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**AN ITERATIVE ALGORITHM FOR THE VOLUME
INTEGRAL METHOD FOR MAGNETOSTATICS PROBLEMS**

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APPLIED MATHEMATICS DEPARTMENT

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ABSTRACT

Volume integral methods for solving nonlinear magnetostatics problems are considered in this paper. The integral method is discretized by a Galerkin technique. Estimates are given which show that the linearized problems are well conditioned and hence easily solved using iterative techniques. Comparisons of iterative algorithms with the elimination method of GFUN3D shows that the iterative method gives an order of magnitude improvement in computational time as well as memory requirements for large problems. Computational experiments for a test problem as well as a double layer dipole magnet are given. Error estimates for the linearized problem are also derived.

INTRODUCTION

Many methods for approximating magnetostatics problems in three dimensions have been investigated. These methods can be categorized as boundary integral, volume integral, differential or combinations of the three [12]. A partial list of computer codes for these problems is given in [10]. General purpose codes have been developed for example by Simkin, et al [1,9]. (TOSCA, a differential formulation, and GFUN3D, a volume integral approach.) Comparisons made by Simkin et al. indicate that GFUN3D and TOSCA exhibit somewhat similar performance on typical engineering applications [2]. That is the cost for accuracy on the order of two or three percent is about the same for the two methods. In addition, the higher accuracy necessary for some engineering applications can not be achieved with either code due to prohibitive computational expense and memory requirement.

The purpose of this paper is to present a new algorithm for implementing the volume integral method which is an order of magnitude improvement over GFUN3D for magnetization calculation with a large number of elements. Discretizing the integral method leads to a matrix equation with a nonlinear parameter μ . GFUN3D iterates for the parameter μ . Each iterative improvement requires Gaussian elimination on a matrix of size $3N \times 3N$ where N is the number of iron elements. The new algorithm (GFUN3DIT) uses matrix iterative techniques instead of Gaussian elimination. To compute magnetization, the GFUN3DIT algorithm only uses relatively few evaluations of $3N \times 3N$ matrices times vectors. For most problems GFUN3DIT requires on the order of $(3N)^2$ operations whereas GFUN3D requires $O((3N)^3)$. Also, since GFUN3DIT only uses matrix multiplication, the $3N \times 3N$ system need not reside in core at any one time. Thus storage requirements for GFUN3DIT are on the order of N in contrast to N^2 for GFUN3D.

MAGNETIC FIELD EQUATIONS AND THE NUMERICAL METHOD

The basic equations for magnetostatics are given below in terms of the magnetic field induction B , intensity H , and current density J

$$\operatorname{div} B = 0 \quad \operatorname{Curl} H = J \quad (1)$$

These are connected by a constitutive relationship

$$B = \mu H \quad (2)$$

where the permeability μ is an experimentally determined function of $|H|$. The magnetization M is defined by

$$M = B - H \quad (3)$$

and is nonzero only in the region Ω_I which contains magnetic material. The field due to sources H_S is defined as the solution of

$$\operatorname{div} H_S = 0 \quad \operatorname{Curl} H_S = J. \quad (4)$$

The field H can then be decomposed into

$$H = H_S + H_M \quad (5)$$

It is well known that

$$H_M = - \frac{\nabla}{4\pi} \int_{\Omega_I} M \cdot \nabla \left(\frac{1}{|x - x'|} \right) dx' \equiv G(M) \quad (6)$$

Here G is considered to be an operator defined on vector fields. The field due to sources H_s is given by

$$H_s = \frac{1}{4\pi} \int_{R^3} J \times \nabla \frac{1}{|x-x'|} dx' \quad (7)$$

H_s can be calculated analytically or numerically for many current densities J . In this paper we shall assume that H_s is known and consider approximating M . Combining the above identities gives that M satisfies

$$\left(\frac{1}{\mu-1} M - GM \right) = H_s. \quad (8)$$

To approximate M we shall discretize equation (8) using a Galerkin technique. Let S_h denote a finite element subspace of approximating functions defined on the region Ω_I . Examples of the construction of finite element subspaces can be found in for example [5]. For our purposes we shall consider the basic subspace S_h defined by partitioning the region Ω_I into volume elements of size h , and then setting S_h to be the linear space of functions which are piecewise constant on each element. Let $\bar{S}_h$ be subspace of vector functions which have the property that each component of any function in $\bar{S}_h$ is in S_h . The Galerkin approximation of M is the function M_h in $\bar{S}_h$ satisfying

$$\int_{\Omega_I} \left(\frac{1}{\mu-1} M_h - GM_h \right) \cdot \Psi \, dx = \int_{\Omega_I} H_s \cdot \Psi \, dx \quad \text{for all } \Psi \text{ in } \bar{S}_h. \quad (9)$$

Of course (9) is still a nonlinear problem. One way of solving the nonlinear system (9) is by successive linearization of M . Typically, start with a guess for μ say $\bar{\mu}$, solve the linear system defined by the equations

$$\int_{\Omega_I} \left(\frac{1}{\bar{\mu}-1} \bar{M}_h - G\bar{M}_h \right) \cdot \Psi \, dx = \int_{\Omega_I} H_s \cdot \Psi \, dx \quad \text{for all } \Psi \in \bar{S}_h \quad (10)$$

then use $\bar{M}_h$ to define a new "improved" guess for μ . It is shown in the appendix that the equations (10) have a unique solution, indeed the corresponding linear system is symmetric positive definite. In addition, the condition number α for the linear system corresponding to (10) is bounded by

$$\alpha \leq \frac{x_m + 1}{x_s} \quad (11)$$

where

$$x_m = \max \frac{1}{\bar{\mu}-1} \quad x_s = \min \frac{1}{\bar{\mu}-1} . \quad (12)$$

The condition number α determines the rate of convergence of iterative methods for solving the linear system. As an example, consider the conjugate gradient method for solving (10). The user must supply an initial guess for $\bar{M}_h$. After k steps of the iteration the resulting approximation of $\bar{M}_h$ has an error which is reduced by a factor of at least $\min(e^{-k/\alpha}, 4e^{-k/\sqrt{\alpha}})$. For example, if $2 < \bar{\mu} < 25$, forty iterations give an improvement of three orders of magnitude. Hence iterative methods will always be faster than Gaussian elimination for large problems.

By changing the iteration procedure, the algorithm for solving the nonlinear problem (9) can be made more efficient. The idea is to iterate for μ and M_h simultaneously and avoid oversolving (10) for any given $\bar{\mu}$. Start with an initial guess for μ and M_h . Reasonable guesses for μ and M_h are given by

$$\mu^0 \equiv \mu(|H_s|) \quad \text{and} \quad M_h^0 \equiv \frac{\mu^0 - 1}{\mu^0} H_s \quad (13)$$

Note that the guesses μ^0 and M_h^0 tend asymptotically to the solution μ and M_h in the saturation limit (i.e. when $H_s \rightarrow \infty$ and $\mu \rightarrow 1$). The iterative algorithm for solving (9) is then given inductively. If μ^i and M_h^i are given, apply a few conjugate gradient iterations for the linear system (10) with $\bar{\mu} = \mu^i$ and initial guess $\bar{M}_h = M_h^i$. The result of that conjugate gradient iteration defines M_h^{i+1} . Define $\mu^{i+1} \equiv \mu(|H_s + G M_h^{i+1}|)$. It is important not to iterate too much (oversolve) for the solution of the linear system (10) for any fixed $\bar{\mu}$ since the value of $\bar{\mu}$ only approximates the value of μ . Symbolically the above algorithm can be described by

Algorithm: (GFUN3DIT)

(a) Set

$$\begin{aligned}\mu^0 &= \mu(|H_s|) \\ M_h^0 &= \frac{\mu^0 - 1}{\mu^0} H_s\end{aligned}$$

(b) Inductively define

$$\begin{aligned}M_h^{i+1} &\leftarrow \text{result of } k \text{ conjugate gradient iterations of (10) or (14) with} \\ &\quad \bar{\mu} = \mu^i \text{ and initial guess } M_h^i \\ \mu^{i+1} &\equiv \mu(|H_s + G M_h^{i+1}|)\end{aligned}$$

Intuition suggests that one should iterate with the system with best possible conditioning. To improve conditioning, we consider the scaled linear system corresponding to the equations

$$\int_{\Omega_I} \left[\left(\frac{1}{\bar{\mu}} - \frac{\bar{\mu}-1}{\bar{\mu}} G \right) \bar{M}_h \right] \cdot \Psi \, dx = \int_{\Omega_I} \frac{\bar{\mu}-1}{\bar{\mu}} H_S \cdot \Psi \, dx. \quad (14)$$

The conjugate gradient algorithm can be applied directly to the linear system corresponding to (14), see [4]. In the appendix, it is shown that the relevant condition number for (14) is bounded by $\max \bar{\mu}$. Thus conjugate gradient iteration for (14) will converge rapidly even in the saturation limit when $\bar{\mu}$ is close to one.

The success of the above iteration scheme is due in part to the use of Galerkin techniques for discretization. The importance of this approach is that it leads to symmetric positive definite systems. In contrast, iterative methods with the nonsymmetric systems used in GFUN3D were unsuccessful [8]. In addition, for the linearized systems (10), the condition number and discretization error can be easily estimated.

IMPLEMENTATION AND NUMERICAL RESULTS

Computer implementation of the numerical method of this paper involves further approximation. First, the gram elements for system (14) must be computed. Since $\bar{\mu}^i$ is constant over volume elements subdividing Ω_I , the calculation of the gram matrix only requires computing the integrals

$$G_{i,j,k,\ell} = \int_{\tau_i} \frac{\partial}{\partial x_j} \int_{\tau_k} \frac{\partial}{\partial x_\ell} \frac{1}{|x-x'|} dx' \, dx$$

where τ_i and τ_k are volume elements. For our purposes it suffices to approximate $G_{i,j,k,\ell}$ by the symmetric quadrature formula

$$G_{i,j,k,\ell} = V(\tau_i) \frac{\partial}{\partial x_j} \int_{\tau_k} \frac{\partial}{\partial x_\ell} \frac{1}{|x-x'|} dx' + V(\tau_k) \frac{\partial}{\partial x_\ell} \int_{\tau_i} \frac{\partial}{\partial x_j} \frac{1}{|x-x'|} dx'$$

where $V(\tau)$ is the volume of the region τ . The first (resp. second) integral is evaluated at a node x in τ_i (resp. τ_k). Finally, the integrals on the right hand side of (14) are replaced by first order quadrature approximations.

As expected, the computer algorithm implementing GFUN3DIT exhibits a significant reduction in CPU time and memory requirements when compared to GFUND3D. Nonlinear magnetostatics calculations with 100 iron elements ran 15 times faster with GFUN3DIT. For greater than one hundred elements, the calculation exhibited the expected asymptotic properties; that is, the computation time for GFUN3DIT increased quadratically in contrast to the cubic increase for GFUND3D. The reduction in memory requirements for GFUN3DIT were so substantial that memory requirements are no longer a limiting factor for problem complexity.

Computational algorithms using the iterative technique of this paper can be made even faster for some problems with special geometry. If the iron can be described by a regular mesh, the matrix corresponding to (10) is of discrete convolution form (non cyclic). By using Fast Fourier Transform techniques, the matrix can be evaluated using far fewer operations. A program using these ideas has been developed at Brookhaven for 3D magnetostatic problems with cylindrical shell geometry. 6000 element runs with this program cost the same as 125 element runs with the Gaussian elimination program.

To experimentally investigate the convergence properties of the algorithm test runs were compared with analytical results for a two dimensional linear problem. Specifically, an annular iron region with constant permeability immersed in a constant field was considered. The iron region was subdivided into NR radial and NT angular divisions. The results of these experiments are given in Tables 1 and 2. The tables give the maximum norm error between calculated and analytical values of the magnetization. In addition, the error in the zeroth order harmonic coefficient in the center of the magnet is also given. Approximation of harmonic coefficients

$$\langle u, v \rangle_{\partial\Omega} = \int_{\partial\Omega} uv \, ds.$$

Let n be the outward normal defined on $\partial\Omega$. We then have the integral relation

$$(\Psi, \nabla\phi)_{\Omega} = \langle \Psi \cdot n, \phi \rangle_{\partial\Omega} - (\operatorname{div} \Psi, \phi)_{\Omega} \quad (\text{A.3})$$

From (6) and (A.3) we have that

$$GM = -\frac{\nabla}{4\pi} \int_{\partial\Omega_I} M \cdot n \frac{dx'}{|x-x'|} + \frac{\nabla}{4\pi} \int_{\Omega_I} \frac{\operatorname{div} M}{|x-x'|} dx'$$

and we consider the scalar potential

$$\phi_M = \frac{1}{4\pi} \int_{\partial\Omega_I} (M \cdot n) \frac{dx'}{|x-x'|} - \frac{1}{4\pi} \int_{\Omega_I} \frac{\operatorname{div} M}{|x-x'|} dx' \quad (\text{A.4})$$

The first integral is a simple layer potential and the second is a volume potential. Classical results of potential theory [11] imply that ϕ_M satisfies the differential equations

$$\Delta\phi_M = \operatorname{div} M \quad \text{in} \quad \Omega_I \quad (\text{A.5})$$

$$\Delta\phi_M = 0 \quad \text{in} \quad \Omega_I^c \quad (\text{A.6})$$

$$[\phi_M] = 0 \quad \text{on} \quad \partial\Omega_I$$

$$\left[\frac{\partial\phi_M}{\partial n} \right] = -M \cdot n \quad \text{on} \quad \partial\Omega_I$$

$$\phi_M \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty$$

Here Δ denotes the Laplacian, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$, Ω_I^c is the region exterior to Ω_I , and $[]$ denotes the jump across the boundary¹.

Let N be another vector function defined on Ω_I , multiplying (A.5) and (A.6) by ϕ_N and using (A.3) gives

$$(\nabla\phi_M, \nabla\phi_N)_{\Omega_I} + (\nabla\phi_M, \nabla\phi_N)_{\Omega_I^c} = (-\operatorname{div} M, \phi_N)_{\Omega_I} + \langle M \cdot n, \phi_N \rangle_{\partial\Omega_I}$$

Using (A.3) once again on $(-\operatorname{div} M, \phi_N)_{\Omega_I}$ and noting that $\nabla\phi_M = -GM$ gives

$$(GM, GN)_{R^3} = - (M, GN)_{\Omega_I} \quad (A.7)$$

Equation (A.7) immediately implies that the linear system corresponding to (10) is symmetric positive definite. (A.7) also implies that

$$(GM, GM)_{\Omega_I} \leq (M, M)_{\Omega_I} \quad (A.8)$$

from which Result 1 follows.

¹ The jump across the boundary is defined by

$$[f] \equiv f^+ - f^-$$

$$f^+(x) = \lim_{\substack{\zeta \rightarrow x \\ \zeta \in \Omega_I^c}} f(\zeta) \quad \text{and} \quad f^-(x) = \lim_{\substack{\zeta \rightarrow x \\ \zeta \in \Omega_I}} f(\zeta)$$

To estimate the relevant condition number for (14) one must bound the forms $((\frac{1}{\mu-1} - G) M, M)_{\Omega_I}$ and $(\frac{\mu}{\mu-1} M, M)_{\Omega_I}$ [4]. Using (A.8), it easily follows that

$$(\min \frac{1}{\mu}) (\frac{\mu}{\mu-1} M, M)_{\Omega_I} \leq ((\frac{1}{\mu-1} - G) M, M)_{\Omega_I} \leq (\frac{\mu}{\mu-1} M, M)_{\Omega_I}$$

Error estimates follow readily from the approximation properties of the subspaces. Indeed using the fact that the solution of the linearized problem (10) satisfies (10) for all functions Ψ we have that the error e satisfies

$$((\frac{1}{\mu-1} - G) e, \Psi) = 0 \quad \text{for } \Psi \text{ in } \bar{S}_h. \quad (\text{A.9})$$

Using (A.1) gives for Ψ in $\bar{S}_h$

$$(e, e) \leq \frac{1}{x_s} (\frac{1}{\mu-1} - G) e, e - \Psi \leq \frac{x_M^{+1}}{x_s} (e, e)^{1/2} \inf_{\Psi \in \bar{S}_h} (e - \Psi, e - \Psi)^{1/2}$$

and the approximation properties of $\bar{S}_h$ imply

$$(e, e)^{1/2} = O(h) \quad (\text{A.10})$$

which proves Result 2.

To prove Result 3 we shall need regularity for the problem

$$(\frac{1}{\mu-1} - G) M = H_s \quad (\text{A.11})$$

By Result 1 and the Reiz Representation Theorem, (A.11) has a solution N for any given H_s in $L^2(\Omega_I)^3$. Let $H^s(\Omega_I)$ denote the Sobolev space of order s on Ω_I , (see [7]). The norm in $H^s(\Omega_I) \times H^s(\Omega_I) \times H^s(\Omega_I)$ is denoted by $\|\cdot\|_s$. We

shall see that

$$\|M\|_1 \leq C \|H_s\|_1 \quad (\text{A.12})$$

for some positive constant C.

We note that (A.11) implies that

$$M + (\mu-1) \nabla \phi_M = (\mu-1) H_s \quad \text{on } \Omega_I$$

Using (A.5), (A.6) and obvious manipulations gives that ϕ_M satisfies the differential equations

$$\nabla \cdot \mu \nabla \phi_M = \operatorname{div} (\mu-1) H_s \quad \text{in } \Omega_I$$

$$\Delta \phi_M = 0 \quad \text{in } \Omega_I^C$$

$$[\phi_M] = 0 \quad \text{on } \partial\Omega_I$$

$$\mu \frac{\partial \phi_M^-}{\partial n} - \frac{\partial \phi_M^+}{\partial n} = (\mu-1) H_s \cdot n \quad \text{on } \partial\Omega_I$$

Elliptic regularity for interface problems [3] implies that

$$\|\phi_M\|_2 \leq C \|H_s\|_1 \quad (\text{A.13})$$

and hence (A.12) holds.

We next estimate $\|e\|_{-1}$ by a standard duality argument. By definition

$$\|e\|_{-1} = \sup_{\psi \in C^\infty(\Omega_I)} \frac{(e, \psi)_{\Omega_I}}{\|\psi\|_1}.$$

For $\psi \in C^\infty(\Omega_I)$ let χ solve

$$\left(\frac{1}{\mu-1} - G\right) \chi = \Psi.$$

Then

$$(e, \Psi)_{\Omega_I} = \left(\left(\frac{1}{\mu-1} - G\right) e, \chi\right)_{\Omega_I} \leq \frac{\inf}{\phi \varepsilon S_h} \left(\left(\frac{1}{\mu-1} - G\right) e, \chi - \phi\right)_{\Omega_I}$$

from which it immediately follows using (A.12) that

$$\|e\|_{-1} = O(h^2). \quad (\text{A.14})$$

We now establish Result 3. From (6) we have that the error using M_h instead of M is given by

$$|GM(x) - GM_h(x)| = \left| -\frac{\nabla}{4\pi} \int_{\Omega_I} M \cdot \nabla \left(\frac{1}{|x-x'|} \right) dx' \right|.$$

For points outside of Ω_I

$$|GM(x) - GM_h(x)| \leq \frac{1}{4\pi} \|M\|_{-1} \left\| \nabla \left(\frac{1}{|x-x'|} \right) \right\|_2 = O(h^2)$$

which proves Result 3.

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Table 1: Computational Error vs. Mesh Size

NR	NT	E_M	RE_M	μ	E_H
3	16	.186	.06	10	.55
6	32	.047	.014	10	.16
12	64	.027	.0077	10	.005
3	16	.35	.095	20	.072
6	32	.113	.028	20	.02
12	64	.068	.016	20	.005

E_M = maximum norm error in magnetization

RE_M = relative maximum norm error

$$RE_M = \frac{E_M}{\max_{x \in \Omega_I} |M(x)|}$$

E_H = error in zeroth order harmonic coefficient.

Table 2: Computational Error vs. Permeability
(NR = 6, NT = 32)

μ	E_M	RE_M	A_0	A_0^C
5	.025	.011	.625	.633
10	.047	.014	.395	.411
20	.113	.028	.228	.247
40	.255	.057	.123	.146
4000	2.61	.435	.0013	.027

A_0 = first harmonic coefficient

A_0^C = first harmonic coefficient approximation

TABLE 3: Computational results for the double layer magnet, $\mu = 1259$

NR	NT	B_0	B_2/B_0	B_4/B_0	B_6/B_0
3	16	14.86	7.36×10^{-5}	1.32×10^{-6}	1.31×10^{-8}
6	32	15.3	1.3×10^{-4}	1.13×10^{-6}	-7.88×10^{-9}
12	64	15.36	1.47×10^{-4}	1.07×10^{-6}	-1.05×10^{-8}
Fourier Analysis		15.40	1.50×10^{-4}	1.047×10^{-6}	-1.16×10^{-8}

FIGURE 1 : Two layer magnet cross section

Iron Region
Conductor Region

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