of the depression of $I_0$ for small critical energies is a depletion of counts from soft photons from the beam-beam interaction. Soft photons are mainly generated in the (longitudinaly and transverse) tails of the bunch that generates the deflecting field. In a transverse beam-beam scan "Beamstrahlung vs. Offset" this means that the tails of the scans are depleted compared to simulated scans "Total emitted Energy vs. Offset".

# 4 Scaling relations for the Beamstrahlung Flux

We now want to exploit the previous results to obtain scaling relations of the beamstrahlung flux with the bunch length and the number of particles in the interacting beams.

To this end we note that the transverse electric field of a gaussian target bunch running toward the negative $z$-axis is given by

$$E(x,y,z,t) = -\frac{2eN_{target}}{4\pi\varepsilon_0}F(x,y)\frac{1}{\sqrt{2\pi}\sigma_z}e^{-\frac{(z+ct)^2}{2\sigma_z^2}} \tag{14}$$

where $F(x,y)$ denotes the transverse dependence. For a round beam it is given by the well known result $F(x,y) = (1 - e^{r^2/2\sigma^2})/r$ with $r^2 = x^2 + y^2$

and for the general elliptic case given by [7]

$$F(X_1, X_2) = \frac{\sqrt{\pi}}{\sqrt{2(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})}} \left\{ w \left[ \frac{X_1 + iX_2}{\sqrt{2(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})}} \right] - \epsilon^{-\frac{1}{2}\Sigma^2_{i,j}(\sigma^{-1})_{ij} X_i X_j} w \left[ \frac{(\sigma_{22} - i\sigma_{12})X_1 + i(\sigma_{11} + i\sigma_{12})X_2}{\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\sqrt{2(\sigma_{11} - \sigma_{22} + 2i\sigma_{12})}} \right] \right\} \quad (15)$$

where $\sigma_{ij}$ are the matrix elements of the covariance matrix of the target beam, i.e. $\sigma_{11} = \sigma_x^2$, etc.

The electric field a source particle travelling to the positive $z$-axis experiences as it passes through the target bunch is given by substituting $z = ct$ in the expression for the electric field. Now the bending radius $\rho$ experienced by the source particle can be determined from $F\,dt = \Delta p_\perp = p\,d\Theta = pcdt/\rho$ where the force on the source particle is given by $F = 2\epsilon E$ (assuming equal contributions from the electric and magnetic fields) and $p = \gamma mc$ is the momentum of the source particle. Solving for $1/\rho$ we obtain

$$\frac{1}{\rho(x, y, t)} = \frac{2eE}{pc} = \frac{2r_e}{\gamma_{source}} \left( \frac{4\pi\varepsilon_0 E(x, y, z = ct, t)}{\epsilon} \right) . \quad (16)$$

Inserting $E$ in the right hand side and organizing terms, we get

$$\frac{1}{\rho(x, y, t)} = \Theta(x, y) \frac{1}{\sqrt{2\pi}(\sigma_z/2)} \epsilon^{-\frac{(ct)^2}{2(\sigma_z/2)^2}} \quad (17)$$

with $\Theta(x, y) = -\frac{2r_e}{\gamma} F(x, y)$. This expression can be interpreted easily. As the source particle traverses the target bunch its total deflection angle $\Theta(x, y) = \int_{-\infty}^{\infty} \frac{c\,dt}{\rho(x,y,t)}$ is entirely determined by its initial offset. However, the length of the target bunch determines at what rate the deflection occurs. In the light of eq. 6 this obviously determines the critical energy of the emitted synchrotron radiation. Note that in eq. 17 $\sigma_z/2$ appears, since the two bunches approach each other with the speed of light. Thus the lengths of the target bunch appears effectively halved.

In order to estimate the scaling relations of the beamstrahlung flux w note that in the regime $\varepsilon_c/\varepsilon_0 \leq 1$ $I_0(\gamma_0, \varepsilon_c/\varepsilon_0)$ can be approximated by $I_0 \approx 0.699(\varepsilon_c/\varepsilon_0)^4$ (with errors of about 3 %). Of course we can rewrite this in terms of the bending radius and get $I_0 \approx 0.699(\rho_0/\rho)^4$ where $\rho_0$ is the bending radius at which the critical energy equals the cutoff energy of the Cerenkov monitor. From eq. 12 we can now easily determine the total number of Cerenkov photons emitted by one electron with an offset $x, y$ relative to the target bunch as

$$N_c(x,y) \propto \int_{-\infty}^{\infty} dt\, I_0 \;\propto\; \int_{-\infty}^{\infty} dt \left(\frac{1}{\rho(x,y,t)}\right)^4 . \tag{18}$$

Inserting $1/\rho$ and evaluating the integral we are led to the following scaling relation

$$N_c(x,y) \propto \Theta^4(x,y)\frac{N_{target}^4 N_{source}}{\sigma_z^3\,\gamma_{source}^4} . \tag{19}$$

Here it becomes clear that the magnitude of the Cerenkov photon flux is mainly determined by the number of particles per bunch, the bunch length, and the particle energy. The transverse profile, however, is mainly determined by the transverse beamsizes entering in $\Theta$.

In order to investigate the influence of the transverse beamsizes we have to average over the incoming transverse (source-) electron distribution. This leads in a straight forward manner to a simulation algorithm, which will be the topic of a forthcoming note.

# 5 References


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[7 ] V. Ziemann, in preparation

# 6 Appendix: Evaluation of $I_0(\gamma_0, y_0)$

In this appendix we will reduce the double integral appearing in eq. 10 to simple integrals. $I_0(\gamma_0, y_0)$ is defined as follows

$$\begin{aligned} I_0(\gamma_0, y_0) &= \frac{1}{y_0}\int_{y_0}^{\infty} dy \Big\{\frac{28}{9}\ln(2\gamma_0) - \frac{218}{27} - \frac{28}{9}\ln(y_0) + \frac{28}{9}\ln(y)\Big\} \\ &\qquad \times \Big(1 - \frac{y_0}{y}\Big)^2 \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x) \qquad (20) \\ &= \frac{1}{y_0}\Big\{\frac{28}{9}\ln(2\gamma_0) - \frac{218}{27} - \frac{28}{9}\ln(y_0)\Big\} I_1(y_0) + \frac{28}{9}\frac{1}{y_0} I_2(y_0) \end{aligned}$$

with

$$\begin{aligned} I_1(y_0) &= \int_{y_0}^{\infty} dy \Big(1 - \frac{y_0}{y}\Big)^2 \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x) \qquad (21) \\ I_2(y_0) &= \int_{y_0}^{\infty} dy \ln(y) \Big(1 - \frac{y_0}{y}\Big)^2 \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x)\ . \end{aligned}$$

Now consider $I_1(y_0)$ and expand

$$\begin{aligned} I_1(y_0) &= \int_{y_0}^{\infty} dy \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x) - 2y_0 \int_{y_0}^{\infty} \frac{dy}{y} \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x) \\ &\quad + y_0^2 \int_{y_0}^{\infty} \frac{dy}{y^2} \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x)\ . \qquad (22) \end{aligned}$$

These integrals can be evaluated using

$$\begin{aligned} \int_{y_0}^{\infty} dy\, y^{\alpha} \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x) &= \frac{y^{\alpha+1}}{\alpha+1} \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x)\Big|_{y_0}^{\infty} \\ &\quad - \int_{y_0}^{\infty} dy \frac{1}{\alpha+1}\big( - \mathrm{K}_{5/3}(y)\big) \qquad (23) \\ &= \frac{1}{\alpha+1}\Big\{F_{\alpha+1} - y_0^{\alpha+1} G_0(y_0)\Big\} \qquad \text{for } \alpha \neq -1 \\ \int_{y_0}^{\infty} \frac{dy}{y} \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x) &= \ln(y) \int_y^{\infty} dx\, \mathrm{K}_{5/3}(x)\Big|_{y_0}^{\infty} \\ &\quad - \int_{y_0}^{\infty} dy \ln(y)\big( - \mathrm{K}_{5/3}(y)\big) \\ &= -\ln(y_0) G_0(y_0) + F_0(y_0) \qquad \text{for } \alpha = -1 \end{aligned}$$

with

$$
\begin{aligned}
G_0(y_0) &= \int_{y_0}^{\infty} dy\, \mathrm{K}_{5/3}(y) \\
F_0(y_0) &= \int_{y_0}^{\infty} dy\, \ln(y) \mathrm{K}_{5/3}(y) \\
F_n(y_0) &= \int_{y_0}^{\infty} dy\, y^n \mathrm{K}_{5/3}(y) \quad \text{for } n \neq 0 \, .
\end{aligned}
\tag{24}
$$

These expressions allow us to write $I_1(y_0)$ as

$$
I_1(y_0) = 2y_0 \ln(y_0) G_0(y_0) + F_1(y_0) - 2y_0 F_0(y_0) - y_0^2 F_{-1}(y_0) \, . \tag{25}
$$

The integral $I_2(y_0)$ can be tackeled in very much the same way. We first expand the square

$$
\begin{aligned}
I_2(y_0) &= \int_{y_0}^{\infty} dy\, \ln(y) \int_{y}^{\infty} dx\, \mathrm{K}_{5/3}(x) \\
&\quad -2y_0 \int_{y_0}^{\infty} dy\, \frac{\ln y}{y} \int_{y}^{\infty} dx\, \mathrm{K}_{5/3}(x) \\
&\quad +y_0^2 \int_{y_0}^{\infty} dy \frac{\ln y}{y^2} \int_{y}^{\infty} dx\, \mathrm{K}_{5/3}(x) \, .
\end{aligned}
\tag{26}
$$

In order to use partial integration again we need the following integrals [5]

$$
\begin{aligned}
\int dx\, \frac{\ln x}{x} &= \frac{1}{2} [\ln x]^2 \\
\int dx\, \frac{\ln x}{x^m} &= -\frac{1}{(m-1)x^{m-1}} \left[ \ln x + \frac{1}{m-1} \right] \, .
\end{aligned}
\tag{27}
$$

Now we can tackle the three integrals. They can be written as

$$
\begin{aligned}
\int_{y_0}^{\infty} dy\, \ln(y) \int_{y}^{\infty} dx\, \mathrm{K}_{5/3}(x) &= \int_{y_0}^{\infty} dy\, y [\ln(y) - 1]\, \mathrm{K}_{5/3}(y) \\
&\quad -y_0 [\ln(y_0) - 1] \int_{y_0}^{\infty} dx\, \mathrm{K}_{5/3}(x) \\
\int_{y_0}^{\infty} dy \frac{\ln y}{y} \int_{y}^{\infty} dx\, \mathrm{K}_{5/3}(x) &= -\frac{1}{2} [\ln(y_0)]^2 \int_{y_0}^{\infty} dx\, \mathrm{K}_{5/3}(x) \\
&\quad +\frac{1}{2} \int_{y_0}^{\infty} dy [\ln(y)]^2 \mathrm{K}_{5/3}(y)
\end{aligned}
\tag{28}
$$

$$\int_{y_0}^{\infty} dy \frac{\ln y}{y^2} \int_{y}^{\infty} dx\, \mathrm{K}_{5/3}(x) = \frac{\ln(y_0)+1}{y_0} \int_{y_0}^{\infty} dx\, \mathrm{K}_{5/3}(x) - \int_{y_0}^{\infty} dy \frac{\ln(y)+1}{y} \mathrm{K}_{5/3}(y) \; .$$

Combining these expressions and inserting into the expression for $I_2(y_0)$ we obtain

$$I_2(y_0) = \left(2 + [\ln(y_0)]^2\right) y_0 G_0(y_0) + H_1(y_0) - y_0 H_0(y_0) - y_0^2 H_{-1}(y_0) \quad (29)$$

where we have introduced the abbreviations

$$\begin{aligned} H_1(y_0) &= \int_{y_0}^{\infty} dy\, y[\ln(y) - 1]\, \mathrm{K}_{5/3}(y) \\ H_0(y_0) &= \int_{y_0}^{\infty} dy\, [\ln(y)]^2\, \mathrm{K}_{5/3}(y) \\ H_{-1}(y_0) &= \int_{y_0}^{\infty} dy\, \frac{\ln(y)+1}{y}\, \mathrm{K}_{5/3}(y) \; . \end{aligned} \quad (30)$$

Collecting terms we finally arrive at the expression for $I_0(\gamma_0, y_0)$ quoted in eq. 13.

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