MASTER

UNITARITY RESTRICTIONS ON SEMI-CLASSICAL APPROXIMATIONS
TO CERTAIN FUNCTIONAL INTEGRALS[†]

H. M. Fried
Physics Department
Brown University

Providence, Rhode Island, U. S. A. 02912

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ABSTRACT

Estimates of physically relevant quantities (cross sections, multiplicities, correlations) in eikonal models of high-energy field theories lead to a class of semi-classical solutions to non-linear Euler equations. The requirement of unitarity restricts these possible solutions to a unique function of energy and impact parameter.

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Stationary phase or saddle-point approximations to functional integrals lead naturally to non-linear Euler equations, with the degree of non-linearity depending on the specific problem. For solutions to such equations there is a type of quasi-topological invariant familiar from complex variable theory. This invariant does not depend on the strength of singularities, but only on their location. This invariant is an integer, long familiar from the work of Cauchy and Riemann, and is used to denote the branch, n , of the particular multi-sheeted function under consideration. Thus n may be thought of as that integer which characterizes a solution as long as appropriate boundaries of singularities are not crossed.

The question then arises: How is one to specify the appropriate branch of a particular solution to the Euler equations? The purpose of these remarks is to suggest, for physical problems involving the scattering and production of particles and/or resonances, that the requirement of Unitarity may be used to determine the branch; and to illustrate this possibility with a simple and concrete example drawn from a problem in high-energy eikonal physics.¹

In essence, this problem may be formulated as a semi-classical approximation to that functional integral representing the total cross section (as well as multiplicities, etc.) for the scattering of a pair of very high-energy particles, with the elastic scattering amplitude defined essentially as the shadow of absorption corresponding to inelastic particle production. The calculation is set up within the context of an eikonal pionization approximation, with the scattering amplitude $T(s,t)$ given in terms of an eikonal function $\chi(b,s)$,

$$T(s,t) = i \frac{s}{2} \int d^2b e^{i\vec{q} \cdot \vec{b}} [1 - e^{i\chi(b,s)}] \quad (1)$$

where $q^2 = -t \ll s$. This eikonal is to correspond to a pair of nucleons (heavy horizontal lines) that scatter by the exchange of an increasing number of virtual vector mesons (vertical lines) which themselves contain the exchanges of all possible numbers of virtual scalar mesons (light horizontal lines). At asymptotic energies, ladder graphs provide the dominant contribution, and only these are included.

$$i\chi = \text{[Diagram 1]} + \sum \text{[Diagram 2]} + \sum \text{[Diagram 3]} + \dots = i\chi_1 + i\chi_2 + i\chi_3 + \dots$$

The retention of only $i\chi_1 + i\chi_2$ generates an eikonal with absorptive behavior²
 $i\chi \sim -s^\alpha (\ln s)^{-1} \exp[-\beta b] \quad \text{or}^3 \sim s^{\alpha'} (\ln s)^{-1} \exp[-\beta' b^2 / \ln s]$, from which follows the physically observed effect of rising total cross sections,

$$\sigma_{TOT} = 2 \operatorname{Re} \int d^2b [1 - e^{i\chi}] \quad (2)$$

which in this case yields

$$\sigma_{TOT} \sim \ln^2 s \quad (3)$$

Estimates of the remaining χ_n are difficult to obtain, but have been attempted.⁴ There, one found suggestions of very strong cancellations, generated by all remaining $i\chi_n$, such that the result (3) is replaced by a σ_{TOT} which vanishes as a power of s . One difficulty with these estimates is that they essentially assume an expansion can be made in the nucleon-vector meson coupling constant to yield a specific $i\chi_n$; and then, at fixed n , the $s \rightarrow \infty$ limit is effectively taken, after which the summation on all n is performed. This method may well be suspect if the coefficient of g^{2n} , in the n th order expansion, increases sharply with energy.

A different method of estimation has subsequently been given,¹ in which functional integral representations were written for the eikonal and all relevant cross sections, and then approximated in a semi-classical way. From that analysis there was obtained a non-linear Euler equation for a semi-classical field $\tilde{\phi}_0(k)$, in terms of impact parameter b , mass m of the vertical mesons, mass μ of the horizontal mesons, coupling constants g and λ , and the rapidity $Y \sim \ln s$:

$$\tilde{\phi}_0(k) = - \frac{g^2 \lambda}{(2\pi)^2} \frac{1}{k^2 + \mu^2} \int \frac{d^2 q}{q^2 + m^2} e^{i\vec{q} \cdot \vec{b}} \cdot \frac{e^{R(q/m)}}{(q-k_\perp)^2 + m^2} \quad (4)$$

with

$$R(q/m) = \lambda \int d^4 p \tilde{\phi}_0(p) [m^2 + (q-p_\perp)^2]^{-1} \quad ,$$

a somewhat complicated pair of equations.

However, for the case of physical interest corresponding to large impact parameter, $mb \gg 1$, this non-linear functional relationship may be approximated by a single non-linear algebraic equation. One expands $R(q/m) \approx R_0 + i\vec{q} \cdot \vec{R}_1 - q_i q_j R_2^{ij}$, and assumes $q_{\max} \ll m$, to obtain

$$R_1^i = 0, \quad R_2^{ij} = R_2 \delta_{ij}, \quad R_2 = R_0 / 6m^2 \quad ,$$

and

$$R_0^2 = -i\xi \exp[R_0 - \eta/R_0] \quad (5)$$

with

$$\xi = 3(g\lambda)^2 \pi Y / 2m^2, \quad \eta = \frac{3}{2}(mb)^2 \quad .$$

If solutions to (5) can be found which have the property $\text{Re}(R_0) > 0$, necessary for the convergence of the q -integral of (4), $\tilde{\phi}_0$ can then be written as

$$\tilde{\phi}_0(k) \approx -i \left(\frac{m^2 R_0}{\lambda \pi Y} \right) (k^2 + \mu^2)^{-1} (k_\perp^2 + m^2)^{-1} \quad (6)$$

The eikonal function corresponding to this semi-classical approximation is then

given by

$$i\chi = \left(\frac{m}{\pi\lambda}\right)^2 \frac{R_0}{Y} - \left(\frac{m}{\pi\lambda}\right)^2 \frac{R_0^2}{2Y} - \frac{1}{2} \ln(1-R_0) - \frac{1}{2} R_0 .$$

For such a non-linear relation as (5), one may expect dependence upon the branch of the possible solutions. With $R_0 \equiv \rho \exp[i(\theta+2\pi n)]$, where ρ , θ , and n denote magnitude, phase ($-\pi \leq \theta \leq +\pi$) and branch, respectively, substitution into (5) yields the pair of relations

$$(\rho - \eta/\rho) \cos\theta = \ln(\rho^2/\xi) \quad (7)$$

$$(\rho + \eta/\rho) \cos\theta = 4\pi n + 2\theta - \pi/2 \quad (8)$$

The solutions to (7) and (8) depend upon n , and they are qualitatively different for different values of n .

The requirement of Unitarity, easily given in this eikonal context, is now invoked to provide one more (non-linear) relation, in order to uniquely fix n , or rather to fix that value of n which should be used in different b and Y regions. In the notation of (7) and (8), Unitarity requires that

$$\rho^2 = \rho^2 \cos^2\theta - \rho \cos\theta + aY [\ln(1+\rho^2-2\rho\cos\theta) + 2\rho\cos\theta], \quad (9)$$

where $a = (\lambda\pi/2m)^2$. As Y , and b , increase it follows that the necessary n values must also increase; and in the limit of exceedingly large Y , one can, in effect, treat n essentially as a continuous variable. One thus finds

$$\rho^2 \sim aY \ln Y, \quad \theta \sim \pi/2, \quad n \sim \frac{1}{4\pi} [aY \ln Y]^{\frac{1}{2}}$$

in the region of relatively small impact parameter, $\eta < Y \ln Y$; but that

$$\rho \sim \alpha Y + \beta \ln Y, \quad \theta \rightarrow \theta_0 < \pi/2, \quad n \sim Y$$

with $\alpha = 2a \cos\theta_0 \csc^2\theta_0$, $\beta^{-1} = 2 \cos\theta_0$ in the region of larger impact parameter $\eta \leq \eta_{\max} = (\alpha Y)^2 \lesssim \rho^2$. The maximum η corresponds to $m b_{\max} \sim Y$, that impact parameter at which one expects the Froissart bound to become operative. As η exceeds $\eta_{\max}$, there is a discontinuous change as ρ falls sharply below $\xi \sim Y$, exhibiting $\rho \sim 0(1)$ and $\theta \sim 0$ for any n . Finally, for the largest impact parameters, $\eta > \xi > \rho^2$, ρ falls off very rapidly with increasing Y , $\rho \sim Y^{\frac{1}{2}} \exp[-\frac{a}{4\pi} \eta Y]$ while $n \sim \eta/\rho$ and $\theta \sim \pi/2$.

The result of this analysis is to reproduce, in a somewhat cumbersome way, the intuitive Froissart distribution in terms of an absorptive eikonal function which effectively vanishes for values of impact parameter which exceed Y/m ; and from this there follows (among other physical predictions) an increase of cross sections of the form (3). However, the calculated behavior of σ_{TOT} is really not the point of this exercise, for hadronic physics is far more complicated than sketched here. What is (to our knowledge) unique, and what may be important, is the use of Unitarity

as a restriction on the class of acceptable semi-classical solutions needed in connection with a particular functional integral. It may be hoped that this technique will find other applications in related problems.

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