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CALCULATION OF A TRACKING FORCE

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CALCULATION OF A TRACKING FORCE

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ABSTRACT

The original calculation (1975-76) of an electromagnetic mechanism for beam-channel tracking is presented. The system model is not realistic because only a flat profile channel which completely encloses the beam is considered. However, a clear demonstration of the existence of a tracking force is made, and its order of magnitude and duration are estimated. This calculation serves as a point of departure for the more recent studies which incorporate a realistic model of the beam head and conductivity profile.

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TEXT OF A 1976 MEMO ON TRACKING

We make an analytical calculation of the channel tracking force with a highly idealized model for the beam and conducting channel.

The equation of motion of a single beam particle of charge q and mass m is

$$\frac{d^2 \underline{r}_1}{dt^2} = \frac{q}{\gamma m} (\underline{E}_1 + \underline{\tilde{e}}_2 \times \underline{B}_1) \quad , \quad (1)$$

where $\underline{1}$ denotes components transverse to the direction of propagation. Conductivity σ is low so the magnetic field is centered on the beam axis--it supplies a pinching force but no tracking force. The electric field is antipinching, but gives a tracking force if the channel is not uniform.

We determine $\underline{E}_1$ in the electrostatic approximation:

$$\underline{E}_1 = - \underline{\nabla}_1 \phi \quad , \quad (2)$$

$$\nabla_1^2 \phi = - 4\pi\rho \quad , \quad (3)$$

$$\left(\frac{\partial \rho}{\partial t} \right)_z + \underline{\nabla}_1 \cdot \sigma \underline{E} = \left(\frac{\partial \rho_b}{\partial t} \right)_z \quad . \quad (4)$$

Here σ is conductivity and ρ_b is beam charge density. These equations are valid provided the electric charge relaxation time is long compared with the magnetic diffusion time, i.e.,

$$\tau_e = \frac{1}{4\pi\sigma} \gg \frac{4\pi\sigma a^2}{c^2} \approx \tau_m \quad , \quad (5)$$

where a is the scale beam radius.

The beam is treated in the "rigid beam approximation." Let $\underline{y} = y \underline{\tilde{e}}_y$ be a transverse displacement of the beam axis; then to first order in y

$$\rho_b = \rho_{b_0} (|\underline{r} - \underline{y}|) \approx \rho_{b_0}(r) - \underline{y} \cdot \underline{\nabla}_1 \rho_{b_0} = \rho_{b_0} - y \frac{\partial \rho_{b_0}}{\partial r} \sin \theta \quad . \quad (6)$$

ρ_{b0} is the unperturbed (monopole) beam charge density and it is a function of z and t as well as r . The equation of motion for y is found by averaging the single particle force (1) over the beam profile:

$$\frac{d^2 y}{dt^2} = \frac{q}{\gamma m} \langle E_{\perp} \rangle = - \frac{q}{\gamma m} \frac{1}{Q_b} \int d^2 r \rho_b \nabla_{\perp} \phi . \quad (7)$$

Q_b is the beam charge per cm.

It is convenient to introduce the longitudinal variable $x = t - z/c$, which plays the role of time in the pulse. Time derivatives are transformed according to

$$\left(\frac{\partial \rho}{\partial t} \right)_z = \left(\frac{\partial \rho}{\partial x} \right)_z \quad (8a)$$

$$\frac{dy}{dt} = \left(\frac{\partial y}{\partial t} \right)_z + c \left(\frac{\partial y}{\partial z} \right)_t = c \left(\frac{\partial y}{\partial x} \right)_x . \quad (8b)$$

Henceforth we use x and z instead of t and z as independent variables.

Since ρ_b is the fundamental source for the fields we decompose ϕ and ρ into monopole and dipole components:

$$\phi = \phi_0 + \phi_1 \sin \theta , \quad (9a)$$

$$\rho = \rho_0 + \rho_1 \sin \theta . \quad (9b)$$

The conductivity may be similarly decomposed, but in the present calculation it is considered to be preferentially centered around the system axis--hence σ_r vanishes. Equations (2) to (4) yield;

$$\underline{E}_{\perp} = - \tilde{e}_r \frac{\partial \phi_0}{\partial r} - \left(\tilde{e}_r \frac{\partial \phi_1}{\partial r} \sin \theta + \tilde{e}_{\theta} \frac{\phi_1}{r} \cos \theta \right) , \quad (10)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \phi_0}{\partial r} = - 4\pi \rho_0 , \quad (11)$$

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r \phi_1 = - 4\pi \rho_1 , \quad (12)$$

$$\frac{\partial \rho_0}{\partial x} + 4\pi \sigma \rho_0 - \frac{\partial \sigma}{\partial r} \frac{\partial \phi_0}{\partial r} = \frac{\partial \rho_{b0}}{\partial x} , \quad (13)$$

$$\frac{\partial \rho_1}{\partial x} + 4\pi \sigma \rho_1 - \frac{\partial \sigma}{\partial r} \frac{\partial \phi_1}{\partial r} = \frac{\partial}{\partial x} \left(-y \frac{\partial \rho_{b0}}{\partial r} \right) . \quad (14)$$

The electric field must be well behaved at $r = 0$ and $r = \infty$, in particular $(\partial/\partial r) (r\phi_1)_{\infty} = 0$.

The average over angle in Eq. (7) is readily performed, giving

$$\left(\frac{\partial^2 y}{\partial z^2} \right)_x = \frac{q}{\gamma m c^2} \frac{\pi}{Q_b} \int_0^{\infty} dr r \left(y \frac{\partial \rho_{b0}}{\partial r} \frac{\partial \phi_0}{\partial r} - \frac{\rho_{b0}}{r} \frac{\partial}{\partial r} r \phi_1 \right) . \quad (15)$$

Equation (15) closely resembles the well known rigid beam equation used when magnetic forces are dominant ($\phi \rightarrow -A$). However, the respective roles played by the monopole and dipole fields are different in the electric regime considered here. For example, ϕ_0 provides a destabilizing term in Eq. (15) (which is compensated by ϕ_1). Furthermore, a simple hose equation is obtained even when ϕ_0 vanishes.

Equation (15) may be written in another useful form if we eliminate ϕ in favor of ρ on the right hand side. After several integrations by parts and substitution from Eqs. (11) and (12) we get

$$\frac{\partial^2 y}{\partial z^2} = - \frac{q}{\gamma m c^2} \frac{2\pi}{Q_b} \int dr Q_b(r) \left(y \frac{\partial \rho_0}{\partial r} + \rho_1 \right) , \quad (16)$$

with

$$Q_b(r) \equiv \int_0^r dr' 2\pi r' \rho_{b0}(r') . \quad (17)$$

To proceed with the calculation, a specific conductivity profile must be assumed. We take

$$\sigma(r, x) = \begin{cases} \sigma(x) & 0 < r < R , \\ 0 & r > R . \end{cases} \quad (18)$$

The beam charge density may have any profile which lies completely within radius R and depends on x only through the overall magnitude $Q_b(x)$, i.e., the shape and scale radius are independent of x .

From Eqs. (13) and (14), the charge densities in the channel are

$$\rho_0 = \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx'' 4\pi\sigma\right) \frac{\partial \rho_{b0}}{\partial x'}, \quad (19)$$

$$\rho_r = \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx'' 4\pi\sigma\right) \frac{\partial}{\partial x'} \left(-y \frac{\partial \rho_{b0}}{\partial r}\right). \quad (20)$$

There is also a charge layer at the channel radius (R). Its azimuthally symmetric component does not affect the beam, but the asymmetrical component [$\rho \propto \sigma(r - R) \sin \theta$] is the source of the tracking force.

We define the dipole surface charge per cm^2

$$\Delta = \int_{R-}^{R+} dr' \rho_r. \quad (21)$$

Then using expressions (19) and (20) Eq. (16) is written

$$\begin{aligned} \frac{\partial^2 y}{\partial z^2} = & -\frac{q}{\gamma mc^2} \frac{2\pi}{Q_b} \left\{ Q_b \Delta + \int_0^{R-} dr Q_b(r) \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx'' 4\pi\sigma\right) \right. \\ & \left. \times \frac{\partial}{\partial x'} \left[y(x) \frac{\partial \rho_{b0}}{\partial r} - y(x') \frac{\partial \rho_{b0}}{\partial r} \right] \right\}. \quad (22) \end{aligned}$$

The integration over r in Eq. (22) is elementary. We define a mean betatron frequency

$$\begin{aligned} \kappa_B^2 = & -\frac{q}{\gamma mc^2} 2\pi \int_0^R dr \frac{Q_b(r)}{Q_b} \frac{\partial \rho_{b0}}{\partial r} \\ = & \frac{q}{\gamma mc^2} \frac{(2\pi)^2}{Q_b} \int dr r \rho_{b0}^2. \quad (23) \end{aligned}$$

For a beam of flat profile and radius a , it reduces to

$$\kappa_B^2 = \frac{2q Q_b}{\gamma m c^2 a^2}$$

as expected. Substituting κ_B^2 and the charge relaxation time $\tau_e = (4\pi\sigma)^{-1}$ in Eq. (22) yields

$$\frac{\partial^2 y}{\partial z^2} = -\frac{q}{\gamma m c^2} (2\pi\Delta) + \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx''/\tau_e\right) \frac{\partial}{\partial x'} \kappa_B^2 [y(x) - y(x')] \quad (24)$$

The second term on the right has the form of the standard hose equation, except that τ_e appears instead of τ_m .

It remains to evaluate Δ . The easiest way to do this is to note that the rate of change of Δ equals the radial component of the dipole current at R :

$$\frac{\partial \Delta}{\partial x} = -\sigma \left(\frac{\partial \phi_1}{\partial r} \right)_{R-} \quad (25)$$

The dipole potential inside the channel is found by integrating Eq. (12) twice in r :

$$\frac{1}{r} \frac{\partial}{\partial r} r \phi_1 = - \int_{-\infty}^r dr' 4\pi \rho_1 = + 4\pi\Delta - \int_{R-}^r dr' 4\pi \rho_1 \quad (26)$$

$$\phi_1 = \frac{4\pi\Delta r}{2} - \frac{4\pi}{r} \int_0^r dr' r' \int_{R-}^{r'} dr'' \rho_1 \quad (27)$$

Substituting from Eq. (20) for ρ_1 , we may integrate over r'' and r' to get

$$\phi_1 = 2\pi\Delta r + \frac{2}{r} \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx''/\tau_e\right) \frac{\partial}{\partial x'} y Q_b(r) \quad (28)$$

and

$$\frac{\partial \phi_1}{\partial r} (R-) = 2\pi\Delta - \frac{2}{R^2} \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx''/\tau_e\right) \frac{\partial}{\partial x'} y Q_b \quad (29)$$

Substituting expression (29) in Eq. (25) we have

$$\frac{\partial \Delta}{\partial x} + \frac{\Delta}{2\tau_e} = \frac{1}{\pi R^2} \frac{1}{2\tau_e} \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx''/\tau_e\right) \frac{\partial}{\partial x'} y Q_b \quad (30)$$

An elementary integration yields

$$\Delta = \frac{1}{\pi R^2} \int_{-\infty}^x dx' \frac{\exp\left(\int_x^{x'} dx''/2\tau_e\right)}{2\tau_e} \int_{-\infty}^{x'} dx'' \exp\left(\int_{x'}^{x''} dx'''/\tau_e\right) \frac{\partial}{\partial x''} y Q_b. \quad (31)$$

A more convenient form for Δ is found with some elementary manipulation of the integrals in Eq. (31); note that we may write

$$\Delta = \frac{\exp\left(-\int_{-\infty}^x dx'/2\tau_e\right)}{\pi R^2} \int_{-\infty}^x dx' \frac{\exp\left(-\int_{-\infty}^{x'} dx''/2\tau_e\right)}{2\tau_e} \int_{-\infty}^{x'} dx'' \times \exp\left(\int_{-\infty}^{x''} dx'''/\tau_e\right) \frac{\partial}{\partial x''} y Q_b. \quad (32)$$

After an integration by parts one has

$$\Delta = \frac{\exp\left(-\int_{-\infty}^x dx'/2\tau_e\right)}{\pi R^2} \left[-\exp\left(-\int_{-\infty}^x dx'/2\tau_e\right) \int_{-\infty}^x dx' \exp\left(\int_{-\infty}^{x'} dx''/\tau_e\right) \frac{\partial}{\partial x'} y Q_b + \int_{-\infty}^x dx' \exp\left(\int_{-\infty}^{x'} dx''/2\tau_e\right) \frac{\partial}{\partial x'} y Q_b \right]. \quad (33)$$

Gathering terms:

$$\Delta = \frac{1}{\pi R^2} \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx''/2\tau_e\right) - \exp\left(\int_x^{x'} dx''/\tau_e\right) \frac{\partial}{\partial x'} Q_b y. \quad (34)$$

Inserting expression (34) into Eq. (24) gives

$$\frac{\partial^2 y}{\partial z^2} = -\left(\frac{q}{\gamma mc^2}\right) \frac{2\pi}{\pi R^2} \int_{-\infty}^x dx' \left[\exp\left(\int_x^{x'} dx''/2\tau_e\right) - \exp\left(\int_x^{x'} dx''/\tau_e\right) \right] \frac{\partial}{\partial x'} Q_b y + \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx''/\tau_e\right) \frac{\partial}{\partial x'} \kappa_\beta^2 [y(x) - y(x')]. \quad (35)$$

A "tracking frequency" is defined by

$$\kappa_T^2 = \frac{2qQ_b}{\gamma mc^2 R^2} \approx \frac{a^2}{R^2} \kappa_\beta^2. \quad (36)$$

we have finally

$$\begin{aligned} \frac{\partial^2 y}{\partial z^2} = & - \int_{-\infty}^x dx' \left[\exp\left(\int_x^{x'} dx''/2\tau_e\right) - \exp\left(\int_x^{x'} dx''/\tau_e\right) \right] \frac{\partial}{\partial x'} \kappa_T^2 y \\ & + \int_{-\infty}^x dx' \exp\left(\int_x^{x'} dx''/\tau_e\right) \frac{\partial}{\partial x'} \kappa_B^2 [y(x) - y(x')] . \end{aligned} \quad (37)$$

To obtain the magnitude of the tracking force from Eq. (37) we consider the force acting on a beam with constant displacement in x ; then

$$\frac{\partial^2 y}{\partial z^2} = -y \int_{-\infty}^x dx' \left[\exp\left(\int_x^{x'} dx''/2\tau_e\right) - \exp\left(\int_x^{x'} dx''/\tau_e\right) \right] \frac{\partial \kappa_T^2}{\partial x'} . \quad (38)$$

This result is simplified by expansion in a series of derivatives if $(\tau_e \partial/\partial x)$ is considered small; integrating by parts repeatedly we get

$$\frac{\partial^2 y}{\partial z^2} = -y \left[(2-1) \tau_e \frac{\partial \kappa_T^2}{\partial x} - (2^2-1) \tau_e \frac{\partial}{\partial x} \tau_e \frac{\partial \kappa_T^2}{\partial x} + \dots \right] \quad (39)$$

$$\approx -y \tau_e \frac{\partial \kappa_T^2}{\partial x} . \quad (40)$$

The tracking force, although usually small compared with the pinch force, is quite large compared with external influences on propagation. Hence the beam tends to follow the conductivity axis.

A growth rate for the high frequency hose instability is also derived from the formalism. Going to the continuous beam limit (κ_B^2 , κ_T^2 , and τ_e constant) we may take

$$y \propto e^{-i(\Omega z + \omega x)} . \quad (41)$$

Then Eq. (37) becomes

$$\begin{aligned}
 -\Omega^2 e^{-i\omega x} &= -\kappa_T^2 \int_{-\infty}^x dx' \left[\exp\left(\frac{x' - x}{2\tau_e}\right) - \exp\left(\frac{x' - x}{\tau_e}\right) \right] (-i\omega) e^{-i\omega x'} \\
 &+ \kappa_B^2 \int_{-\infty}^x dx' \exp\left(\frac{x' - x}{\tau_e}\right) i\omega e^{-i\omega x'} = -\kappa_T^2 e^{-i\omega x} (-i\omega) \\
 &\times \left(\frac{1}{\frac{1}{2\tau_e} - i\omega} - \frac{1}{\frac{1}{\tau_e} - i\omega} \right) + \kappa_B^2 e^{-i\omega x} \frac{i\omega}{\frac{1}{\tau_e} - i\omega} . \quad (42)
 \end{aligned}$$

After some arrangement of terms the dispersion relation takes the form*

$$\frac{\Omega^2}{\kappa_B^2} = \frac{-i\omega\tau_e}{1 - i\omega\tau_e} \left(1 + \frac{\kappa_T^2}{\kappa_B^2} \frac{1}{1 - 2i\omega\tau_e} \right) . \quad (43)$$

Except for the small term proportional to κ_T^2/κ_B^2 , Eq. (43) has exactly the form of the simple high frequency hose dispersion relation. The new term does not appear to be either damaging nor very helpful. At very low frequency one has growth rate in x

$$-i\omega\tau_e = \frac{\Omega^2}{\kappa_B^2 + \kappa_T^2} , \quad (44)$$

i.e., a small improvement. When magnetic effects are included we expect τ_e to be replaced with $\tau_e + \tau_m$ in the dominant terms of Eq. (43). This result has been demonstrated¹ for a channel of infinite radius.

*The rhs of Eq. (43) is the electrostatic force recently (1976) calculated by M. Rosenbluth.

REFERENCE

1. E. P. Lee, "Hose Instability at Arbitrary Conductivity," Lawrence Livermore National Laboratory Report UCID-16734, March 7, 1975.

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