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CONTINUUM REGULARIZATION OF GAUGE THEORY WITH FERMIONS*

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ABSTRACT

The continuum regularization program is discussed in the case of d -dimensional gauge theory coupled to fermions in an arbitrary representation. Two physically equivalent formulations are given. First, a Grassmann formulation is presented, which is based on the two-noise Langevin equations of Sakita, Ishikawa and Alfaro and Gavela. Second, a non-Grassmann formulation is obtained by regularized integration of the matter fields within the regularized Grassmann system. Explicit perturbation expansions are studied in both formulations, and considerable simplification is found in the integrated non-Grassmann formalism.

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Chapter 1

Introduction

1.1 The Quest for a Satisfactory Regularization Scheme

The problem of ultraviolet divergences has plagued quantum field theory ever since its inception. A regularization scheme is therefore a prerequisite for meaningful mathematical computations and physical interpretations. While a naive momentum-cutoff may be sufficient for simple scalar theories, for theories that possess symmetries, such as gauge theory, the situation is more intricate: In addition to suppressing the ultraviolet divergences, a successful regularization scheme should also preserve the symmetries of the unregularized theory. To meet these requirements, a number of gauge-invariant regularization schemes have been proposed for gauge theory. Of these, the most practical to date has been the lattice [1], on which numerical simulation of hadron physics has seen considerable success. However, for many reasons, including loss of continuum topology on the lattice, the idea of continuum-regularized quantum field theory persists.

Among continuum regularization schemes the most popular to date is dimensional regularization [2], which is adequate for preserving gauge symmetries, but nevertheless suffers from the deficiency that at present it can only be perturbatively implemented. On the other hand, early attempts to regularize gauge theory nonperturbatively using higher covariant derivatives in the action failed, as pointed out in 1972 by Lee and Zinn-Justin [3], due to the fact that the regularization of

the propagators was cancelled by higher-derivative vertices at the one-loop level. Slavnov [4] has since proposed a hybrid action scheme in the Faddeev-Popov approach [5] to covariant gauge-fixing. Defects in this scheme have recently been pointed out by Warr [6], who proposes a even more complicated scheme which is not actually a gauge invariant regularization. Any such action scheme must be considered perturbative until the Gribov ambiguity [7], intrinsic to the Faddeev-Popov approach, is resolved. Two geometric approaches have also been proposed by Asorey and Mitter [8], and by Singer [9], but, in both cases, no detailed perturbative analysis is given.

The idea of a viable continuum nonperturbative regularization scheme for gauge theory remains attractive because it may enable us to address interesting problems such as continuum confinement and general nonperturbative properties of quantum field theory. With the introduction of stochastic quantization by Parisi and Wu [10] in 1981 as an alternate formulation of quantum theory, we see renewed efforts in this direction. Stochastic quantization reduces quantum field theory to a stochastic process governed by a Langevin equation [11] with Gaussian noise fields, which usually runs in an auxiliary Markov time. This new non-action formulation is particularly suitable for the study of gauge theory, as it allows the covariant nonperturbative Zwanziger gauge-fixing [12-14], which circumvents the Gribov ambiguity.

The first continuum-regularization scheme utilizing the stochastic formulation was proposed by Breit, Gupta and Zaks [15]. Their scheme is based on smear-

ing in Markov time, which presumably does not affect any symmetry in physical spacetime. This non-Markovian regularization is not a satisfactory nonperturbative scheme for QCD_4 , however, since the superficial quadratic divergences force a bottomless action for the noise [15,16]. Additionally, Markov time smearing is incompatible [16] with Zwanziger gauge-fixing, and with other gauge-fixing as well [17].

1.2 The Continuum Regularization Scheme of Bern, Halpern, Sadun and Taubes

A more recent proposal motivated by the stochastic formulation is the presumably nonperturbative covariant-derivative continuum-regularization scheme developed by Bern, Halpern, Sadun and Taubes [18-20]. The scheme regularizes gauge theory by a Markovian gauge-invariant displacement of the noise. In distinction to the Markov time smearing mentioned above, all the technical advantages of a Markov process are retained, including closed form equilibrium equations, which are second-order Schwinger-Dyson (SD) equations, and Zwanziger gauge-fixing if desired. The resulting regularized gauge theory is manifestly Lorentz invariant, gauge invariant, ghost-free and ultraviolet finite to all orders.

Although the original target of this continuum-regularization scheme is 4-dimensional Yang-Mills, the principles of the scheme are in fact adequate for the regularization of any quantum field theory of interest in d -dimensional spacetime. Moreover, the SD formulation, which was studied originally as the equilibrium (large Markov time) equivalent of the stochastic formulation, is found to be of

wider application. For example, the regularized SD equations, which do not contain any Markov time-dependence, are still valid even if the corresponding regularized stochastic process fails to equilibrate. As we shall see in subsequent chapters, the regularized SD equations may be considered as a *regularization* of formal *unregularized* statements obtainable from the conventional action formulations, and is an appropriate regularization scheme in its own right.

In general, the regularization is in terms of a regulator $R(\Delta)$, which is a function of an appropriate covariant spacetime Lapacian Δ . At the level of $(d + 1)$ -dimensional regularized Langevin systems with auxiliary Markov time, $R(\Delta)$ appears as a covariant displacement of the noise, while at the level of d -dimensional regularized SD equations $R(\Delta)$ enters in a covariant *regularized functional Laplacian*.

This general continuum regularization program has been successfully applied to the scalar prototype [19] and gauge theory [18,20-24]. In a series of papers, Bern *et al.* give detailed perturbative analysis and study renormalization aspects of these theories both at the Langevin [22] and SD [23] levels. Gauge invariance is explicitly verified in perturbation theory by the vanishing one-loop gluon mass [20,24], and the usual one-loop β -function is also obtained [21]. More recent advances include regularization of supersymmetry [27] and coordinate-invariant regularization [28-31], including phase-space regularization and general geometric interpretation of the program, with explicit applications to general non-linear sigma model [28,29] and Euclidean gravity [30]. A brief overview of the general regularization program

will be given in Chapter 2.

1.3 Continuum Regularization of Gauge Theory with Fermions

The main topic of this dissertation is the application of the continuum regularization program to d -dimensional gauge theory coupled to fermions in an arbitrary representation. The starting point of the analysis is the two-noise equations developed by Sakita [32], Ishikawa [33] and Alfaro and Gavela [34] (SIAG equations). These stochastic equations have been developed in their unregularized form to overcome the technical difficulties encountered in earlier attempts to extend the stochastic formulation to fermions by using a naive one-noise Langevin equation [15]. The technical advantages of the SIAG equations includes manifest gauge-invariance and weak-coupling equilibration. In Chapter 3, continuum regularization of gauge theory with fermions will be given in terms of Markovian-regularized SIAG equations, and the equivalent regularized Schwinger-Dyson equations. This treatment is called a Grassmann formulation because the fermion fields ψ , $\bar{\psi}$ and the fermionic noise fields η_1 , $\bar{\eta}_1$, η_2 and $\bar{\eta}_2$ are anticommuting Grassmann fields.

In Chapter 4, I study the alternate non-Grassmann formulation of regularized gauge theory with fermions, in which the Grassmann fermion fields are *integrated* out. Although this formulation is physically equivalent to the Grassmann formulation of Chapter 3, the weak-coupling expansion of the non-Grassmann formulation is much simpler. In fact, this simplification is expected for any regularized formulation with the quadratic matter fields integrated. As a further example, the

integrated regularized formulation of scalar electrodynamics is also given.

In Chapter 5, I focus on the relation of the integrated systems of Chapter 4 to the regularized Grassmann SIAG systems of Chapter 3. A λ -family of regularized SIAG systems is given, with the case $\lambda = 1$ corresponding to the systems of Chapter 3. It is demonstrated that the large λ limit of this SIAG λ -family is the integrated non-Grassmann formulation. Techniques introduced are equivalent to *regularized integration* of matter field *within* the regularized theories. The case of scalar electrodynamics is also included as an illustration. As will be discussed in Chapter 6, these techniques are useful in many other applications [29,31].

Finally, I give some concluding remarks in Chapter 6, and discuss the outlook for the general continuum regularization program.

Chapter 2

Overview of the Continuum Regularization Program

2.1 Langevin and Schwinger-Dyson Formulations

The continuum regularization scheme of Bern *et al.* is a covariant-derivative regularization designed to preserve continuum symmetries and topology. As mentioned in Chapter 1, there are two *equivalent* formulations for the regularization of a d -dimensional field theory: (a) Regularized Markovian Langevin systems in $(d+1)$ -dimensions, or (b) regularized Schwinger-Dyson (SD) equations in d -dimensions. Since the scheme is not an action regularization, it does not suffer from the problem of Lee and Zinn-Justin [3] encountered in the *action* covariant-derivative regularization. Further details of this point may be found in Refs. [20,25].

Each of the two equivalent regularized formulations enjoys distinctive advantages. The Langevin approach has the major advantage of being similar to a standard action formalism, so many of the standard techniques of action-based quantum field theory can be applied. Furthermore, the perturbative expansion of regularized Langevin systems in terms of regularized Langevin tree diagrams is more intuitive compared to the SD pictures and diagrams. However, these advantages are counterbalanced by the need to introduce an unphysical Markov time, resulting in more complicated computations. On the other hand, SD equations are formulated in physical spacetime, and computation of specific SD diagrams tend to

be simpler than their Langevin counterparts since no integration over Markov time is needed. Moreover, SD equations are well-defined without regard for stochastic equilibration or choice of stochastic calculus. In particular, the SD equations are applicable in situations where the Langevin formulation is questionable. For example, formulations in Minkowski space, the “naive” fermion equations mentioned in Chapter 3, and in cases where the naive action formulations are unbounded, such as Euclidean gravity studied in Refs. [28-31].

In general, for a given formal theory, defined as an action functional integral with certain symmetries, the program always proceeds in two basic steps:

- 1) Obtain a formally equivalent stochastic or SD formulation.
- 2) Apply covariant-derivative regularization

$$R(\Delta) = \begin{cases} \exp(\Delta/\Lambda^2), & \text{Euclidean} \\ (1 - \Delta/\Lambda^2)^{-m}, & \text{Euclidean-Minkowski} \end{cases} \quad (2.1)$$

where Λ is the ultraviolet cutoff, R is the regulator and Δ is the relevant covariant spacetime Laplacian. The exponential or heat-kernel [24] regulator is guaranteed to regularize any Euclidean theory, while the power-law regularization [19,20] is expected to succeed as well for m not less than some theory-dependent critical power. In addition, the power-law regulator presumably allows Euclidean-Minkowski rotation at finite cutoff.

In this chapter, I will give a brief overview of the regularization program using concrete examples from scalar theory and gauge theory. Special emphasis will be

given to regularization mechanisms, both at the Langevin and SD levels.

2.2 Scalar Prototype

Since it exhibits most of the ideas in the simplest context, I begin the discussion with the scalar prototype [19]. For a theory whose formal action is $S[\phi]$, the regularized Langevin system is given by

$$\dot{\phi}(x, t) = \frac{\delta S}{\delta \phi}(x, t) + \int (dy) R(\square)_{xy} \eta(y, t) , \quad (2.2a)$$

$$\langle \eta(x, t) \eta(y, \tau) \rangle_\eta = 2\delta^d(x - y) \delta(t - \tau) . \quad (2.2b)$$

Here the overdot is Markov time derivative, $(dy) \equiv d^d y$, the regulator R is a function of $\square = \partial_\mu \partial_\mu$, and $R = 1$ regains the Parisi-Wu formulation [10]. As in the unregularized Parisi-Wu formulation, the connection to the standard formulation is established by the prescription for computing Euclidean field averages of any functional $F[\phi]$ as the large equal Markov time limit of the stochastic averages,

$$\langle F[\phi(\cdot)] \rangle = \lim_{t \rightarrow \infty} \langle F[\phi(\cdot, t)] \rangle_\eta . \quad (2.3)$$

The regularization is trivial in this case since the covariant Laplacian Δ of eq. (2.1) reduces to the ordinary Laplacian $\square$. Nevertheless, as we shall see below, this simple example is sufficient to illustrate the general mechanism of regularization.

The regularized Langevin system (2.2) can be used to perturbatively solve quantum field theory. In general, the action $S[\phi]$ consists of a kinetic term plus

an interaction potential V ,

$$S = \int (dx) \left\{ \frac{1}{2} \phi (-\partial^2 + m^2) \phi + V(\phi) \right\}. \quad (2.4)$$

The regularized Langevin equation (2.2a) can then be rewritten in the integral form

$$\phi(x, t) = \int_{-\infty}^t dt' (dy) G(x - y; t - t') \left[-\frac{dV}{d\phi}(y, t') + \int (dz) R(\square)_{yz} \eta(z, t') \right] \quad (2.5)$$

in terms of the Langevin Green function

$$G(x - y; t - t') = \theta(t - t') \int (dp) e^{-ip \cdot (x - y)} e^{-(p^2 + m^2)(t - t')}, \quad (2.6)$$

where $(dp) \equiv d^d p / (2\pi)^d$. In eq. (2.5), the initial Markov time is taken to be $-\infty$ to ensure that the system is at equilibrium at any finite Markov time.

Diagrammatically, the Langevin Green function is represented in Fig. 2.1 as a wavy line with a solid arrow indicating the direction of decreasing Markov time, which is a feature of the *retarded* nature of the Green function (2.6). The regulator R is represented by a straight line as shown in Fig. 2.2, and the noise

$$G(x - y; t - t') = \text{wavy line from } x \text{ to } y \text{ with a solid arrow pointing from } x \text{ to } y$$

Fig. 2.1 Langevin Green function.

$$R = \text{straight line}$$

Fig. 2.2 Scalar regulator.

$$\phi^{(0)} = \overset{G}{\text{wavy line with arrow}} \overset{R}{\text{straight line}} \times \eta$$

Fig. 2.3 Zeroth-order Langevin field.

η is denoted by a cross as usual. The simplest Langevin tree diagram is that of Fig. 2.3, which corresponds to the zeroth-order Langevin field

$$\phi^{(0)}(x, t) = \int_{-\infty}^t dt' \int dy G(x - y; t - t') \int (dz) R(\square)_{yz} \eta(z, t'). \quad (2.7)$$

By a standard iteration procedure, interaction vertices are generated by the term $dV/d\phi$ in eq. (2.5). For $V \propto \phi^N$, each vertex consists of a single incoming Langevin Green function and $N - 1$ outgoing Langevin Green functions, as depicted in Fig. 2.4. As an explicit example, the regularized Langevin tree expansion through order λ^2 is shown in Fig. 2.5 for $V = \lambda\phi^3/3!$.

The regularized Langevin diagrams themselves, which are representation of n -point Green functions $\langle \phi(x_1, t) \cdots \phi(x_n, t) \rangle$, are constructed by contracting the

Fig. 2.4 A Langevin vertex with $N - 1$ outgoing Green functions.

Fig. 2.5 Regularized Langevin trees.

regularized Langevin trees according to the rule (2.2b). Using the example of Fig. 2.5, diagrams for the two-point function through order λ^2 are shown in Fig. 2.6. The important observation here is that the stochastic formulation has found the ultraviolet structure of the theory, which we have then regularized: Since at least one contraction is necessary to form a loop, and each contraction carries a factor of R^2 , it follows that each loop contains at least one power¹ of R^2 and is therefore regularized by suitable choices of R .

It is then a matter of power counting to establish the following results to all orders in weak coupling for any polynomial interaction $V \propto \phi^N$ in d -dimensions: The n -point Green functions of the theory are finite with the power-law regulator in (2.1) for $m \geq [(d+2)/4]$, where $[x]$ is the greatest integer less than or equal to x . With heat-kernel regularization, all finite-derivative composite operators are successfully regularized in any dimension.

Fig. 2.6 Regularized Langevin diagrams.

The equivalent d -dimensional regularized SD equations are

$$0 = \langle LF[\phi] \rangle, \quad (2.8a)$$

$$L = - \int (dx) \frac{\delta S}{\delta \phi(x)} \frac{\delta}{\delta \phi(x)} + \Delta, \quad (2.8b)$$

$$\Delta = \int (dx)(dy) R_{xy}^2(\square) \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(y)}, \quad (2.8c)$$

where $F[\phi]$ is any field functional, and the regulator appears in the regularized functional Laplacian Δ . The SD equations (2.8) describe the stochastic formulation (2.2) at equilibrium. Alternately, it may be considered as a regularization of the formal statement

$$0 = \int \mathcal{D}\phi \int (dx) \frac{\delta}{\delta \phi(x)} \left\{ e^{-S} \frac{\delta}{\delta \phi(x)} F[\phi] \right\} \quad (2.9)$$

that can be obtained from the unregularized action formalism. The weak-coupling expansion of the SD eqs. (2.8) proceeds through regularized SD diagrams [19,20], which may also be obtained from the regularized Langevin diagrams by performing all integrations over Markov time.

Since the rules for constructing these regularized SD diagrams have been discussed in detail in Refs. [19], I restrict myself to some brief remarks. A general principle to keep in mind is that a generic term $\phi^N[\delta^M/\delta\phi^M]$ ($M \leq 2$) in the SD operator L in eq. (2.8) generates a vertex with M incoming and N outgoing field lines. In particular, the solid-line factors [19] are generated by the kinetic term of the action S , while the interaction vertices are generated by the interaction term in $[\delta S/\delta\phi]\delta/\delta\phi$. The loop structure is controlled by the regularized functional Laplacian Δ , which connects two field lines ($M = 2$) through the factor R^2 , thus provides regularization to every loop.

2.3 Gauge Theory

The generalization to gauge theory is straightforward. In close analogy with the scalar prototype, the regularized Parisi-Wu system for gauge theory is

$$\dot{A}_\mu^a = -\frac{\delta S_{\text{YM}}}{\delta A_\mu^a}(x, t) + D_\mu^{ab} Z^b(x, t) + \int (dy) R_{xy}^{ab}(\Delta) \eta_\mu^b(y, t), \quad (2.10a)$$

$$\langle \eta_\mu^a(x, t) \eta_\nu^b(y, \tau) \rangle_\eta = 2\delta^{ab} \delta_{\mu\nu} \delta^d(x - y) \delta(t - \tau). \quad (2.10b)$$

Here S_{YM} is the Yang-Mills action, and Z^a is Zwanziger's [12] nonperturbative gauge-fixing, and is usually chosen to be $\alpha^{-1} \partial \cdot A^a$ for computational purposes.

The regulator $R(\Delta)$, now a function of the covariant Laplacian $\Delta \equiv D_\mu D_\mu$, is a covariant displacement of the noise.

The field-dependence of the regulator $R(\Delta)$ gives rise to new features not encountered in the scalar prototype. Firstly, instead of a single regulator propagator, the regulator has to be expanded in weak-coupling as a series of *regulator strings*, as shown in Fig. 2.7, in terms of *regulator vertices* [20,24]. As we shall see below, these regulator vertices are crucial in maintaining gauge invariance. Secondly, the field-dependence of $R(\Delta)$ requires a choice of stochastic calculus [36].

Fig. 2.7 Regulator strings in gauge theory.

The regularized SD equations for gauge theory are

$$0 = \langle L_{\text{YM}} F[A] \rangle, \quad (2.11a)$$

$$L_{\text{YM}} = \int (dx) \left[-\frac{\delta S_{\text{YM}}}{\delta A_\mu^a(x)} + D_\mu^{ab} Z^b(x) \right] \frac{\delta}{\delta A_\mu^a(x)} + \Delta(\gamma), \quad (2.11b)$$

$$\begin{aligned} \Delta(\gamma) \equiv & \int (dx)(dy) [R^2(\Delta)]_{xy}^{ab} \frac{\delta}{\delta A_\mu^b(y)} \frac{\delta}{\delta A_\mu^a(x)} \\ & + \gamma \int (dx)(dy)(dz) R_{yz}^{bc} \frac{\delta R_{yz}^{ba}}{\delta A_\mu^c(z)} \frac{\delta}{\delta A_\mu^a(x)}. \end{aligned} \quad (2.11c)$$

The $\gamma = 1$ form of the regularized functional Laplacian $\hat{\Delta}$ is equivalent to the stochastic system (2.10) with Stratonovich calculus, while the simpler $\gamma = 0$ form arises with Ito calculus. Ito calculus is preferred for explicit computations, since terms that are proportional to the field derivative of the regulator, $\delta R/\delta A$, are omitted. Nevertheless, gauge-invariant regularization is achieved for all γ [20,24]: Finiteness of Green functions in d -dimensions requires $m \geq [(d+1)/2]$ for the power-law regulator in (2.1) when $\gamma \neq 0$, and $m \geq [(d+3)/4]$ when $\gamma = 0$. With heat-kernel regulator, all finite derivative composite operators are regularized in any dimension.

Zwanziger gauge-fixing may be understood at the stochastic level as a damping term that prevents runaway behavior tangential to the gauge direction in the Langevin eq. (2.10). Moreover, the corresponding term in the SD formulation,

$$\int (dx) D_{\mu}^{ab} Z^b(x) \frac{\delta}{\delta A_{\mu}^a(x)} = - \int (dx) Z^a(x) G^a(x) , \quad (2.12)$$

is proportional to the generator of infinitesimal gauge transformation

$$G^a \equiv D_{\mu}^{ab} \frac{\delta}{\delta A_{\mu}^b} \quad (2.13)$$

that vanishes on any gauge-invariant functional F_{GI} . The Zwanziger term therefore does not affect any physical gauge-invariant quantities.

As an illustration of the regulator's role in maintaining gauge invariance, I reproduce in Fig. 2.8 the three $\gamma = 0$ diagrams that contribute to d -dimensional $\text{SU}(N)$ one-loop gluon mass, originally computed with heat-kernel regularization

Fig. 2.8 Zero gluon mass in d -dimensions.

by Bern, Halpern and Kalivas [24]. The first two diagrams are regularized ordinary [20] diagrams, while the dotted box of the third diagram encloses a regulator vertex. The contribution of each diagram to zero-momentum vacuum polarization (in units of $\delta_{\mu\nu}\delta^{ab}(\Lambda^2/2)^{d/2-1}[2Ng^2/d(4\pi)^d]$) is also shown, and it is easily checked that the gluon remains massless in all dimensions.

For further details of various aspects of regularized gauge theory the readers are referred to the original papers [18-24]. A brief introduction may also be found in Ref. [25].

Footnotes: Chapter 2

¹ A similar observation in a different context was made by Niemi and Wijewardhana [35] in 1982.

Chapter 3

Grassmann Formulation of Regularized Gauge Theory with Fermions

The extension of the continuum regularization program outlined in Chapter 2 to include fermions is straightforward. As usual, the regularization may be studied either at the $(d+1)$ -dimensional stochastic level, via Markovian-regularized Langevin systems, or at the d -dimensional level of the regularized SD equations. In this chapter, details are provided at both levels, with special emphases on fermionic features.

As mentioned in Chapter 1, the two-noise equations developed by Sakita, Ishikawa and Alfaro and Gavela (SIAG equations [32-34]) have emerged as an adequate fermionic extension of the Parisi-Wu program. These equations provide an almost bosonic stochastic description of fermions, and have been applied, with stochastic regularization by Markov-time smearing [15,16], to the study of anomalies in background gauge fields [37-40] and vacuum polarization in QED [40]. These SIAG equations are adopted here as an adequate vehicle for our regularization at the stochastic level. It should be emphasized however that other satisfactory stochastic formulations of fermions exist, such as that developed for numerical purposes in Ref.[41], and these also may be studied with our regularization. I also continue to employ Zwanziger's gauge-fixing, which in conjunction with our regularization provides an apparently non-perturbative description of QCD.

The organization of this chapter is as follows. Starting in section 3.1 at the $(d + 1)$ -dimensional level, the regularized Langevin systems are given for d -dimensional gauge theory coupled to fermions in any representation. The weak coupling expansion of these systems is discussed, and finally summarized in section 3.2 by a set of tree rules for the construction of the regularized Langevin diagrams to all orders.

In section 3.3, the rules are applied in a computation of the leading fermionic contribution to the gluon vacuum polarization in four dimensions (QCD). The fermionic contribution to the gluon mass is zero, verifying gauge-invariance of the regularized systems. The leading term exhibits the Zwanziger non-transversality (obtained also with dimensional regularization) previously observed in the Yang-Mills contribution [42,20,21]. The smooth approach of the Zwanziger gauge-fixed two-point Green function to the ordinary Landau gauge value [13] is also verified. For completeness, the analogous results for scalar electrodynamics [20] are also given.

Section 3.4 is an exposition of the regularization at the d -dimensional level, in terms of the regularized SD equations. Fermionic contributions to renormalization constants are computed in the SD renormalization scheme [19,21]. Finally, an alternate “naive” regularized SD formulation of gauge theory with fermions is discussed, which does not correspond to a stochastic system that equilibrates.

In section 3.5, regularized fermions in background gauge field are studied.

These systems provide adequate modelling for axial and chiral anomalies, but cannot be considered totally satisfactory as models of gauge coupling: In contrast to the regularized dynamical gauge theories, the problem of Lee and Zinn-Justin is not removed in background fields.

3.1 Regularized Langevin Systems for Gauge Theory with Fermions

In this section, covariant-derivative regularization is discussed at the $(d+1)$ -dimensional stochastic level for d -dimensional $SU(N)$ gauge theory coupled to Dirac fermions¹ in an arbitrary representation R of the gauge group. The Euclidean action is

$$S = \int (dx) \left[\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi}_i^A (\bar{D}_{ij}^{AB} + \delta^{AB} \delta_{ij} m) \psi_j^B \right] \quad (3.1)$$

where $F_{\mu\nu}^a(A)$ is the usual Yang-Mills field strength, as a function of the gauge field A_μ^a . Our fermionic notation is as follows. The Dirac fields ψ_i^A , $\bar{\psi}_j^B$ carry spinor sub-indices and capital letters which run over the representation, while the Dirac matrices $(\gamma_\mu)_{ij}$ and representation matrices $(T^a)^{AB}$ satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \gamma_\mu^\dagger = \gamma_\mu, \quad (3.2a)$$

$$[-iT^a, -iT^b] = f^{abc}(-iT^c), \quad (T^a)^\dagger = T^a, \quad (3.2b)$$

$$\text{Tr}[T^a T^b] = C_R \delta^{ab}, \quad (3.2c)$$

where C_R is the Dynkin index of representation R . Furthermore,

$$D_\mu^{ab} \equiv \delta^{ab} \partial_\mu + g f^{abc} A_\mu^c \quad (3.3a)$$

$$(\vec{\mathcal{D}})_{ij}^{AB} \equiv (\gamma_\mu)_{ij} \vec{D}_\mu^{AB} , \quad \vec{D}_\mu^{AB} \equiv \delta^{AB} \vec{\partial}_\mu + ig A_\mu^a (T^a)^{AB} \quad (3.3b)$$

$$(\vec{\mathcal{D}}')_{ij}^{AB} \equiv (\gamma_\mu)_{ij} (\vec{D}')_\mu^{AB} , \quad (\vec{D}')_\mu^{AB} \equiv \delta^{AB} \vec{\partial}_\mu - ig A_\mu^a (T^a)^{AB} \quad (3.3c)$$

are the relevant covariant derivatives.

An adequate vehicle for the regularization of such theories at the stochastic level is the set of regularized and Zwanziger gauge-fixed SIAG-Langevin systems,

$$\dot{A}_\mu^a(x, t) = -\frac{\delta S}{\delta A_\mu^a}(x, t) + D_\mu^{ab} Z^b(x, t) + \int (dy) R_{xy}^{ab} \eta_\mu^b(y, t) , \quad (3.4a)$$

$$\begin{aligned} \dot{\psi}_i^A(x, t) = & (\vec{\mathcal{D}}_x^2 - m^2)_{ij}^{AB} \psi_j^B(x, t) + \int (dy) (R_{ij}^{AB})_{xy} (\eta_2^B)_j(y, t) \\ & - (\vec{\mathcal{D}}_x - m)_{ij}^{AB} \int (dy) (R_{jl}^{BC})_{xy} (\eta_1^C)_l(y, t) \end{aligned} \quad (3.4b)$$

$$- ig Z^a (T^a)^{AB} \psi_i^B(x, t) ,$$

$$\begin{aligned} \dot{\bar{\psi}}_i^A(x, t) = & \bar{\psi}_j^B(x, t) (\vec{\mathcal{D}}_x^2 - m^2)_{ji}^{BA} + \int (dy) (\bar{\eta}_1^B)_j(y, t) (R_{ji}^{BA})_{yx} \\ & + \int (dy) (\bar{\eta}_2^C)_l(y, t) (R_{ij}^{CB})_{yx} (\vec{\mathcal{D}}_x + m)_{ji}^{BA} \\ & + ig Z^a (T^a)^{BA} \bar{\psi}_i^B(x, t) , \end{aligned} \quad (3.4c)$$

in which the various Gaussian noise fields satisfy

$$\langle \eta_\mu^a(x, t) \eta_\nu^b(x', t') \rangle_\eta = 2 \delta^{ab} \delta_{\mu\nu} \delta(t - t') \delta(x - x') \quad (3.5a)$$

$$\langle (\eta_\alpha^A)_i(x, t) (\bar{\eta}_\beta^B)_j(x', t') \rangle_\eta = \delta_{\alpha\beta} \delta^{AB} \delta_{ij} \delta(t - t') \delta(x - x') . \quad (3.5b)$$

Here $\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2$ are Grassmann variables which anticommute among one another.

For computational purposes, $Z^a = \alpha^{-1} \partial \cdot A^a$ is chosen for the Zwanziger gauge-

fixing, and it will be checked below at the Schwinger-Dyson level that gauge-invariant quantities are independent of the gauge-fixing.

The Yang-Mills regulator $R(\Delta)$, mentioned in Chapter 2, has been discussed in detail in Refs. [18,20,24]. Here, a fermionic regulator $[\mathcal{R}_{ij}^{AB}(\mathcal{D}^2)]_{xy}$ has also been introduced, which is a function of the covariant fermionic Laplacian $\mathcal{D}^2$. The explicit form of the matrix elements of this Laplacian

$$[(\mathcal{D}^2)_{ij}^{AB}]_{xy} = (\bar{\mathcal{D}}_x^2)_{ij}^{AB} \delta(x-y), \quad (3.6a)$$

$$(\bar{\mathcal{D}}_x^2)_{ij}^{AB} = \delta^{AB} \delta_{ij} \partial_x^2 + g \Gamma_{ij}^{(1)AB}(x) + g^2 \Gamma_{ij}^{(2)AB}(x), \quad (3.6b)$$

$$\Gamma_{ij}^{(1)AB} \equiv i(T^a)^{AB} \left\{ (-i\sigma_{\mu\nu})_{ij} (\partial_\mu A_\nu^a) + \delta_{ij} [(\partial \cdot A^a) + 2A^a \cdot \partial] \right\}, \quad (3.6c)$$

$$\Gamma_{ij}^{(2)AB} \equiv -(T^a T^b)^{AB} \left\{ -i(\sigma_{\mu\nu})_{ij} A_\mu^a A_\nu^b + \delta_{ij} A^a \cdot A^b \right\}, \quad (3.6d)$$

$$\sigma_{\mu\nu}^i = \sigma_{\mu\nu} \equiv \frac{i}{2} [\gamma_\mu, \gamma_\nu] \quad (3.6e)$$

will be useful below. The theory is safely regularized to all orders in any dimension with heat-kernel regularization [24] for both regulators: $R = \exp(\Delta/\Lambda^2)$, $\mathcal{R} = \exp(\mathcal{D}^2/\Lambda^2)$. When explicit Feynman rules are required however for four-dimensional applications (QCD), I will choose minimal power-law regularization

$$R = (1 - \Delta/\Lambda^2)^{-2}, \quad \mathcal{R} = (1 - \mathcal{D}^2/\Lambda^2)^{-1} \quad (3.7)$$

as in Ref. [20].

To study the weak-coupling expansions of the Langevin systems, consider the

set of equivalent integral equations,

$$A_\mu^a(x, t) = \int_{-\infty}^t dt' \int (dy) G_{\mu\nu}^{ab}(x - y, t - t') [V_\nu^b(y, t') + J_\nu^b(y, t') + \frac{1}{\alpha} Y_\nu^b(y, t') + \int (dz) R_{\nu z}^{bc} \eta_\nu^c(z, t')] , \quad (3.8a)$$

$$\begin{aligned} \psi_i^A(x, t) = & \int_{-\infty}^t dt' \int (dy) G_{ij}^{AB}(x - y, t - t') \\ & \times \left\{ \left[\Gamma_{jh}^{BC}(y, t') - \frac{1}{\alpha} \tilde{Y}_{jh}^{BC}(y, t') \right] \psi_h^C(y, t') \right. \\ & - (\tilde{D}_\nu - m)_{jh}^{BC} \int (dz) (R_{kl}^{CD})_{\nu z} (\eta_1^D)_l(z, t') \\ & \left. + \int (dz) (R_{jh}^{BC})_{\nu z} (\eta_2^C)_k(z, t') \right\} , \end{aligned} \quad (3.8b)$$

$$\begin{aligned} \bar{\psi}_i^A(x, t) = & \int_{-\infty}^t dt' \int (dy) G_{ji}^{BA}(x - y, t - t') \\ & \times \left\{ \left[(\Gamma^I)_{kj}^{CB}(y, t') + \frac{1}{\alpha} \tilde{Y}_{kj}^{CB}(y, t') \right] \bar{\psi}_k^C(y, t') \right. \\ & + \int (dz) (\bar{\eta}_2^D)_l(z, t') (R_{lk}^{DC})_{\nu y} (\tilde{D}_\nu + m)_{kj}^{CB} \\ & \left. + \int (dz) (\bar{\eta}_1^C)_k(z, t') (R_{kj}^{CB})_{\nu y} \right\} . \end{aligned} \quad (3.8c)$$

In these equations, the usual gauge-field Green function

$$\begin{aligned} G_{\mu\nu}^{ab}(x - y, t - t') = & \delta^{ab} \theta(t - t') \int (dp) e^{-ip \cdot (x - y)} \\ & \times \left[T_{\mu\nu} e^{-p^2(t - t')} + L_{\mu\nu} e^{-p^2(t - t')/\alpha} \right] , \end{aligned} \quad (3.9)$$

and the (SIAG-bosonized) fermionic Green function

$$G_{ij}^{AB}(x - y, t - t') = \delta^{AB} \delta_{ij} \theta(t - t') \int (dp) e^{-ip \cdot (x - y)} e^{-(p^2 + m^2)(t - t')} \quad (3.10)$$

have been employed. The interaction terms are defined as

$$V_\nu^b \equiv -g f^{bcd} [\partial_\sigma (A_\sigma^c A_\nu^d) - (\partial_\sigma A_\nu^c) A_\sigma^d + (\partial_\nu A_\sigma^c) A_\sigma^d] - g^2 f^{bcd} f^{cfe} A_\sigma^f A_\nu^e A_\sigma^d, \quad (3.11a)$$

$$Y_\nu^b \equiv g f^{bcd} A_\nu^d (\partial \cdot A^c), \quad (3.11b)$$

$$J_\nu^b \equiv -ig \bar{\psi}^A \gamma_\nu (T^b)^{AB} \psi^B, \quad (3.11c)$$

$$\Gamma_{ij}^{AB} \equiv [g \Gamma^{(1)} + g^2 \Gamma^{(2)}]_{ij}^{AB}, \quad (3.11d)$$

$$\tilde{Y}_{ij}^{AB} \equiv ig \delta_{ij} (\partial \cdot A^a) (T^a)^{AB}, \quad (3.11e)$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are defined in eq. (3.6), and $(\Gamma^i)_{kj}^{cB} \equiv (\Gamma_{jk}^{Bc})^*$.

As usual, it is also necessary to expand the regulators into regulator strings. Such expansions have been discussed in detail in Refs. [18,20,24] for the Yang-Mills regulator. In the same way, corresponding expansions of the fermionic regulators are obtained,

$$\begin{aligned} R(\not{D}^2) &= \left[1 - \frac{\not{D}^2}{\Lambda^2} \right]^{-1} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{1 - \square/\Lambda^2} \left(\frac{g \Gamma^{(1)} + g^2 \Gamma^{(2)}}{\Lambda^2} \right) \right]^n \frac{1}{1 - \square/\Lambda^2}. \end{aligned} \quad (3.12)$$

The objects $\Gamma^{(1)}$ and $\Gamma^{(2)}$ therefore serve a dual role, entering first in the SIAG structure of eq. (3.8), and now also as the one- and two-gluon regulator vertices, which connect regulator strings.

The integral equations (3.8) may then be more or less conventionally expanded to all orders as Langevin tree graphs. In the next section, 3.2, the Langevin-Feynman rules for the construction of these tree graphs to all orders will be given.

3.2 Regularized Langevin Tree Rules and Diagrams

The pure Yang-Mills part of these rules has been discussed in Refs. [18,20]. Here we study only the fermionic additions. In Fig. 3.1, the new rules are given explicitly for the regulators of eq. (3.7), which are minimal in $d = 4$. Throughout these rules, thick arrows ($\longrightarrow$) indicate the direction of decreasing Markov-time, while thin arrows ($\longrightarrow$) track the direction of fermionic charge flow.

Propagators: The two new fermionic propagators of the theory are shown in Fig. 3.1a: The thin lines are fermionic Green functions G_{ij}^{AB} , and the thick lines are fermionic regulator propagators.

Noise vertices: The one-point noise vertices $\textcircled{1}$, $\textcircled{2}$, $\overline{\textcircled{1}}$, $\overline{\textcircled{2}}$, shown in Fig. 3.1b, are quadrupled relative to familiar cases, since the Grassmann noise is complex and comes in two varieties.

Ordinary SIAG vertices: Ordinary three- and four-point vertices (with no regulator contributions) are shown in Fig. 3.1c: The first three-point vertex carries no Zwanziger gauge-fixing, since it arises from the fermionic part of the $\delta S/\delta A$ term in the Langevin equation for A_μ^a . The rest of the ordinary vertices are peculiar to the SIAG form, arising from the $\bar{\psi}\bar{\psi}^2\psi$, $\bar{\psi}\bar{\psi}^2$ terms, plus the Zwanziger term in the fermion equations.

Joining vertices: These vertices, shown in Fig. 3.1d, join the regulator strings to the rest of a diagram. The first two-point joining vertex, which comes from regulator factors times $\bar{\eta}_1$ and η_2 , occurs regularly in previous work [18-20], but there is an

extra two-point joining vertex $(\bigcirc)$ which comes from the $(\vec{p}-m)R\eta_1, \bar{\eta}_2 R(\vec{p}+m)$ terms. The joining vertex $\bigcirc$ is always connected by a regulator propagator to $\textcircled{1}$ or $\textcircled{2}$, and never to $\textcircled{2}$ or $\textcircled{1}$. Finally, because the extra SIAG derivatives on η_1 and $\bar{\eta}_2$ are covariant, there is also a three-point joining vertex.

Regulator vertices: These two vertices, shown in Fig. 3.1e, arise in familiar fashion from the regulator expansion, and correspond to the factors $\Gamma^{(1)}, \Gamma^{(2)}$ of eqs. (3.6) and (3.12). As in the case of Yang-Mills [18,20], the regulator vertices reflect the non-abelian structure of the regulator, and always play a crucial role in maintaining gauge-invariance. On the other hand, as discussed in Refs. [20,21], the explicit Λ^{-2} factor of the regulator vertices means that they play essentially no role in the study of one-loop renormalization.

(a) Propagators.

(i) Fermion Green function:

$$t_1 \xrightarrow[i]{A} \xrightarrow[p]{B} t_2 = t_1 \xrightarrow[i]{A} \xleftarrow[p]{B} t_2 = \delta^{AB} \delta_{ij} \theta(t_1 - t_2) e^{-(p^2+m^2)(t_1-t_2)}$$

(ii) Fermion regulator propagator:

$$t_1 \xrightarrow[i]{A} \xrightarrow[p]{B} t_2 = t_1 \xrightarrow[i]{A} \xleftarrow[p]{B} t_2 = \delta^{AB} \delta_{ij} \delta(t_1 - t_2) \frac{\Lambda^2}{\Lambda^2 + p^2}$$

(b) Fermion noise vertices.

$$\begin{array}{ll} \xrightarrow[i]{A} \textcircled{1} = (\eta_1^A)_i & \xleftarrow[i]{A} \textcircled{1} = (\bar{\eta}_1^A)_i \\ \xrightarrow[i]{A} \textcircled{2} = (\eta_2^A)_i & \xleftarrow[i]{A} \textcircled{2} = (\bar{\eta}_2^A)_i \end{array}$$

Fig. 3.1a-e Fermion additions to regularized Langevin tree rules. (to be cont'd)

(c) Ordinary SIAG vertices.

$$= -ig(\gamma_\mu)_{ij}(T^a)^{AB}$$

$$= g(T^a)^{AB}[i(\sigma_{\nu\mu})_{ij}q_\nu + \delta_{ij}(p-k)_\mu + \frac{1}{\alpha}\delta_{ij}q_\mu]$$

$$= g(T^a)^{AB}[i(\sigma_{\nu\mu})_{ij}q_\nu + \delta_{ij}(p-k)_\mu - \frac{1}{\alpha}\delta_{ij}q_\mu]$$

$$= -g^2[f^{abc}(\sigma_{\mu\nu})_{ij}(T^c)^{AB} + \delta_{ij}\delta_{\mu\nu}\{T^a, T^b\}^{AB}]$$

Fig. 3.1a-e Fermion additions to regularized Langevin tree rules. (to be cont'd)

To form the Langevin diagrams, the Langevin trees are contracted, using eq. (3.5) as usual. Of course $\textcircled{1}$ $\textcircled{2}$ noise vertices must only contract to $\overline{\textcircled{1}}$ $\overline{\textcircled{2}}$ noise vertices, as shown in Fig. 3.2. A symbol $\boxed{1}$ $\boxed{2}$ is placed at each such contraction. The Grassmann character of the fermionic noise leads naturally to a factor of (-1) for each closed fermion loop and a global sign [43] in the Langevin diagrams. An overall momentum-conservation factor $(2\pi)^d \delta(\sum_{i=1}^n p_i)$ should also be included in Langevin diagrams for n -point Green functions.

As the simplest illustration of these rules, the zeroth-order two-point fermion Green functions are computed. Fig. 3.3a gives the tree-diagrammatic representation of the zeroth order fields ψ and $\bar{\psi}$. Contracting as shown in Fig. 3.3b, one obtains for the regularized $\psi\bar{\psi}$ propagator

$$\begin{aligned}
 & \langle \psi_i^A(p) \bar{\psi}_j^B(q) \rangle^{(0)} \\
 &= 2\delta^{AB} (2\pi)^d \delta(p+q) \left(\frac{\Lambda^2}{\Lambda^2 + p^2} \right)^2 \int_{-\infty}^t dt' e^{-2(p^2+m^2)(t-t')} (i\not{p} + m)_{ij} \quad (3.13) \\
 &= \delta^{AB} (2\pi)^d \delta(p+q) \left(\frac{\Lambda^2}{\Lambda^2 + p^2} \right)^2 \left(\frac{1}{-i\not{p} + m} \right)_{ij},
 \end{aligned}$$

and same result with a minus sign for $\langle \bar{\psi}_j^B(q) \psi_i^A(p) \rangle^{(0)}$, according to the global sign rule above.

Fig. 3.2 Contractions of fermion noise.

SIAG structure: The four diagrams of Fig. 3.4 and the first $6 \times 2 = 12$ (a-f) diagrams of Fig. 3.5 comprise 16 ordinary diagrams, attributable to the SIAG structure, which contain no regulator vertices. In the naive regulator limit, ($R = \mathcal{R} = 1$), It has been checked explicitly that these 16 diagrams combine to form the single usual vacuum polarization Feynman diagram. Our regulator is responsible for the additional $4 \times 2 = 8$ (g-j) diagrams of Fig. 3.5, which contain explicit regulator vertices.

Mass and $p^2 \ln \Lambda^2$ contributions: As discussed in Ref. [20,21], the 4 diagrams of Fig. 3.4 fail to contribute to the gluon mass on dimensional grounds, since they contain no contractions on external lines. On the other hand, as noted above, the explicit Λ^{-2} of the regulator vertices means that the regulator vertex diagrams (g-j) of Fig. 3.5 make no contributions to wavefunction and α renormalizations.

Fig. 3.4a-d Diagrams with vanishing contributions to the gluon mass.

Fig. 3.5a-j Diagrams with non-vanishing individual contributions to the gluon mass. Indices and momenta are given for the examples in the text.

In the computations below, truncation to define $\Pi_{\mu\nu}^{ab}(p)$ at large Λ are accomplished by factoring out two zeroth-order (Zwanziger gauge-fixed) propagators

$$\frac{1}{p^2}[T_{\mu\rho}(p) + \alpha L_{\mu\rho}(p)] \times \frac{1}{p^2}[T_{\nu\sigma}(p) + \alpha L_{\nu\sigma}(p)] \quad (3.14)$$

from the propagator diagrams for $\langle A_\rho^a A_\sigma^b \rangle$.

As an explicit example of an ordinary diagram, the Langevin rules of Fig. 3.1 and Refs. [18,20] are used to write down the expression for diagram 3.5(a),

$$\begin{aligned} & -g^2 C_R \delta^{ab} f \left[\frac{1}{p^2} \left(\frac{\Lambda^2}{\Lambda^2 + p^2} \right)^4 \right] \int (dq) \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 \left(\frac{\Lambda^2}{\Lambda^2 + q^2} \right)^2 \\ & \quad \times \left(\text{Tr}[\gamma_\rho \not{p} \gamma_\sigma \not{q}] + 4q_\rho (2q - \frac{1}{\alpha} p)_\sigma \right) \\ & \quad \times e^{-(q^2 + m^2)(t_1 - t_3)} e^{-(q^2 + m^2)(t_2 - t_3)} e^{-[(p+q)^2 + m^2](t_1 - t_2)} \quad (3.15) \\ & \quad \times \left\{ T_{\mu\rho}(p) T_{\nu\sigma}(p) e^{-p^2(t-t_1)} e^{-p^2(t-t_2)} \right. \\ & \quad \left. + \alpha L_{\mu\rho}(p) L_{\nu\sigma}(p) e^{-p^2(t-t_1)/\alpha} e^{-p^2(t-t_2)/\alpha} \right\}, \end{aligned}$$

where f is the number of flavors. After Markov-time integration, this becomes

$$\begin{aligned} & -\frac{g^2}{4} C_R \delta^{ab} f \left[\frac{1}{p^4} \left(\frac{\Lambda^2}{\Lambda^2 + p^2} \right)^4 \right] \int (dq) \left(\frac{\Lambda^2}{\Lambda^2 + q^2} \right)^2 \left(\text{Tr}[\gamma_\rho \not{p} \gamma_\sigma \not{q}] + 4q_\rho (2q - \frac{1}{\alpha} p)_\sigma \right) \\ & \quad \times \frac{1}{q^2 + m^2} \left\{ \frac{T_{\mu\rho}(p) T_{\nu\sigma}(p)}{Q(1)} + \frac{\alpha^2 L_{\mu\rho}(p) L_{\nu\sigma}(p)}{Q(\alpha)} \right\}, \quad (3.16) \end{aligned}$$

in which I have defined the quantity

$$Q(\alpha) \equiv (q^2 + m^2) + [(p+q)^2 + m^2] + p^2/\alpha. \quad (3.17)$$

Integrating also over the internal momentum q gives, after truncation, the following

contribution to the zero momentum vacuum polarization

$$\Pi_{\mu\nu}^{ab}(0) \Big|_{3.5a} = \frac{1}{4} \delta_{\mu\nu} \mathcal{N}^{ab} (-\Lambda^2 + 2m^2 \ln \frac{\Lambda^2}{m^2} - 3m^2) \quad (3.18)$$

+ terms which vanish as $\Lambda \rightarrow \infty$.

Here I have defined the constant

$$\mathcal{N}^{ab} \equiv g^2 C_R \delta^{ab} f / (4\pi)^2 \quad (3.19)$$

and the neglected terms are of order m^4/Λ^2 times possible logarithms. The $p^2 \ln \Lambda^2$ contribution to $\Pi_{\mu\nu}^{ab}(p)$ may also be computed by differentiation with respect to external momentum². These results are recorded in Table 1.

Diagrams	Contributions to $\Pi_{\mu\nu}^{ab}(p)$
3.4a, b, c, d	$p^2 [T_{\mu\nu} + (1/\alpha) L_{\mu\nu}] \ln(\Lambda^2/m^2)$
3.5a, b, c, d plus $(\mu, a, p) \leftrightarrow (\nu, b, -p)$	$\delta_{\mu\nu} [-2\Lambda^2 + 4m^2 \ln(\Lambda^2/m^2) - 6m^2]$ $+ p^2 [(2/3)T_{\mu\nu} + L_{\mu\nu}] \ln(\Lambda^2/m^2)$
3.5e, f plus $(\mu, a, p) \leftrightarrow (\nu, b, -p)$	$\delta_{\mu\nu} [4\Lambda^2 - 4m^2 \ln(\Lambda^2/m^2) + 4m^2]$ $+ p^2 [-3T_{\mu\nu} - (1 + (2/\alpha))L_{\mu\nu}] \ln(\Lambda^2/m^2)$
3.5g, h, i, j plus $(\mu, a, p) \leftrightarrow (\nu, b, -p)$	$\delta_{\mu\nu} [-2\Lambda^2 + 2m^2]$

Table 1. Order g^2 contributions to leading terms of $\Pi_{\mu\nu}^{ab}(p)$ in units of $\mathcal{N}^{ab} \equiv g^2 C_R \delta^{ab} f / (4\pi)^2$.

Diagrams with identical contributions are grouped together in a row and their sums are listed.

As an example of a diagram with a regulator vertex, the explicit expression for diagram 3.5g is

$$\begin{aligned}
& \frac{ig^2 C_R \delta^{ab} f}{\Lambda^2} \left[\frac{1}{p^2} \left(\frac{\Lambda^2}{\Lambda^2 + p^2} \right)^4 \right] \int (dq) \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \left(\frac{\Lambda^2}{\Lambda^2 + q^2} \right)^2 \left(\frac{\Lambda^2}{\Lambda^2 + (p+q)^2} \right) \\
& \times \text{Tr} \left[\gamma_\rho (i\not{p} + i\not{q} + m) \{ (2q+p)_\sigma + i\sigma_{\gamma\sigma} p_\gamma \} \right] \\
& \times e^{-(q^2+m^2)(t_1-t_2)} e^{-[(p+q)^2+m^2](t_1-t_2)} \\
& \times \left\{ T_{\mu\rho}(p) T_{\nu\sigma}(p) e^{-p^2(t-t_1)} e^{-p^2(t-t_2)} \right. \\
& \quad \left. + \alpha L_{\mu\rho}(p) L_{\nu\sigma}(p) e^{-p^2(t-t_1)/\alpha} e^{-p^2(t-t_2)/\alpha} \right\} \quad (3.20a)
\end{aligned}$$

$$\begin{aligned}
& = -\frac{g^2 C_R \delta^{ab} f}{2\Lambda^2} \left[\frac{1}{p^2} \left(\frac{\Lambda^2}{\Lambda^2 + p^2} \right)^4 \right] \int (dq) \left(\frac{\Lambda^2}{\Lambda^2 + q^2} \right)^2 \left(\frac{\Lambda^2}{\Lambda^2 + (p+q)^2} \right) \\
& \times \text{Tr} \left[\gamma_\rho (\not{p} + \not{q}) \{ (2q+p)_\sigma + i\sigma_{\gamma\sigma} p_\gamma \} \right] \\
& \times \left\{ \frac{T_{\mu\rho}(p) T_{\nu\sigma}(p)}{Q(1)} + \frac{\alpha^2 L_{\mu\rho}(p) L_{\nu\sigma}(p)}{Q(\alpha)} \right\}. \quad (3.20b)
\end{aligned}$$

With truncation,

$$\Pi_{\mu\nu}^{ab}(0) \Big|_{3.5g} = \frac{1}{4} [-\Lambda^2 + m^2] \mathcal{N}^{ab} \delta_{\mu\nu} + \text{terms which vanish as } \Lambda \rightarrow \infty \quad (3.21)$$

is computed. As recorded in Table 1, this diagram, (since it contains a regulator vertex), contributes no $p^2 \ln \Lambda^2$ term.

In this way, all non-zero contributions to $\Pi_{\mu\nu}^{ab}(0)$, and all $p^2 \ln \Lambda^2$ contributions to $\Pi_{\mu\nu}^{ab}(p)$ have been computed. The results are listed with their diagrams in Table

1. The reader may easily verify that the sum of all contributions to $\Pi_{\mu\nu}^{ab}(0)$ is zero, so the gluon remains massless to this order.

Adding all contributions in Table 1, one obtains the total fermionic contribution to the gluon vacuum polarization

$$\begin{aligned} \Pi^{(f)\,ab}_{\mu\nu}(p) &= \mathcal{N}^{ab} p^2 \left(-\frac{4}{3} T_{\mu\nu}(p) - \frac{1}{\alpha} L_{\mu\nu}(p) \right) \ln \frac{\Lambda^2}{m^2} \\ &+ \text{terms finite as } \Lambda \rightarrow \infty. \end{aligned} \quad (3.22)$$

The transverse term in eq. (3.22) is the standard [44] fermionic contribution, while the longitudinal term is peculiar to Zwanziger's gauge-fixing, since the same result is obtained in dimensional regularization of the Zwanziger gauge-fixed theory with the dictionary $\ln \Lambda \leftrightarrow (4-d)^{-1}$. This phenomenon was first observed in similar investigations [42,20] of Yang-Mills theory.

The α^{-1} dependence of the longitudinal term is in a sense an artifact of truncation, however, since the one-loop contribution to the two-point function,

$$\begin{aligned} \langle A_\mu^a(p) A_\nu^b(q) \rangle^{(f)} &= (2\pi)^d \delta(p+q) \left[\frac{T_{\mu\rho}(p) + \alpha L_{\mu\rho}(p)}{p^2} \right] \Pi^{(f)\,ab}_{\rho\sigma}(p) \\ &\times \left[\frac{T_{\sigma\nu}(p) + \alpha L_{\sigma\nu}(p)}{p^2} \right] \\ &+ \text{terms which vanish as } \Lambda \rightarrow \infty \end{aligned} \quad (3.23)$$

$$\begin{aligned} &= (2\pi)^d \delta(p+q) \frac{\mathcal{N}^{ab}}{p^2} \left(-\frac{4}{3} T_{\mu\nu} - \alpha L_{\mu\nu} \right) \ln \frac{\Lambda^2}{m^2} \\ &+ \text{terms finite as } \Lambda \rightarrow \infty \end{aligned}$$

shows a smooth³ approach to the ordinary Landau gauge result as $\alpha \rightarrow 0$. This verifies Zwanziger's formal argument [13] that the gauge-fixing should give standard Landau gauge results as $\alpha \rightarrow 0$. Moreover, as we shall see in the next section,

the usual α -independent fermionic contribution to wavefunction renormalization is obtained.

Although these Zwanziger phenomena have nothing to do with the present regularization scheme, for completeness the leading term in the case of scalar electrodynamics have also been computed, using the Langevin rules of Ref. [20]. The one-loop contribution is

$$\begin{aligned}
 \langle A_\mu(p) A_\nu(q) \rangle &= (2\pi)^d \delta(p+q) \left[\frac{T_{\mu\rho}(p) + \alpha L_{\mu\rho}(p)}{p^2} \right] \\
 &\times \frac{e^2}{(4\pi)^2} p^2 \left[-\frac{1}{3} T_{\rho\sigma} - \frac{1}{2\alpha} L_{\rho\sigma} \right] \ln \frac{\Lambda^2}{m^2} \\
 &\times \left[\frac{T_{\sigma\nu}(p) + \alpha L_{\sigma\nu}(p)}{p^2} \right],
 \end{aligned} \tag{3.24}$$

which shows the same qualitative features discussed above for fermions. The same behavior is also expected in pure Yang-Mills, when the computations are extended beyond $\alpha = 1$ [42,20].

3.4 Regularized Schwinger-Dyson Equations and Diagrams

Following standard methods [18-20], but allowing for the Grassmann character of the fermionic noise, the regularized Langevin systems (3.4) may be recast into a set of regularized Schwinger-Dyson (SD) equations⁴

$$\begin{aligned}
 0 = \int (dx) \left\langle \left\{ \left[-\frac{\delta S}{\delta A_\mu^a(x)} + \int (dy)(dz) R_{\nu x}^{bc} \frac{\delta}{\delta A_\mu^c(z)} R_{\nu x}^{ba} \right] \frac{\delta}{\delta A_\mu^a(x)} \right. \right. \\
 - \left[(\bar{\mathcal{D}}_x - m)_{ij}^{AB} \left(-\frac{\delta S}{\delta \bar{\psi}_j^B(x)} + \int (dy)(R_{xy}^2)^{BC} \frac{\delta}{\delta \bar{\psi}_k^C(y)} \right) \right] \frac{\delta}{\delta \psi_i^A(x)} \\
 - \left[\left(-\frac{\delta S}{\delta \psi_j^B(x)} + \int (dy)(R_{yx}^2)^{CB} \frac{\delta}{\delta \psi_k^C(y)} \right) (\bar{\mathcal{D}}_x + m)_{ji}^{BA} \right] \frac{\delta}{\delta \bar{\psi}_i^A(x)} \\
 \left. \left. - Z^a(x) G^a(x) \right\} F \right\rangle, \quad (3.25)
 \end{aligned}$$

where

$$G^a(x) \equiv D_\mu^{ab} \frac{\delta}{\delta A_\mu^b(x)} + ig(T^a)^{AB} \psi_i^B(x) \frac{\delta}{\delta \psi_i^A(x)} - ig(T^a)^{BA} \bar{\psi}_i^B(x) \frac{\delta}{\delta \bar{\psi}_i^A(x)} \quad (3.26)$$

is the generator of non-abelian gauge transformations. These SD equations provide a d -dimensional description of the regularization scheme which is equivalent⁵ to equilibrium results of the regularized Langevin systems, after all Markov-time integrations are performed.

Aside from the Zwanziger term, the SD equations (3.25) are a regularized form of the unregularized second-order SD equations obtainable in the action form as

$$0 = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \int (dx) \frac{\delta}{\delta A_\mu^a(x)} \left[e^{-S} \frac{\delta}{\delta A_\mu^a(x)} F \right], \quad (3.27a)$$

$$0 = - \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \int (dx) \left[(\bar{\mathcal{D}}_x - m)_{ij}^{AB} \frac{\delta}{\delta \bar{\psi}_j^B(x)} \right] \left[e^{-S} \frac{\delta}{\delta \psi_i^A(x)} F \right], \quad (3.27b)$$

$$0 = - \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \int (dx) \left[\frac{\delta}{\delta \psi_j^B(x)} (\bar{\mathcal{D}}_x + m)_{ji}^{BA} \right] \left[e^{-S} \frac{\delta}{\delta \psi_i^A(x)} F \right]. \quad (3.27c)$$

The SD prescription for regularization is then to add equations (3.27a,b,c) and to regularize each of the three functional Laplacians.

The gauge-invariance of the regularized SD systems (3.25) may be verified as in Ref. [20]: The crucial ingredients of the demonstration are that 1) The Zwanziger gauge-fixing term vanishes on a gauge-invariant quantity F_{GI} , and 2) The other terms in the SD equations are manifestly gauge-invariant.

As discussed in Chapter 2 and Refs. [18,19], the SD diagrammatic method is more efficient than Langevin techniques for diagrams with a large number of Markov-time integrations. The rules for the construction of the SD diagrams may be derived directly from the regularized SD equations (3.25) in the manner of the appendix of Ref. [19], and appendix C of Ref. [20] (or by Markov-time integration of all Langevin diagrams at equilibrium).

As usual, SD diagrams are drawn on the Langevin diagrams. The vertices in the SD formalism are those of the Langevin formalism (Fig. 3.1), but the Langevin Green functions and Markov-time integrations are replaced by SD “pictures”, which provide the momentum denominators called “solid-line factors”.

Since the Yang-Mills SD rules have been thoroughly discussed [20], we concentrate here on the fermionic additions to the rules.

1) The fermionic “simple contraction” [20] factor $\delta^{AB} \delta_{ij} R_0^2(p) / [2(p^2 + m^2)]$ includes an additional factor of 1/2.

2) No factor of 2 is associated to a fermionic regulator vertex cluster with two incoming lines (RVC_2).

These first two rules follow from the absence of the usual factor of 2 in the charged Grassmann contractions (3.5b). Note also the absence of fermionic regulator vertex clusters with one incoming line (RVC_1 's), since the fermion functional derivatives commute with the fermionic regulator.

3) Each fermionic loop gives an additional factor (-1) , and the global sign convention holds.

The solid-line factors are obtained as usual by studying the operator

$$\begin{aligned} K \equiv \int (dp) & \left\{ \frac{\delta S^{(0)}}{\delta A_\mu^a(-p)} \frac{\delta}{\delta A_\mu^a(p)} + \frac{1}{\alpha} p^2 L_{\mu\nu} A_\nu^a(p) \frac{\delta}{\delta A_\mu^a(p)} \right. \\ & + \left[(i\not{p} + m)_{ij}^{AB} \frac{\delta S^{(0)}}{\delta \bar{\psi}_j^B(-p)} \right] \frac{\delta}{\delta \psi_i^A(p)} \\ & \left. + \left[\frac{\delta S^{(0)}}{\delta \bar{\psi}_j^B(-p)} (i\not{p} - m)_{ji}^{BA} \right] \frac{\delta}{\delta \bar{\psi}_i^A(p)} \right\} \end{aligned} \quad (3.28)$$

where $S^{(0)}$ is the free part of the action. Following Ref. [20], the eigenvectors of K for arbitrary α are

$$\begin{aligned} F = & \bar{\psi}_{i_1}^{A_1}(p_1) \dots \bar{\psi}_{i_m}^{A_m}(p_m) \psi_{j_1}^{B_1}(k_1) \dots \psi_{j_n}^{B_n}(k_n) \\ & \times (A^T)_{\mu_1}^{a_1}(q_1) \dots (A^T)_{\mu_r}^{a_r}(q_r) (A^L)_{\nu_1}^{b_1}(r_1) \dots (A^L)_{\nu_s}^{b_s}(r_s) \end{aligned} \quad (3.29)$$

where A^T and A^L are the transverse and longitudinal gauge fields. The solid-line factors are the inverse eigenvalues of K

$$KF = \left(\sum_{i=1}^m (p_i^2 + m^2) + \sum_{i=1}^n (k_i^2 + m^2) + \sum_{i=1}^r q_i^2 + \frac{1}{\alpha} \sum_{i=1}^s r_i^2 \right) F. \quad (3.30)$$

Note that the SIAG structure gives fermionic contributions to the solid-line factors which are totally bosonic, and that the longitudinal gluon terms carry an extra factor α^{-1} .

As an explicit example, the SD evaluation of the regulator vertex diagram 3.5g, which was discussed in section 3.3, will be given below. Because $\alpha \neq 1$, there are two SD diagrams, corresponding to both external gluons transverse or both longitudinal. The transverse-longitudinal cross-terms vanish identically. The case with both external gluons longitudinal is shown in Fig. 3.6 with all relevant indices. The only allowed SD ordering is AB, and the dotted box surrounds the regulator vertex cluster (RVC₂), as discussed in Ref. [20].

The sequence of pictures for this diagram is shown in Fig. 3.7, along with the factor associated with each picture. Collecting all factors from the pictures, a combinatoric factor of unity, a minus sign for the fermion loop, Kronecker-deltas

Fig. 3.6 SD diagram for Fig. 3.5g with longitudinal external gluons. The only valid ordering is AB.

(a): $\frac{1}{2p^2/\alpha}$

(b): $\frac{-ig\gamma_\mu T^c}{(q^2 + m^2) + [(p+q)^2 + m^2] + p^2/\alpha^2}$

(c): $\frac{g}{\Lambda^2} (i\not{p} + i\not{q} + m) [-i\sigma_{\kappa\sigma} p_\kappa - (2q + p)_\sigma] T^d$
 $\times \left(\frac{\Lambda^2}{\Lambda^2 + q^2} \right)^2 \left(\frac{\Lambda^2}{\Lambda^2 + (p+q)^2} \right) \frac{1}{p^2} \left(\frac{\Lambda^2}{\Lambda^2 + p^2} \right)^4$

Fig. 3.7a-c SD pictures for Fig. 3.6.

in color indices and longitudinal projection operators for gluon lines, appropriate traces for the fermion loop and a sum over flavors, the value

$$\begin{aligned}
 & -\frac{g^2 C_R \delta^{ab} f}{2\Lambda^2} \left[\frac{1}{p^4} \left(\frac{\Lambda^2}{\Lambda^2 + p^2} \right)^4 \right] \int (dq) \left(\frac{\Lambda^2}{\Lambda^2 + q^2} \right)^2 \left(\frac{\Lambda^2}{\Lambda^2 + (p+q)^2} \right) \\
 & \times \frac{\text{Tr} [\gamma_\rho (\not{p} + \not{q}) \{ (2q + p)_\sigma + i\sigma_{\kappa\sigma} p_\kappa \}]}{q^2 + (p+q)^2 + 2m^2 + p^2/\alpha} \\
 & \times \alpha^2 L_{\mu\rho}(p) L_{\nu\sigma}(p)
 \end{aligned} \tag{3.31}$$

is found for the SD diagram in Fig. 3.6. This result is precisely the second term in the result eq. (3.20b), obtained from the Langevin system by Markov-time in-

tegration. Exactly the same steps are followed to evaluate the SD diagrams with both external gluons transverse. The only changes are $L \rightarrow T$ and $\alpha = 1$ in the solid-line factors, which gives the first term in eq. (3.20b).

As an application of the Schwinger-Dyson equations (3.25), we turn to a brief discussion of renormalization. The SD renormalization program of Ref. [21] for pure Yang-Mills is easily extended to include fermions. As the simplest example, the counterterm (CT) contribution to the two-point gluon Green function is easily read off from Figure 1 of that reference,

$$\begin{aligned} \text{CT} = & \frac{\delta^{ab}}{p^2} [T_{\mu\rho} + \alpha L_{\mu\rho}(p)] \\ & \times [(1 - Z_A)T_{\rho\sigma}(p) + \frac{1}{\alpha}(1 - Z_A Z_\alpha)L_{\rho\sigma}(p)] \\ & \times [T_{\nu\sigma} + \alpha L_{\nu\sigma}(p)] + O\left(\frac{\ln \Lambda}{\Lambda^2}\right). \end{aligned} \quad (3.32)$$

Requiring that the renormalized (R) contribution (eq. (3.22)) plus the counterterm contribution (eq. (3.32)) equals zero gives immediately the fermionic contributions to the renormalization constants

$$(Z_A - 1)^{(f)} = -\frac{4}{3} \frac{g^2 C_R f}{16\pi^2} \ln(\Lambda^2/m^2) \quad (3.33a)$$

$$(Z_\alpha - 1)^{(f)} = \frac{1}{3} \frac{g^2 C_R f}{16\pi^2} \ln(\Lambda^2/m^2). \quad (3.33b)$$

This is the usual α -independent contribution to the wavefunction renormalization, and the fermionic contribution to the β -function for α

$$\beta_\alpha^{(f)} = -\alpha \frac{2}{3} \frac{g^2 C_R f}{16\pi^2} \quad (3.34)$$

exhibits the expected fixed-point at $\alpha = 0$.

It is reasonable to expect that the usual fermionic contribution to the coupling constant β -function is also obtained.

Finally, it should be emphasized that there are alternative regularized SD formulations for gauge theory with fermions. As mentioned in Chapter 2, SD formulations may exist even when stochastic formulations are questionable. As an example, consider the “naive” regularized SD equations

$$\begin{aligned}
0 = \int (dx) \left\langle \left\{ \left[-\frac{\delta S}{\delta A_\mu^a(x)} + \int (dy)(dz) R_{yz}^{bc} \frac{\delta}{\delta A_\mu^c(z)} R_{yz}^{ba} \right] \frac{\delta}{\delta A_\mu^a(x)} \right. \right. \\
+ \lambda \left[\frac{\delta S}{\delta \bar{\psi}_i^A(x)} - \int (dy) (R_{xy}^2)^{AB}_{ij} \frac{\delta}{\bar{\psi}_j^B(y)} \right] \frac{\delta}{\delta \psi_i^A(x)} \\
- \lambda \left[\frac{\delta S}{\delta \psi_i^A(x)} - \int (dy) (R_{yx}^2)^{BA}_{ji} \frac{\delta}{\psi_j^B(y)} \right] \frac{\delta}{\delta \bar{\psi}_i^A(x)} \\
\left. \left. - Z^a(x) G^a(x) \right\} F \right\rangle, \tag{3.35}
\end{aligned}$$

in which λ , an arbitrary parameter of dimension inverse length, has replaced the SIAG bosonizing-kernel $\mathcal{D} \pm m$ of equations (3.25) and (3.27). The naive equations (3.35) form a λ -family of alternative descriptions of regularized gauge theory with fermions which, however, do not correspond to any known equilibrating stochastic system. The apparent simplicity of the naive equations leads directly to a significant reduction in the number of required SD vertices. A compensating aspect arises, however, in that the corresponding solid line factors pick up matrix structure for the fermions, and are more difficult to handle.

3.5 Background Fields, Anomalies and Currents

Regularized fermions in a background gauge field may be described either a) at the $(d+1)$ -dimensional stochastic level, by dropping the $\dot{A}_\mu^a$ equation, along with the Zwanziger terms, in the SIAG-Langevin system (3.4), or b) at the d -dimensional level of the SD equations (3.25) by dropping terms with gluonic functional derivatives, along with the Zwanziger terms,

$$0 = \int (dx) \left\langle \left\{ \left[(\bar{\mathcal{D}}_x - m)_{ij}^{AB} \left(-\frac{\delta S}{\delta \bar{\psi}_j^B(x)} + \int (dy) (R_{xy}^2)_{jk}^{BC} \frac{\delta}{\bar{\psi}_k^C(y)} \right) \right] \frac{\delta}{\delta \psi_i^A(x)} + \left[\left(-\frac{\delta S}{\delta \psi_j^B(x)} + \int (dy) (R_{yx}^2)_{kj}^{CB} \frac{\delta}{\delta \psi_k^C(y)} \right) (\bar{\mathcal{D}}_x + m)_{ji}^{BA} \right] \frac{\delta}{\delta \bar{\psi}_i^A(x)} \right\} F \right\rangle. \quad (3.36)$$

In either case, the system is essentially a free theory, and the exact fermionic n -point functions are easily obtained. Choosing $F = \psi_i^A(x) \bar{\psi}_j^B(y)$ in eq. (3.36), for example, one obtains

$$\langle \psi_i^A(x) \bar{\psi}_j^B(y) \rangle = \left[(\{\mathcal{D} + m\}^{-1} R^2)_{ij}^{AB} \right]_{xy} \quad (3.37)$$

and the n -point functions are constructed as usual by Wick's theorem. From this result, or an examination of the corresponding Fokker-Planck equation, it follows that the regularized Euclidean effective action⁶

$$S_{\text{eff}} = \int (dx) (dy) \bar{\psi}_i^A(x) \left[(\{\mathcal{D} + m\} R^{-2})_{ij}^{AB} \right]_{xy} \psi_j^B(y) \quad (3.38)$$

provides an equivalent description of the fermionic equilibrium averages.

The fermionic averages $\langle \psi_1 \cdots \psi_n \bar{\psi}_1 \cdots \bar{\psi}_n \rangle$ of the model (as well as any finite-derivative composite operator) are successfully regularized in d dimensions by, say,

the heat-kernel regulator $\mathcal{R} = \exp(\mathcal{D}^2/\Lambda^2)$, which is adopted in the following discussion. On the other hand, the existence of a well-defined equivalent action formulation for the background field model (which does not exist in the case of regularized dynamical gauge fields [19,20]) means that the problem of Lee and Zinn-Justin [3] has not been eliminated, and these background field models cannot be totally satisfactory⁷. I will return to these limitations on the background field models after a brief discussion of axial and chiral anomalies, for which the modelling appears to be adequate.

As a simple example, the axial anomaly in four dimensions,

$$\begin{aligned} \partial_\mu^\alpha \langle \bar{\psi}(x) \gamma_5 \gamma_\mu \psi(x) \rangle &= \int (dz) \text{Tr} \left[\gamma_5 (\not{\partial})_{xy} (\{\mathcal{D} + m\}^{-1} \mathcal{R}^2)_{xz} \right. \\ &\quad \left. - \gamma_5 (\not{\partial})_{xz} (\{\mathcal{D} + m\}^{-1} \mathcal{R}^2)_{xy} \right] \Big|_{y=x} \quad (3.39a) \end{aligned}$$

$$= -2\text{Tr}[\gamma_5 (\mathcal{R}^2)_{xx}] - 2m \langle \bar{\psi}(x) \gamma_5 \psi(x) \rangle, \quad (3.39b)$$

is computed, where the trace is over spinor and color indices. To obtain the final line, I have rewritten $\not{\partial}_x = \not{\mathcal{D}}_x - igA(x)$, and made use of the fact that $\mathcal{R}(\mathcal{D}^2)$ commutes with $\not{\mathcal{D}}$. The first term of the result (3.39b) is the anomaly term due to the presence of the regulator. The heat kernel expansion [24] have been used to obtain the standard result $F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a (-g^2 C_R / 8\pi^2)$ for the anomaly term at large Λ .

The Noether structure of this current is easily understood in the effective action formulation eq. (3.38). Consider the *regularized* infinitesimal axial transformation⁸

$$\psi'(x) - \psi(x) = i \int (dy) \gamma_5 (\mathcal{R}^2)_{xy} \alpha(y) \psi(y) + O(\alpha^2) \quad (3.40a)$$

$$\bar{\psi}'(x) - \bar{\psi}(x) = i \int (dy) \bar{\psi}(y) \alpha(y) (\mathcal{R}^2)_{yx} \gamma_5 + O(\alpha^2), \quad (3.40b)$$

which possesses a finite Jacobian,

$$\left| \frac{\delta(\bar{\psi}', \psi')}{\delta(\bar{\psi}, \psi)} \right| = 1 + 2i \int (dx) \alpha(x) \text{Tr} [\gamma_5 [\mathcal{R}^2(\mathcal{D}^2)]_{xx}] + O(\alpha^2) \quad (3.41)$$

as computed from eq. (3.40). Following Fujikawa [46], eq. (3.39b) is immediately obtained as a Ward identity. It should be emphasized that this regularized application of Fujikawa's idea is not formal.

Moreover, for an effective action with regulator dependence $\mathcal{R}^{-r}$, the Noether transformations $\delta\psi(x) = i\gamma_5 \int (dy)(dz)(\mathcal{R}^r)_{zy} \alpha(y) (\mathcal{R}^{n+1-r})_{yx} \psi(z)$, $\delta\bar{\psi}(x) = i \int (dy) \bar{\psi}(y) \alpha(y) (\mathcal{R}^{n+1})_{yx} \gamma_5$ with $n \geq 0$ give rise to currents $\int (dy) \bar{\psi}(x) \times \gamma_5 \gamma_\mu (\mathcal{R}^{n+1-r})_{xy} \psi(y)$ and corresponding anomaly terms $-2\text{Tr}[\gamma_5 \mathcal{R}^{n+1}]$. Each of these currents is regularized and exhibits the same correct anomaly at large Λ . Note that the case $r = 0$ corresponds to regularized (point-splitting) anomaly computations without regularizing the action.

The background field model is also adequate for chiral anomalies. In this case, the regularized $d = 2l$ dimensional SIAG-Langevin systems may be taken as

$$\begin{aligned} (\dot{\psi}_L)_i^A(x, t) &= (\bar{\mathcal{D}}_x^2 \psi_L)_i^A(x, t) - (\bar{\mathcal{D}}_x)^{AB}_{ij} \int (dy) (\mathcal{R}_{jl}^{BC})_{xy} (\eta_+^C)_l(y, t) \\ &\quad + \int (dy) (\mathcal{R}_{ij}^{AB})_{xy} (\eta_-^B)_j(y, t), \end{aligned} \quad (3.42a)$$

$$\begin{aligned} (\dot{\bar{\psi}}_L)_i^A(x, t) &= (\bar{\psi}_L \bar{\mathcal{D}}_x)_i^A(x, t) + \int (dy) (\bar{\eta}_+^B)_j(y, t) (\mathcal{R}_{ji}^{BA})_{yx} \\ &\quad + \int (dy) (\bar{\eta}_-^C)_l(y, t) (\mathcal{R}_{ij}^{CB})_{yx} (\bar{\mathcal{D}}_x)^{BA}_{ji}, \end{aligned} \quad (3.42b)$$

$$\langle (\eta_{\pm}^A)_i(x, t) (\bar{\eta}_{\pm}^B)_j(x', t') \rangle_{\eta} = \frac{1}{2} \delta^{AB} (1 \pm \gamma_{d+1})_{ij} \delta(x - x') \delta(t - t'), \quad (3.42c)$$

where

$$\psi_L \equiv \frac{1}{2}(1 - \gamma_{d+1})\psi, \quad \bar{\psi}_L \equiv \frac{1}{2}\bar{\psi}(1 + \gamma_{d+1}), \quad (3.43a)$$

$$\gamma_{d+1} \equiv i^{d/2} \prod_{\mu=0}^{d-1} \gamma_{\mu}, \quad \gamma_{d+1}^2 = 1. \quad (3.43b)$$

Alternatively, the Schwinger-Dyson equations

$$\begin{aligned} 0 = \int (dx) \left\langle \left\{ (\bar{\mathcal{D}}^2 \psi_L)_i^A(x) \frac{\delta}{\delta(\psi_L)_i^A(x)} + (\bar{\psi}_L \bar{\mathcal{D}}^2)_i^A(x) \frac{\delta}{\delta(\bar{\psi}_L)_i^A(x)} \right. \right. \\ \left. \left. + \frac{1}{2} \int (dy) [\mathcal{R}_{xy}^2 \bar{\mathcal{D}}_y - \bar{\mathcal{D}}_x \mathcal{R}_{xy}^2]_{ij}^{AB} \frac{\delta}{\delta(\bar{\psi}_L)_j^B(y)} \frac{\delta}{\delta(\psi_L)_i^A(x)} \right\} F \right\rangle \end{aligned} \quad (3.44)$$

may be employed. In either case, the equilibrium fermion averages may be described by the effective action $S_{\text{eff}} = \int (dx)(dy) \bar{\psi}_L(x) (\mathcal{D} \mathcal{R}^{-2})_{xy} \psi_L(y)$. With the heat-kernel regulator $\mathcal{R} = \exp(\mathcal{D}^2/\Lambda^2)$, the results

$$\begin{aligned} \partial_{\mu} \langle \bar{\psi}_L(x) \gamma_{\mu} \psi_L(x) \rangle &= -\text{Tr}[\gamma_{d+1}(\mathcal{R}^2)_{xx}] \\ &\simeq -g^2 [(d/2)! (4\pi)^{d/2}]^{-1} \epsilon_{\mu_1 \mu_2 \dots \mu_d} \text{Tr}[F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} \dots F_{\mu_{d-1} \mu_d}], \end{aligned} \quad (3.45a)$$

$$\begin{aligned} D_{\mu}^{ab} \langle \bar{\psi}_L(x) \gamma_{\mu} T^b \psi_L \rangle &= -\text{Tr}[T^a \gamma_{d+1}(\mathcal{R}^2)_{xx}] \\ &\simeq -g^2 [(d/2)! (4\pi)^{d/2}]^{-1} \epsilon_{\mu_1 \mu_2 \dots \mu_d} \end{aligned} \quad (3.45b)$$

$$\times \text{Tr}[T^a F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} \dots F_{\mu_{d-1} \mu_d}]$$

have been verified, where $F_{\mu\nu} \equiv F_{\mu\nu}^a T^a$. The result eq. (3.45) is the usual covariant [47] form of the singlet and non-abelian chiral anomalies.

The zero modes may be treated more carefully, either by the method of Egorian,

Nissimov and Pacheva [37] at the stochastic level, or directly at the d -dimensional level by including an infrared cutoff in the regulator which appears in the SD equations (e.g. $R = \exp(\mathcal{D}^2/\Lambda^2) - \exp(\mathcal{D}^2/\tilde{\Lambda}^2) \xrightarrow{\Lambda \rightarrow 0} \exp(\mathcal{D}^2/\Lambda^2)[1 - P_0]$, where P_0 is the zero mode projector).

These regularized Noether currents (e.g. $\bar{\psi}\gamma_\mu\psi$, $\bar{\psi}\gamma_5\gamma_\mu\psi$ and the chiral anomalous currents of eq. (3.45)) are not the currents to which background gauge fields couple. As seen, for example, in eq. (3.38), the background fields exhibit high-derivative coupling to the fermions, so that, e.g.

$$\begin{aligned} J_\mu^a[A; x] &\equiv \frac{\delta W[A]}{\delta A_\mu^a(x)} \\ &= - \int (dy)(dz) \left\langle \bar{\psi}_i^A(y) \frac{\delta}{\delta A_\mu^a(x)} \left[(\{\mathcal{D} + m\} R^{-2})_{ij}^{AB} \right]_{yx} \psi_j^B(z) \right\rangle \\ &= \int (dy)(dz) \text{Tr} \left[\left(\frac{\delta}{\delta A_\mu^a(x)} (\{\mathcal{D} + m\} R^{-2})_{yx} \right) \left(\frac{R^2}{\mathcal{D} + m} \right)_{xy} \right] \end{aligned} \quad (3.46)$$

is not regularized⁹. This is the problem of Lee and Zinn-Justin. It follows that the consistent [47] form of anomalies cannot be studied in these background field models. A deeper consequence is that the gauge-invariance of the background field models is, in this sense, only formal.

The correctly regularized gauge-field amplitudes are only obtained with the fully quantized models of the previous sections of this chapter, in which the problem of Lee and Zinn-Justin does not occur, and gauge-invariance is not formal.

It would be interesting to compute the axial and chiral anomalies directly from the regularized Green functions of the fully quantized gauge theories. The chiral

case is of particular interest, since the fully quantized discussion provides an alternative completely regularized approach to the internal consistency of such theories [48,49]. In this connection, it is instructive to note that the chiral anomalous currents of eq. (3.45) are not a priori the currents to which regularized dynamical chiral gauge fields couple.

Footnotes: Chapter 3

- ¹ The case of Weyl fermions, e.g. the Weinberg-Salam model, is also straightforward. Chiral anomalies are discussed in section 3.5.
- ² In fact, there is a non-uniformity in the computation of the $p^2 \ln \Lambda^2$ terms of $\Pi_{\mu\nu}^{ab}$ at $\alpha = 0$, since closer examination reveals dependence on parameters such as $\ln(\alpha\Lambda^2/p^2)$. The results quoted for these terms are, strictly speaking, valid only for $\alpha \neq 0$.
- ³ The non-uniformity in $\Pi_{\mu\nu}^{ab}$ at $\alpha = 0$, mentioned in footnote 2 of this chapter, is presumably washed out in the two-point function itself.
- ⁴ As discussed in Refs. [20,24], the simpler $\gamma = 0$ form of the regularized Yang-Mills functional Laplacian may also be employed. In this case, the minimal regulators are uniformly $R = (1 - \Delta/\Lambda^2)^{-1}$, $R = (1 - \not{D}^2/\Lambda^2)^{-1}$ in four dimensions.
- ⁵ As in Refs. [19,20], a unique weak-coupling SD solution is obtained with the boundary condition that the averages have the usual permutation symmetry (anti-symmetry) among the bosonic (fermionic) fields.
- ⁶ A similar but not identical action was obtained in Ref. [45] by Markov-time smearing of a background field problem.
- ⁷ The Lee and Zinn-Justin problem occurs whenever a well-defined action formulation is available. The problem also occurs in Markov-time smearing of background field problems (see, e.g. footnote 6 of this chapter).

⁸ A finite form of this regularized axial transformation is

$\psi'(x) = \int(dy)[\exp(i\gamma_5 R^2 \alpha)]_{xy} \psi(y)$, $\bar{\psi}'(x) = \int(dy)\bar{\psi}(y)[\exp(i\alpha R^2 \gamma_5)]_{yx}$. With regard to the corresponding vector transformation and Noether current $J_\mu = \bar{\psi}\gamma_\mu\psi$ note that the implied Ward identities at finite Λ include $\langle\partial \cdot J\rangle = 0$, but $\langle\partial \cdot J(x)J_\nu(y)\rangle \neq 0$, so that order Λ^2 terms persist in $\langle J_\mu(x)J_\nu(y)\rangle$. This Noether current therefore is not the current to which regularized dynamical gauge fields couple.

⁹ The statement for $W[A] = \sum_n W_n A^n$ in d -dimensions is that, independent of $R(D^2)$, W_n is not regularized for $n \leq d$.

Chapter 4

Non-Grassmann Formulation of Regularized Gauge Theory with Fermions

In this chapter and the next, the non-Grassmann formulation of regularized gauge theory with Dirac fermions is discussed in some detail.¹ Although the same principles of the continuum-regularization scheme are operative in this *integrated* non-Grassmann formulations, which is physically equivalent to its *unintegrated* Grassmann counterpart in Chapter 3, it is superior in elegance and simplicity. In fact, such simplification is a general consequence of the integration of the matter fields in any regularized continuum gauge theory that couples to quadratic matter. As a further illustration, the integrated regularized formulation of scalar electrodynamics is also given.

For clarity of presentation, these regularized integrated systems will first be stated and discussed in this chapter, while the analysis of their relation with their corresponding unintegrated formalisms is postponed till Chapter 5. As such, these regularized integrated (SD) formulations may be simply interpreted as regularizations of formal action statements, as discussed in previous chapters.

The plan for this chapter is as follows. Section 4.1 contains the statement of the integrated regularized formulation for gauge theory with Dirac fermions, as well as that for scalar electrodynamics. The approach is basically Schwinger-Dyson,

though stochastic equivalents are also given. Section 4.2 discusses the regularized SD rules for the weak-coupling expansion of the integrated systems in d -dimensions, emphasizing the simplicity of the rules relative to the regularized SIAG systems of Chapter 3. Section 4.3 illustrates the simplicity of the rules in a computation of the fermionic contribution to the leading term in the vacuum polarization. The result is transverse in any dimension, in contrast to the Zwanziger-SIAG effect found in Chapter 3 and Ref. [39].

4.1 Regularized Fermions without Grassmann Variables

In this section, the basic Schwinger-Dyson formulation of the new regularization scheme for integrated gauge-theory fermions will be given.

To regularize the d -dimensional gauge theory with Dirac fermions whose Euclidean action is (3.1), the regularized SD equations

$$0 = \left\langle \left[L_{\text{YM}} + ig \int (dx) \text{Tr} [T^a \gamma_\mu \{ (\not{D} + m)^{-1} R^2 \}_{xx}] \frac{\delta}{\delta A_\mu^a(x)} \right] F[A] \right\rangle \quad (4.1)$$

is proposed for computing the averages of any functional $F[A]$ of the gauge field.

Here

$$L_{\text{YM}} \equiv - \int (dx) \left[\frac{\delta S_{\text{YM}}}{\delta A_\mu^a(x)} + Z^b(x) D_\mu^{ba} \right] \frac{\delta}{\delta A_\mu^a(x)} + \Delta \quad (4.2)$$

is the usual regularized SD operator for pure Yang-Mills theory, with regularized functional Laplacian [20,24]

$$\Delta = \int (dx)(dy) (R^2)_{xy}^{ab} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^b(y)} \quad (4.3)$$

as a functional of the gauge-field regulator $R(\Delta)$. The fermionic regulator $\mathcal{R}(\mathcal{D}^2)$ appears in the additional *determinantal term* of eq. (4.1), which generates the regularized fermionic contributions to any gauge-field average.

The prescription for the fermionic averages is

$$\begin{aligned} & \left\langle \exp \left[\int (dx) \{ \bar{\chi}_i^A(x) \psi_i^A(x) + \bar{\psi}_i^A(x) \chi_i^A(x) \} \right] F[A] \right\rangle \\ &= \left\langle \exp \left[\int (dx)(dy) \bar{\chi}_i^A(x) \left[(\{\mathcal{D} + m\}^{-1} \mathcal{R}^2)_{ij}^{AB} \right]_{xy} \chi_j^B(y) \right] F[A] \right\rangle, \quad (4.4) \end{aligned}$$

where $\chi_i^A, \bar{\chi}_j^B$ are Grassmann sources. This is a shorthand² which records the vanishing of Green functions with non-zero fermion number, along with the family of statements

$$\left\langle \psi_i^A(x) \bar{\psi}_j^B(y) F_1[A] \right\rangle = \left\langle \left[(\{\mathcal{D} + m\}^{-1} \mathcal{R}^2)_{ij}^{AB} \right]_{xy} F_1[A] \right\rangle, \quad (4.5a)$$

$$\begin{aligned} & \left\langle \psi_i^A(x) \psi_j^B(y) \bar{\psi}_k^C(u) \bar{\psi}_l^D(v) F_2[A] \right\rangle \\ &= \left\langle \left[(\{\mathcal{D} + m\}^{-1} \mathcal{R}^2)_{id}^{AD} \right]_{xu} \left[(\{\mathcal{D} + m\}^{-1} \mathcal{R}^2)_{jk}^{BC} \right]_{yv} F_2[A] \right\rangle \\ & \quad - \left\langle \left[(\{\mathcal{D} + m\}^{-1} \mathcal{R}^2)_{ik}^{AC} \right]_{xu} \left[(\{\mathcal{D} + m\}^{-1} \mathcal{R}^2)_{jl}^{BD} \right]_{yv} F_2[A] \right\rangle, \quad (4.5b) \end{aligned}$$

and so on with arbitrary $F_n[A]$, where n labels fermion pair number. The fermionic prescription eq. (4.4) expresses all fermionic averages in terms of gauge-field averages, which may then be computed from the SD equations (4.1).

It is easy to check the validity of the SD system eqs. (4.1,4) as a regularization of the theory whose formal action is eq. (3.1). When $\mathcal{R} = R = 1$ and Zwanziger's

Z^a is omitted, the SD equations (4.1) are equivalent to the formal relations

$$0 = \int \mathcal{D}A \int (dx) \frac{\delta}{\delta A_\mu^a(x)} \left[\det[\mathcal{D} + m] e^{-S_{\text{YM}}} \frac{\delta}{\delta A_\mu^a(x)} F[A] \right] \quad (4.6)$$

at the action level: The determinantal term in eq. (4.1) is simply a version of the formal expression $\delta \text{Tr}[\ln(\mathcal{D} + m)]/\delta A_\mu^a$, regularized after the differentiation. Similarly, the fermionic prescription eq. (4.4) is a regularized version of standard unregularized formal relations.

It should also be remarked that any fermionic object gauge-invariant in A , ψ , and $\bar{\psi}$ is given, according to the prescription eq. (4.4), by a gauge-invariant construction in A . Since the SD operator without the Zwanziger gauge-fixing term is gauge-invariant, it follows [20] that *all* gauge-invariant observables $F_{\text{GI}}[A]$, including the fermionic constructions, will be independent of the Zwanziger gauge-fixing.

The “quenched” approximation, in which internal fermions are suppressed, is also easy to write down. One simply omits the determinantal term in the SD equations (4.1), while maintaining the fermionic prescription eq. (4.4).

There are stochastic equivalent prescriptions for the above integrated formulation. Following Ref. [24], the SD eqs. (4.1) are equivalent to

$$\dot{A}_\mu^a(x, t) = - \frac{\delta S_{\text{YM}}}{\delta A_\mu^a}(x, t) + D_\mu^{ab} Z^b(x, t) + \int (dy) R_{xy}^{ab} \eta_\mu^b(y, t) \quad (4.7a)$$

$$+ ig \text{Tr}[T^a \gamma_\mu \{(\mathcal{D} + m)^{-1} R^2\}_{xx}(t)] - \int (dy)(dz) R_{yz}^{bc} \frac{\delta R_{xy}^{ab}}{\delta A_\mu^c(z)},$$

$$\langle \eta_\mu^a(x, t) \eta_\nu^b(x', t') \rangle = 2\delta^{ab} \delta_{\mu\nu} \delta(x - x') \delta(t - t'), \quad (4.7b)$$

with the Stratonovich calculus [36]. In this case, both the determinantal term and the final RVC_1 counterterm [24] correspond to loops in the Langevin trees. Alternatively, the RVC_1 counterterm may be omitted with the Ito calculus [36], which simply eliminates all functional derivatives of R (all RVC_1 's).

The case of fermions illustrates the steps to follow in integrating out any quadratic matter fields coupled to the gauge field. As a further example, it is easy to verify that the SD equations

$$0 = \int(dx) \left\langle \left[-\frac{\delta S^{(0)}}{\delta A_\mu(x)} + \partial_\mu Z(x) + \int(dy) R_{xy}^2(\square) \frac{\delta}{\delta A_\mu(y)} + ie\{\Delta^{-1} R^2(\Delta), D_\mu\}_{xx} \right] \frac{\delta}{\delta A_\mu(x)} F[A] \right\rangle \quad (4.8)$$

describe regularized scalar electrodynamics [20] after integration of the charged scalars. Here

$$S^{(0)} = \frac{1}{4} \int(dx) F_{\mu\nu} F_{\mu\nu}, \quad \Delta \equiv D_\mu D_\mu = (\partial_\mu - ieA_\mu)^2, \quad (4.9)$$

and the curly bracket in the (last) determinantal term denotes anticommutator.

Similarly, with J a complex source, the prescription

$$\begin{aligned} & \left\langle \exp \left[\int(dx) \{ J^*(x) \phi(x) + \phi^*(x) J(x) \} \right] F[A] \right\rangle \\ &= \left\langle \exp \left[- \int(dx)(dy) J^*(x) \left[\Delta^{-1} R^2(\Delta) \right]_{xy} J(y) \right] F[A] \right\rangle \quad (4.10) \end{aligned}$$

gives the charged scalar averages.

4.2 Regularized Schwinger-Dyson Diagrams

The regularized SD rules for the weak-coupling expansion of the integrated SD systems (4.1,4) are given in this section. For simplicity, I choose the canonical Zwanziger term $Z^a = \alpha^{-1} \partial \cdot A^a$, and heat kernel regularization [24]

$$R = \exp(\Delta/\Lambda^2), \quad \mathcal{R} = \exp(\not{D}^2/\Lambda^2), \quad (4.11)$$

for which the systems are regularized³ to all orders in d -dimensions.

Since the fermionic prescription eq. (4.4) reduces fermionic averages to gauge-field averages, we concentrate first on the gauge-field SD rules which follow from the SD eqs. (4.1). The pure Yang-Mills SD rules for the operator L_{YM} have been discussed in Refs. [18,20], so only the additional rules necessary to include the contributions of the determinantal term to the gauge-field averages are discussed here.

Solid-line factors remain purely gluonic [20], but extra *composite SD vertices* (fermion loops) are generated by the determinantal term. Since this term has one hanging derivative, the (non-local) coefficient of each power of the gauge field A in the weak-coupling expansion of $\text{Tr}[T^a \gamma_\mu \{(\not{D} + m)^{-1} \mathcal{R}^2\}_{xx}]$ corresponds to a SD composite vertex with one incoming gluon line.

Study of the composite SD vertices begins with the expansion of the fermion propagator

$$(\not{D} + m)^{-1} = \sum_{n=0}^{\infty} [(\not{D} + m)^{-1} \{-igT^a A^a\}]^n (\not{D} + m)^{-1}, \quad (4.12a)$$

and the square of the fermion regulator

$$\begin{aligned} \mathcal{R}^2 = \exp(2\mathcal{D}^2/\Lambda^2) &= \exp(2\Box/\Lambda^2) + \sum_{n=2}^{\infty} \int_0^1 \prod_{j=1}^n d\beta_j \delta(1 - \sum_{k=1}^n \beta_k) \\ &\times \exp(2\beta_1\Box/\Lambda^2)V \exp(2\beta_2\Box/\Lambda^2)V \cdots V \exp(2\beta_n\Box/\Lambda^2), \end{aligned} \quad (4.12b)$$

$$[V_{ij}^{AB}]_{xy} \equiv \frac{2}{\Lambda^2} [g\Gamma^{(1)}(x) + g^2\Gamma^{(2)}(x)]_{ij}^{AB} \delta(x-y), \quad (4.12c)$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are the one- and two-gluon regulator vertices defined in eq. (3.6). Figures 4.1 and 4.2 give the diagrammatic interpretation of the expansion

$$[(\mathcal{D} + m)^{-1}]_{ij}^{AB} = \frac{A}{i} \longrightarrow \frac{B}{j} + \frac{A}{i} \longrightarrow \begin{array}{c} \uparrow \\ \text{wavy line} \\ \uparrow \end{array} \longrightarrow \frac{B}{j} + \frac{A}{i} \longrightarrow \begin{array}{c} \uparrow \\ \text{wavy line} \\ \uparrow \end{array} \longrightarrow \begin{array}{c} \uparrow \\ \text{wavy line} \\ \uparrow \end{array} \longrightarrow \frac{B}{j} + O(g^3)$$

Fig. 4.1 Fermion propagator strings.

$$\begin{aligned} [\exp(\mathcal{D}^2/\Lambda^2)]_{ij}^{AB} &= \frac{A}{i} \longrightarrow \frac{B}{j} + \frac{A}{i} \longrightarrow \begin{array}{c} \uparrow \\ \text{wavy line} \\ \uparrow \end{array} \longrightarrow \frac{B}{j} + \frac{A}{i} \longrightarrow \begin{array}{c} \uparrow \\ \text{wavy line} \\ \uparrow \end{array} \longrightarrow \begin{array}{c} \uparrow \\ \text{wavy line} \\ \uparrow \end{array} \longrightarrow \frac{B}{j} \\ &+ \frac{A}{i} \longrightarrow \begin{array}{c} \uparrow \\ \text{wavy line} \\ \uparrow \end{array} \longrightarrow \begin{array}{c} \uparrow \\ \text{wavy line} \\ \uparrow \end{array} \longrightarrow \frac{B}{j} + O(g^3) \end{aligned}$$

Fig. 4.2 Fermion heat kernel as fermion regulator strings.

$$\begin{aligned}
[(\not{D}+m)^{-1} \exp(\not{D}^2/\Lambda^2)]_{ij}^{AB} = & \text{Diagram 1} + \text{Diagram 2} \\
& + \text{Diagram 3} + O(g^2)
\end{aligned}$$

Fig. 4.3 Regularized fermion propagator strings (RFP strings).

sions (4.12a) and (4.12b) as, respectively, *fermion propagator strings* and *fermion regulator strings*. The product $(\not{D}+m)^{-1}\mathcal{R}^2$ of the two expansions, as a sum of *regularized fermion propagator strings* (*RFP strings*), is shown in Fig. 4.3.

The β -parameters in any RFP string are integrated from zero to one, subject to the constraint that their sum is unity [24]. As in Chapter 3, the thin arrows on the fermion lines in these diagrams indicate the charge flow, while the solid arrows on the gluon lines track the direction of allowed (gauge field) SD ordering (or, equivalently, the direction of decreasing Markov time in the Langevin formulation). Fig. 4.4 gives the momentum-space diagrammatic rules for constructing the RFP strings in terms of *RFP vertices*.

The composite SD vertices are simply RFP loops (Fig. 4.5b), *closed* with a *determinantal vertex* $igT^a\gamma_\mu$, shown in Fig. 4.5a. Each RFP loop is a SD *vertex unit* [20], requiring *no SD ordering of the RFP vertices within a loop*. This feature results in a general reduction of the number of SD pictures for each SD diagram

(a) Fermion propagator:

$$\begin{array}{c} A \\ i \end{array} \xrightarrow[p \rightarrow]{} \begin{array}{c} B \\ j \end{array} = \delta^{AB} \left[\frac{1}{-i\not{p} + m} \right]_{ij}$$

(b) Fermion regulator propagator:

$$\begin{array}{c} A \\ i \end{array} \xrightarrow[\beta]{p} \begin{array}{c} B \\ j \end{array} = \delta^{AB} \delta_{ij} e^{-2\beta p^2 / \Lambda^2}$$

(c) RFP vertices:

$$\begin{array}{c} A \\ i \end{array} \xrightarrow{\quad} \text{---} \xrightarrow{\quad} \begin{array}{c} B \\ j \end{array} = \delta^{AB} \delta_{ij}$$

$$\begin{array}{c} A \\ i \end{array} \xrightarrow{\quad} \begin{array}{c} B \\ j \end{array} = -ig(\gamma_\mu)_{ij}(T^a)^{AB}$$

$\mu \quad a$

$$\begin{array}{c} A \\ i \end{array} \xrightarrow{p} \begin{array}{c} B \\ j \end{array} = 2g(T^a)^{AB} [i(\sigma_{\nu\mu})_{ij} q_\nu + \delta_{ij}(p-k)_\mu] / \Lambda^2$$

$q \quad \mu \quad a$

$$\begin{array}{c} A \\ i \end{array} \xrightarrow{\quad} \begin{array}{c} B \\ j \end{array} = -2g^2 [f^{abc}(\sigma_{\mu\nu})_{ij}(T^c)^{AB} + \delta_{\mu\nu} \delta_{ij} \{T^a, T^b\}^{AB}] / \Lambda^2$$

$\mu \quad a \quad \nu \quad b$

Fig. 4.4a-c Diagrammatic rules for RFP string construction.

relative to the regularized SIAG systems of Chapter 3. Note that the first loop (tadpole) in the expansion of Fig. 4.5b is zero since $\text{Tr}[\gamma_\mu T^a] = 0$, so internal fermions begin to contribute at order g^2 . With the rules for L_{YM} in Refs. [20,24], this completes the SD rules for the computation of any gauge field average.

Fig. 4.5a–b Expansion of RFP loops.

Counting the determinantal vertex with the RFP vertices of Fig. 4.4, Five vertices have been introduced in total. This should be compared with the regularized SIAG systems of Chapter 3, in which nine vertices were necessary. As we shall see below, this results in a serious reduction of the number of diagrams at a given

$$\begin{aligned}
\langle \psi_i^A(p) \bar{\psi}_j^B(q) \rangle &= \begin{array}{c} A \\ \longrightarrow \\ i \end{array} \longrightarrow \begin{array}{c} B \\ j \end{array} \\
&+ \left\langle \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \\ \text{diagram 4} \end{array} \right\rangle + O(g^3) \\
&= (2\pi)^d \delta^{AB} \delta(p+q) \left[\frac{e^{-2p^2/\Lambda^2}}{-i\not{p} + m} \right]_{ij} \\
&+ \begin{array}{c} \text{diagram 5} \\ \text{diagram 6} \\ \text{diagram 7} \\ \text{diagram 8} \end{array} + O(g^3)
\end{aligned}
\tag{a}$$

$$\tag{b}$$

Fig. 4.6a–b Diagrams for $\langle \psi_i^A(p) \bar{\psi}_j^B(q) \rangle$: (a) is the RFP string expansion and (b) shows the SD gauge-field averages.

order in weak coupling.

Finally, to compute the fermionic averages, each factor of $(\not{D} + m)^{-1} \mathbb{R}^2$ in the prescription eq. (4.4) is expanded as *open* (in general) RFP strings, and then the gauge-field SD rules above are used to compute the resulting gauge-field averages. As an illustration, the diagrams for $\langle \psi_i^A(p) \bar{\psi}_j^B(q) \rangle$ through order g^2 are exhibited in Fig. 4.6. In this figure, a thick line without an arrow (Fig. 4.7) represents the heat-kernel gauge field regulator propagator [24]. Fig. 4.6a shows the expansion of the two-point fermionic construction as open RFP strings, while Fig. 4.6b shows the subsequent gauge-field averaging, according to the gauge-field SD rules above.

For example, the value

$$\begin{aligned}
 & -g^2(2\pi)^d \delta(p+q) e^{-2p^2/\Lambda^2} (T^a T^a)^{AB} \int (dk) \left[\frac{T_{\mu\nu}(k) + \alpha L_{\mu\nu}(k)}{k^2} \right] e^{-2k^2/\Lambda^2} \\
 & \times \left[\frac{1}{-i\not{p} + m} \gamma_\mu \frac{1}{-i\not{p} + i\not{k} + m} \gamma_\nu \frac{1}{-i\not{p} + m} \right]_{ij} \quad (4.13)
 \end{aligned}$$

is obtained for the first loop diagram of Fig. 4.6b.

$$\begin{array}{ccc}
 a & p & b \\
 \hline
 \mu & \beta & \nu
 \end{array} = \delta^{ab} \delta_{\mu\nu} e^{-\beta p^2/\Lambda^2}$$

Fig. 4.7 Gauge-field regulator propagator.

4.3 Fermionic Contribution to the Vacuum Polarization

As an illustration of the simplicity of the integrated formulation, and as an explicit check on the gauge-invariance of the regularization, the SD rules of the previous section are used to compute the leading fermionic contribution to the gauge-field vacuum polarization in d -dimensions. In particular, since the regularized Yang-Mills contribution to the gluon mass is zero [18,20,24], the fermionic mass contribution must also vanish.

There are altogether $2 \times 2 = 4$ diagrams with one internal fermion loop which contribute to the vacuum polarization $\Pi_{\mu\nu}^{ab}(p)$. Two of these are shown in Fig. 4.8, while the other two are trivially obtained from those of Fig. 4.8 by interchanging (a, μ, p) with $(b, \nu, -p)$. The dotted box [20] in each diagram indicates here that the entire RFP loop is treated as a vertex unit. The small number of diagrams

is noteworthy, since the regularized SIAG formulation of Chapter 3 required 24 diagrams at this level.

Fig. 4.8a-b Second order vacuum polarization diagrams.

As an explicit example, the SD diagram 4.8b, which contains a fermionic regulator vertex, is evaluated below. The case with both external gluons transverse is shown with all relevant indices in Fig. 4.9, and the sequence of SD pictures for the diagram is given in Fig. 4.10. The RFP-loop factor $\Gamma_{\sigma\sigma}^{cd}(p)$ of Fig. 4.10 is readily

Fig. 4.9 Diagram 4.8b with indices; the external gluons are transverse.

computed from the rules of Fig. 4.4,

$$\Gamma_{\rho\sigma}^{cd}(p) = \int_0^1 d\beta \int (dq) e^{-2\beta(p+q)^2/\Lambda^2} e^{-2(1-\beta)q^2/\Lambda^2} \quad (4.14)$$

$$\times ig \text{Tr} \left[\left\{ T^c \gamma_\rho \frac{1}{-i\not{q} + m} \right\} \frac{2igT^d}{\Lambda^2} \left\{ -i\sigma_{\nu\sigma} p_\nu - (2q + p)_\sigma \right\} \right] .$$

Collecting the factors from the SD pictures, transverse projectors for gluon lines, and appropriate sums in color, flavor, and Dirac indices, the value

$$-\frac{g^2}{\Lambda^2} C_R d_\gamma \delta^{ab} f \frac{e^{-2p^2/\Lambda^2}}{p^4} \int_0^1 d\beta \int (dq) e^{-2\beta(p+q)^2/\Lambda^2} e^{-2(1-\beta)q^2/\Lambda^2} \frac{1}{q^2 + m^2} \quad (4.15)$$

$$\times [2q_\rho q_\sigma + p_\rho q_\sigma + p_\sigma q_\rho - (p \cdot q) \delta_{\rho\sigma}] T_{\mu\rho}(p) T_{\nu\sigma}(p) ,$$

is found for the SD diagram in Fig. 4.9, where $d_\gamma = 2^{[d/2]}$ is the dimension of Dirac matrices in d -dimensions.

Fig. 4.10 SD pictures for Fig. 4.9.

Diagram 4.8b with external longitudinal gluons is similarly evaluated. After truncation [20], one obtains the total contribution of diagram 4.8b (plus its interchange) to the zero momentum vacuum polarization

$$\Pi_{\mu\nu}^{ab}(0)\Big|_{4.8b} = \frac{2}{d}\delta_{\mu\nu}\mathcal{M}^{ab}\Lambda^{d-2}\int_0^\infty dy \frac{y^{d/2}e^{-2y}}{y+m^2/\Lambda^2}, \quad (4.16)$$

where

$$\mathcal{M}^{ab} \equiv \frac{g^2 C_R d_\gamma \delta^{ab} f}{(4\pi)^{d/2} \Gamma(d/2)}. \quad (4.17)$$

The same contribution with opposite sign is obtained from the ordinary diagram 4.8a (plus its interchange), so the gluon remains massless to this order in all dimensions.

The leading p^2 contribution to $\Pi_{\mu\nu}^{ab}(p)$ may also be computed by differentiation with respect to external momentum. Adding the contribution of diagram 4.8a and 4.8b, together with the $(\mu, a, p) \leftrightarrow (\nu, b, -p)$ interchanges, one obtains the total fermionic contribution to the vacuum polarization in d -dimensions⁴

$$\Pi_{\mu\nu}^{(f)ab}(p) = -\left(\frac{d-2}{6}\right)\mathcal{M}^{ab}p^2 T_{\mu\nu}(p)\Lambda^{d-4}\int_0^\infty dy \frac{y^{(d-4)/2}e^{-2y}}{y+m^2/\Lambda^2} + O(p^4). \quad (4.18)$$

Note that the result eq. (4.18) is transverse in all dimensions. This is not surprising here, since the fermionic-Zwanziger terms, which cause the fermionic non-transversality phenomenon in Chapter 3 and Ref. [39], are not present in the integrated formulation. I shall return to this point in the next chapter. The standard result $-(g^2 C_R \delta^{ab} f / 12\pi^2) p^2 T_{\mu\nu} \ln(\Lambda^2/m^2)$ is easily obtained from eq. (4.18) when $d = 4$.

Footnotes: Chapter 4

- ¹ Lattice analogues of non-Grassmann formulation of gauge theory with fermions have been studied in Ref. [41].
- ² The shorthand eq. (4.4) also includes the usual [20,21] global sign convention, implied by the SD boundary condition that the averages have the usual permutation symmetries and antisymmetries.
- ³ The minimal SD regulators for $d = 4$ dimensions (QCD) are $R = (1 - \Delta/\Lambda^2)^{-1}$, $\mathcal{R} = (1 - \not{D}^2/\Lambda^2)^{-1}$, as in Chapter 3.
- ⁴ The regulator vertex diagram 4.8b (or its interchange) contributes to renormalization (terms $O(p^2) \times$ (growing with Λ)) only when $d > 4$.

Chapter 5

Integration of Quadratic Matter in Regularized Formulations

In this chapter, I study the relation of the integrated systems of Chapter 4 to the regularized Grassmann SIAG systems of Chapter 3. In section 5.1, a λ -family of regularized SIAG systems is given, with the case $\lambda = 1$ corresponding to the systems of Chapter 4. Section 5.2 discusses some indications that the large λ limit of this SIAG λ -family is the integrated formulation. A proof of this correspondence is given in section 5.3. The techniques introduced in this section are equivalent to *regularized integration* of matter fields *within* the regularized theories. Finally, section 5.4 discusses the regularized integration technique in a number of other cases, including scalar electrodynamics.

5.1 A λ -Family of Regularized SIAG Systems

We begin our analysis with the λ -family of regularized SIAG equations

$$\dot{A}_\mu^a(x, t) = -\frac{\delta S}{\delta A_\mu^a}(x, t) + D_\mu^{ab} Z^b(x, t) + \int (dy) R_{xy}^{ab} \eta_\mu^b(y, t), \quad (5.1a)$$

$$\begin{aligned} \dot{\psi}_i^A(x, t) = & \lambda(\vec{\mathcal{D}}_x^2 - m^2)_{ij}^{AB} \psi_j^B(x, t) + \int (dy) (R_{ij}^{AB})_{xy} (\eta_2^B)_j(y, t) \\ & - \lambda(\vec{\mathcal{D}}_x - m)_{ij}^{AB} \int (dy) (R_{jl}^{BC})_{xy} (\eta_1^C)_l(y, t) \\ & - ig Z^a (T^a)^{AB} \psi_i^B(x, t), \end{aligned} \quad (5.1b)$$

$$\begin{aligned}
\dot{\bar{\psi}}_i^A(x, t) = & \lambda \bar{\psi}_j^B(x, t) (\bar{\mathcal{D}}_x^2 - m^2)_{ji}^{BA} + \int (dy) (\bar{\eta}_1^B)_j(y, t) (IR_{ji}^{BA})_{yx} \\
& + \lambda \int (dy) (\bar{\eta}_2^C)_l(y, t) (IR_{lj}^{CB})_{yx} (\bar{\mathcal{D}}_x + m)_{ji}^{BA} \\
& + ig Z^a (T^a)^{BA} \bar{\psi}_i^B(x, t),
\end{aligned} \tag{5.1c}$$

plus the usual Gaussian noise averages (3.5) and Ito calculus. Our primary interest, however, will be in the equivalent λ -family of SD equations

$$\begin{aligned}
0 = & \left\langle \left\{ \Delta + \int (dx) \left(\left[-\frac{\delta S_{YM}}{\delta A_\mu^a(x)} - ig \bar{\psi}_i^A(x) (T^a)^{ab} (\gamma_\mu)_{ij} \psi_j^B(x) \right] \frac{\delta}{\delta A_\mu^a(x)} \right. \right. \right. \\
& - \lambda \left[(\bar{\mathcal{D}}_x - m)_{ij}^{AB} \left(-\frac{\delta S}{\delta \bar{\psi}_j^B(x)} + \int (dy) (IR_{xy}^2)_{jk}^{BC} \frac{\delta}{\delta \bar{\psi}_k^C(y)} \right) \right] \frac{\delta}{\delta \psi_i^A(x)} \\
& - \lambda \left[\left(-\frac{\delta S}{\delta \psi_j^B(x)} + \int (dy) (IR_{yx}^2)_{kj}^{CB} \frac{\delta}{\delta \psi_k^C(y)} \right) (\bar{\mathcal{D}}_x + m)_{ji}^{BA} \right] \frac{\delta}{\delta \bar{\psi}_i^A(x)} \\
& \left. \left. \left. - Z^a(x) G^a(x) \right) \right\} F[A, \psi, \bar{\psi}] \right\rangle,
\end{aligned} \tag{5.2}$$

where $G^a(x)$ is the generator of non-abelian gauge transformations defined in eq. (3.26).

The only difference between eqs. (5.1,2) and the regularized SIAG systems of eq. (3.4,5) is the insertion of an extra factor of the dimensionless constant $\lambda > 0$ in the SIAG kernel $(\bar{\psi} \pm m)$. Such a modification of the kernel does not affect the equilibrium theory at the level of the formal unregularized system, as seen in an examination of the (formal) Fokker-Planck equations without Zwanziger gauge-fixing. Alternatively, when Zwanziger gauge-fixing and the regulators are removed,

the SD family (5.2) corresponds to the λ -family of formal statements

$$0 = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \int (dx) \frac{\delta}{\delta A_\mu^a(x)} \left[e^{-S} \frac{\delta}{\delta A_\mu^a(x)} F \right], \quad (5.3a)$$

$$0 = - \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \int (dx) \left[\lambda (\vec{\mathcal{D}}_x - m)_{ij}^{AB} \frac{\delta}{\delta \bar{\psi}_j^B(x)} \right] \left[e^{-S} \frac{\delta}{\delta \psi_i^A(x)} F \right], \quad (5.3b)$$

$$0 = - \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \int (dx) \frac{\delta}{\delta \psi_j^B(x)} \lambda (\vec{\mathcal{D}}_x + m)_{ji}^{BA} \left[e^{-S} \frac{\delta}{\delta \bar{\psi}_i^A(x)} F \right]. \quad (5.3c)$$

at the action level.

On the other hand, in the presence of the regulator (and/or the Zwanziger gauge-fixing), eq. (5.2) defines a λ -family of *distinct* systems, each of which is a gauge-fixed regularization of the formal theory. As will be argued in section 5.3, the large λ limit is the simplest of the family, and in fact corresponds to the integrated formulation of Chapter 4.

The introduction of the parameter λ in eq. (5.2) leads to a few systematic changes, wherever the SIAG kernel enters, relative to the $\lambda = 1$ SD rules of Chapter 3.

λ -modified SD rules

(1) Solid line factors: Every fermionic term $\lambda(p^2 + m^2)$ in a solid line factor now carries an extra factor λ .

(2) Ordinary vertices: All fermionic vertices which originate from $\delta S / \delta \psi$ or $\delta S / \delta \bar{\psi}$ now carry an extra factor of λ . This includes the second, third and fourth ordinary SIAG vertices in Fig. 3.1c of Chapter 3, except for their α^{-1} fermionic-Zwanzi-

ger terms. We may therefore assign an overall λ factor to these vertices, while modifying the fermionic-Zwanziger parameter $\alpha^{-1} \rightarrow (\lambda\alpha)^{-1}$. At least formally, this implies the vanishing of the fermionic Zwanziger contributions at large λ (and $\alpha \neq 0$) relative to other contributions. I shall return to this below.

(3) Joining vertices: The second and third joining vertices in Fig. 3.1d of Chapter 3 now carry an extra factor λ .

(4) Contraction: The fermionic simple contraction factor

$\delta^{ab}\delta_{ij}R_0^2(p)/[2\lambda(p^2 + m^2)]$ now carries an additional factor λ^{-1} .

There are no changes in the regulators or their strings when $\lambda \neq 1$.

5.2 Indications of a Large λ Correspondence

This λ -family of regularized SD rules are now employed to discuss early indications that the regularized SIAG λ -family eq. (5.2) approaches the integrated regularization as $\lambda \rightarrow \infty$. It is trivial to check directly that the resulting regularized λ -tree graphs are independent of λ , and in agreement with the prescription (4.4).

A non-trivial indication of the large λ correspondence comes from a study of the one-loop fermionic contribution to the gluon vacuum polarization in four dimensions. Following the computation of Chapter 3, the result for the leading

term at arbitrary λ is

$$\begin{aligned} \Pi^{(f)ab}_{\mu\nu}(p) &= \frac{-g^2 C_R \delta^{ab} f}{16\pi^2} p^2 \left[\frac{4}{3} T_{\mu\nu}(p) + \frac{1}{\alpha\lambda} L_{\mu\nu}(p) \right] \ln \frac{\Lambda^2}{m^2} \\ &+ \text{terms finite as } \Lambda \rightarrow \infty, \end{aligned} \quad (5.4)$$

or

$$\begin{aligned} \langle A_\mu^a(p) A_\nu^b(q) \rangle^{(f)} &= (2\pi)^4 \delta(p+q) \frac{-g^2 C_R \delta^{ab} f}{16\pi^2} \frac{1}{p^2} \left(\frac{4}{3} T_{\mu\nu} + \frac{\alpha}{\lambda} L_{\mu\nu} \right) \ln \frac{\Lambda^2}{m^2} \\ &+ \text{terms finite as } \Lambda \rightarrow \infty. \end{aligned} \quad (5.5)$$

It is noteworthy that the smooth approach at large λ to the conventional transverse result is in agreement with the result (4.13) of the integrated regularization: The vanishing of the fermionic-Zwanziger contributions at large λ , anticipated in the λ -modified SD rule (2) above, is not formal.

A further indication of the correspondence is found on examination of the 24 SIAG diagrams of this computation at arbitrary external momentum. After cancellation of λ -factors from fermionic vertices against λ^{-1} factors from large λ fermionic solid-line factors, the four diagrams of Fig. 3.4 in Chapter 3 vanish at large λ , while the other twenty diagrams add to exactly the four diagrams (Fig. 4.8) of the integrated scheme.

5.3 Integration of Regularized Quadratic Matter at Large λ

This section contains the proof that the SIAG λ -family eq. (5.2) is equivalent at large λ to the integrated regularization eqs. (4.1,4).

The proof involves two stages. a) A *no-growth theorem*, that all regularized Green functions are bounded by constants at large λ . This part of the proof is given to all orders in weak coupling, though a non-perturbative proof would be preferable. b) A direct *non-perturbative solution of the SD equations* (5.2) at large λ , assuming the no-growth theorem. The techniques of this procedure are equivalent to the integration of matter fields within the regularized formulations.

No-growth theorem

The intuitive basis for the no-growth theorem is that the unregularized and un-gauge-fixed equilibrium theory is independent of λ , and so exhibits no growth. Dimensional regularization (and Faddeev-Popov gauge-fixing) in SD equations analogous to eq. (5.2), for example, would give λ -independent results for the *sums* of SD diagrams at a given order in weak coupling. Since our regulator reproduces dimensional regularization in weak coupling, we should obtain λ -independence at least at large cutoff Λ , except possibly for Zwanziger effects. In fact, the no-growth theorem below is independent of R and $\mathcal{R}$, and no difficulties are found with Zwanziger's gauge-fixing.

The proof of the no-growth theorem follows immediately from λ -power counting of the SD diagrams. From the λ -SD rules of section 5.1 there are only two possible sources of growth in λ . These are the ordinary fermionic vertices of rule (2), and the joining vertices of rule (3), each of which carries a factor λ . In both cases, compensating factors of λ^{-1} are easily found: When a SD picture contains a rule

(2)-vertex, the preceding SD picture contains a fermionic solid line factor, which carries an explicit λ^{-1} at large λ . Further, if a SD picture contains a rule (3)-joining vertex, that picture also contains (after some length of regulator string) a rule (4)-contraction factor, with an explicit λ^{-1} . The total power of λ in any SD diagram is therefore never positive, so no growth is possible at large λ .

Solution of SD equations at large λ

The first step in this stage is to record the relation implied by the SD λ -family eq. (5.2) for any functional $F[A]$ of the gauge field alone

$$0 = \left\langle \left[L_{\text{YM}} - ig \int (dx) \bar{\psi}_i^A(x) (T^a)^{AB} (\gamma_\mu)_{ij} \psi_j^B(x) \frac{\delta}{\delta A_\mu^a(x)} \right] F[A] \right\rangle. \quad (5.6)$$

This equation is true at all λ . The strategy is to show that the fermionic current term $\bar{\psi} T^a \gamma_\mu \psi$ of eq. (5.6) may be replaced, at large λ , by the determinantal term of eq. (4.1).

The next step is the observation that the SD λ -family (5.2) has an explicit factor of λ multiplying all fermionic terms (except the fermionic-Zwanziger gauge-fixing). The no-growth theorem tells us that these terms must therefore be set to zero

$$\begin{aligned} & \int (dx)(dy) \left\langle \left\{ \left[(\not{D}^2 - m^2)_{ij}^{AB} \right]_{xy} \psi_j^B(y) \frac{\delta}{\delta \psi_i^A(x)} \right. \right. \\ & \quad \left. \left. + \left[(\not{D}^2 - m^2)^T_{ij}^{AB} \right]_{xy} \bar{\psi}_j^B(y) \frac{\delta}{\delta \bar{\psi}_i^A(x)} \right\} G[A, \psi, \bar{\psi}] \right\rangle_{(0)} \\ & = 2 \int (dx)(dy) \left\langle \left[(\{\not{D} - m\} R^2)_{ij}^{AB} \right]_{xy} \frac{\delta}{\delta \bar{\psi}_j^B(y)} \frac{\delta}{\delta \psi_i^A(x)} G[A, \psi, \bar{\psi}] \right\rangle_{(0)} \end{aligned} \quad (5.7)$$

in the large λ limit, in order to prevent the growth of the other terms. In eq. (5.7), the superscript T denotes transpose while the subscript (0) on an equation is a reminder that the equation is only true in the large λ limit.

The λ -independent equations (5.6) and (5.7) together are accurately termed *the SD equations at large λ* . Notice that the fermionic-Zwanziger terms do not appear in the SD equations at large λ , as anticipated in sections 4.3, 5.1, and 5.2. These equations are clearly true at the unregularized and un-gauge-fixed action level, and so form—in their own right—an adequately gauge-fixed and regularized version of the original theory.

It is a remarkable fact that eq. (5.7) is a disguised version of the integrated fermionic prescription eq. (4.4). To see this, it is necessary to expand $G[A, \psi, \bar{\psi}]$ in fermionic moments. As a first example, consider the case of one fermion pair,

$$G_1[A, \psi, \bar{\psi}] = \int (du)(dv) [\Omega_1^{-1}]_{Clu; Dmv}^{Aiz; Bju} \psi_l^C(u) \bar{\psi}_m^D(v) F_1[A], \quad (5.8)$$

in which Ω_1^{-1} is defined by

$$\int (du)(dv) [\Omega_1^{-1}]_{Clu; Dmv}^{Aiz; Bju} [\Omega_1]_{Eks; Fmw}^{Clu; Dmv} = \delta_B^A \delta_k^i \delta(x-z) \delta_F^B \delta_n^j \delta(y-w), \quad (5.9a)$$

$$\begin{aligned} [\Omega_1]_{Clu; Dmv}^{Aiz; Bju} &\equiv [(\mathcal{D}^2 - m^2)_{\bar{u}}^{AC}]_{xu} \delta^{BD} \delta_{jm} \delta(y-v) \\ &+ \left[\{(\mathcal{D}^2 - m^2)^T\}_{jm}^{BD} \right]_{vo} \delta^{AC} \delta_{\bar{u}} \delta(x-u). \end{aligned} \quad (5.9b)$$

The operator Ω_1^{-1} is well defined since Ω_1 has no zero eigenvalues when $m \neq 0$.

Substituting G_1 into eq. (5.7), one obtains

$$\begin{aligned} \langle \psi_i^A(x) \bar{\psi}_j^B(y) F_1[A] \rangle_{(0)} &= 2 \int (du)(dv) \langle [\Omega_1^{-1}]_{Clu; Dmv}^{Aix; Bju} [(\{\not{D} - m\} R^2)_{lm}^{CD}]_{uv} \rangle_{(0)} \\ &= \langle [(\{\not{D} + m\}^{-1} R^2)_{ij}^{AB}]_{xy} F_1[A] \rangle_{(0)}, \end{aligned} \quad (5.10)$$

where the identity

$$[(\{\not{D} - m\} R^2)_{lm}^{CD}]_{uv} = \frac{1}{2} \int (dy)(dz) [\Omega_1]_{Eky; Fvx}^{Clu; Dmv} [(\{\not{D} + m\}^{-1} R^2)_{kn}^{DP}]_{yz} \quad (5.11)$$

has been used to obtain the last step in eq. (5.10). The large λ result eq. (5.10) is precisely the one-pair fermionic prescription eq. (4.5a) of the integrated regularization.

As a special case of eq. (5.10), we have

$$-ig \langle \bar{\psi}(x) T^a \gamma_\mu \psi(x) \frac{\delta F[A]}{\delta A_\mu^a(x)} \rangle_{(0)} = ig \langle \text{Tr} [T^a \gamma_\mu (\{\not{D} + m\}^{-1} R^2)_{xx}] \frac{\delta F[A]}{\delta A_\mu^a(x)} \rangle_{(0)}, \quad (5.12)$$

which states that, at large λ , the fermion current term in eq. (5.6) is the determinantal term in eq. (4.1) of the integrated regularization of Chapter 4. This completes the derivation of the integrated SD equations (4.1) as the large λ limit of the regularized SIAG λ -family eq. (5.2).

In general, with G_n an n -fermion pair moment, eq. (5.7) relates an n -pair Green function to a Green function with $(n-1)$ pairs (from the fermionic functional Laplacian). The complete moment expansion in the sector with zero fermion number may therefore be obtained by induction based on the result eq. (5.10). The explicit

moments G_n are

$$G_n[A, \psi, \bar{\psi}] = \int \prod_{r=1}^n (dx_r)(dy_r) [\Omega_n^{-1}]_{C_1 l_1 x_1, \dots, C_n l_n x_n; D_1 m_1 y_1, \dots, D_n m_n y_n}^{A_1 i_1 u_1, \dots, A_n i_n u_n; B_1 j_1 v_1, \dots, B_n j_n v_n} \times \psi_{l_1}^{C_1}(x_1) \cdots \psi_{l_n}^{C_n}(x_n) \bar{\psi}_{m_1}^{D_1}(y_1) \cdots \bar{\psi}_{m_n}^{D_n}(y_n) F_n[A], \quad (5.13)$$

in which the operator Ω_n is defined by

$$[\Omega_n]_{C_1 l_1 x_1, \dots, C_n l_n x_n; D_1 m_1 y_1, \dots, D_n m_n y_n}^{A_1 i_1 u_1, \dots, A_n i_n u_n; B_1 j_1 v_1, \dots, B_n j_n v_n} \equiv \sum_{r=1}^n [D_n]_{C_r l_r x_r; D_r m_r y_r}^{A_r i_r u_r; B_r j_r v_r}, \quad (5.14a)$$

$$[D_n]_{C_r l_r x_r; D_r m_r y_r}^{A_r i_r u_r; B_r j_r v_r} \equiv [\Omega_1]_{C_r l_r x_r; D_r m_r y_r}^{A_r i_r u_r; B_r j_r v_r} \times \prod_{p \neq r}^n \delta_{C_p}^{A_p} \delta_{l_p}^{i_p} \delta(u_p - x_p) \delta_{D_p}^{B_p} \delta_{m_p}^{j_p} \delta(v_p - y_p). \quad (5.14b)$$

Although further details are omitted for brevity, the result of this straightforward inductive procedure is precisely the integrated fermionic prescription eq. (4.4) in the sector with zero fermion number.

Finally, the content of eq. (5.7) is exhausted in a similar inductive process which implies the vanishing of any Green function with non-zero fermion number. In this way, one completes the proof that the large λ limit of the SIAG λ -family eq. (5.2) is exactly the integrated regularized SD system eqs. (4.1,4).

The integration of the SD system eq. (5.2) at finite λ has also been studied, but the situation in this case seems prohibitively complex.

5.4 Remarks on the Regularized Integration Technique

As another example of the regularized integration technique, consider the λ -

family of regularized SD equations for scalar electrodynamics

$$\begin{aligned}
0 = \int(dx) \left\langle \left\{ \left[-\frac{\delta S^{(0)}}{\delta A_\mu(x)} + \int(dy) R_{xy}^2(\square) \frac{\delta}{\delta A_\mu(y)} \right. \right. \right. \\
+ ie[\phi(x) D_\mu^* \phi^*(x) - \phi^*(x) D_\mu \phi(x)] \left. \frac{\delta}{\delta A_\mu(x)} \right. \\
+ \lambda \left[-\frac{\delta S}{\delta \phi(x)} + \int(dy) R_{xy}^2(\Delta^*) \frac{\delta}{\delta \phi(y)} \right] \frac{\delta}{\delta \phi^*(x)} \quad (5.15) \\
+ \lambda \left[-\frac{\delta S}{\delta \phi^*(x)} + \int(dy) R_{xy}^2(\Delta) \frac{\delta}{\delta \phi^*(y)} \right] \frac{\delta}{\delta \phi(x)} \\
\left. \left. \left. - Z(x) G(x) \right\} F[A, \phi, \phi^*] \right\rangle,
\end{aligned}$$

the $\lambda = 1$ case of which was given in Ref. [20]. The implied SD equations at large λ are

$$\begin{aligned}
0 = \int(dx) \left\langle \left[-\frac{\delta S^{(0)}}{\delta A_\mu(x)} + \partial_\mu Z(x) + \int(dy) R_{xy}^2(\square) \frac{\delta}{\delta A_\mu(y)} \right. \right. \\
+ ie[\phi(x) D_\mu^* \phi^*(x) - \phi^*(x) D_\mu \phi(x)] \left. \frac{\delta}{\delta A_\mu(x)} F[A] \right\rangle, \quad (5.16a)
\end{aligned}$$

$$\begin{aligned}
0 = \int(dx) \left\langle \left[-\frac{\delta S}{\delta \phi(x)} \frac{\delta}{\delta \phi^*(x)} - \frac{\delta S}{\delta \phi^*(x)} \frac{\delta}{\delta \phi(x)} \right. \right. \\
+ 2 \int(dy) R_{xy}^2(\Delta) \frac{\delta}{\delta \phi^*(y)} \frac{\delta}{\delta \phi(x)} \left. \right] G[A, \phi, \phi^*] \Big\rangle_{(0)}, \quad (5.16b)
\end{aligned}$$

where eq. (5.16b) follows from a no-growth theorem. As in section 5.3, a matter-field moment expansion¹ of (5.16b) yields the integrated prescription eq. (4.10), the on-shell pair form of which is

$$\langle \phi(x) \phi^*(y) F_1[A] \rangle_{(0)} = \langle [\Delta^{-1} R^2]_{xy} F_1[A] \rangle_{(0)}. \quad (5.17)$$

Then, eq. (5.16a) at large λ is recognized as the SD eq. (4.8) of the integrated regularized formulation of Chapter 4.

I also remark on the “naive” formulation of gauge theory fermions discussed in Chapter 3, in which a dimensionful parameter λ naturally appears. Assuming a no-growth theorem, this system is also equivalent at large λ to the integrated regularized formulation eq. (4.1,4).

Finally, an integrated formulation for chiral fermions may be devised with a determinantal term $ig \int (d\mathbf{x}) \text{Tr}[T^a \gamma_\mu (\frac{1-\gamma_5}{2}) \{\mathcal{D}^{-1} R^2(\mathcal{D}^2)\}_{xx}] \delta / \delta A_\mu^a(\mathbf{x})$ in the SD operator of eq. (4.1). With attention to zero modes, such a formulation should be adequate for the study of perturbative chiral anomalies. In the case of theories with global anomalies [49] however, an integrated formulation is problematic (reflecting problems in the definition of chiral determinants), and should be approached through a non-perturbative study of the large- λ regularized SIAG systems.

Footnotes: Chapter 5

¹ For example, the one-pair moment operator is $[\Omega_1]_{u,v}^{x,y} = \Delta_{xu} \delta(y-v) + \Delta_{vy} \delta(x-u)$.

Chapter 6

Conclusions

In this dissertation, the continuum regularization program was successfully applied to gauge theory with fermions. As an explicit check of gauge invariance, the gluon mass was shown to vanish at one loop. It was also demonstrated that integration of quadratic matter fields in regularized gauge theory leads to considerable simplification in the formalism. In particular, for a given process in weak-coupling, there are much fewer diagrams in the integrated formulation than the unintegrated formulation.

While the regularized Grassmann SIAG formulation is a straightforward extension of previous work on pure Yang-Mills [18-24,26], the integrated regularized non-Grassmann formulation introduces the important method of regularized integration into the general regularization program. Indeed, the large λ regularized integration technique discussed in Chapter 5 can easily be extended to all quadratic fields in any regularized theory. A recent application of this new technique is the derivation of *coordinate-space* regularization from *phase-space* regularization of coordinate-invariant theories [28-31]. Analogous to the determinantal term in the fermion case (Chapter 4,5), a *composite superconnection term* results from regularized integration of momenta in DeWitt-measure coordinate-invariant theories, further details of which may be found in Refs. [28-31].

With the recent advances in applying the general scheme to supersymmetric [27] and coordinate-invariant theories [28-31], the one-loop perturbative test for the general applicability of the program is more or less completed. Moreover, a geometric interpretation of the regularization is also given [29,31], by which the applications to scalar and gauge theories are viewed as flat-space and flat-superspace cases within the general framework. Based on these successes, we are confident that the general scheme gives a regularized nonperturbative description of any quantum field theory of interest. The next logical step is to exploit the continuum-nonperturbative features of the program, which are absent in other regularization schemes, as discussed in Chapter 1. In this connection, I mention a recent nonperturbative analysis of scalar theory by Doering [50], which is based on an independently developed regulator identical to the scalar prototype regulator of Bern *et al.* discussed in Chapter 1. We hope that more efforts will be made in this direction, especially for more intricate theories with local symmetries. This will certainly lead to a better understanding of quantum field theory in general, and a more complete knowledge of gauge theory in particular.

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