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ANALYTIC COHERENT STATES FOR GENERALIZED POTENTIALS^{*}

by

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We give a prescription for finding "coherent states" in generalized potentials. By "coherent states" we mean states which in time follow the motion that a classical particle would. This prescription is based upon finding those "natural classical variables" which vary as the sine and the cosine of the classical $\omega_c t$. As an example, the symmetric Rosen-Morse potential is discussed in detail.

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I. INTRODUCTION

This is work^{1,2} which I (MMN) and Mike Simmons have been doing at Los Alamos. It's hopefully going to be short, sweet, and to the point. The title is "Analytic Coherent States for Generalized Potentials," and I have divided the talk into sections. In the next section I'm going to review in some detail the properties of the coherent states for the harmonic oscillator. Then I'm going to describe what I will call the "classical motion generalization." By that I mean I will define "coherent states" not only for the simple harmonic oscillator but for particles in different potentials. These coherent states should follow the classical motion of a particle in such a classical potential. Finally, to show that our proposed generalization is indeed a good one, I am going to present a specific example which we have analytically beaten to death, closing with a discussion.

[This written form of my presentation is modified from a tape recording of the talk which was actually given at the conference. In places I will insert further comments, sometimes concerning results which have been obtained since then. These will be surrounded by square brackets.]

II. HARMONIC OSCILLATOR COHERENT STATES

You probably all are familiar with the simple harmonic oscillator problem. It is one of the first you see in your beginning course on quantum mechanics. The eigenstates ψ_n are the solutions to the eigenvalue equation

$$H\psi_n = E_n\psi_n = -\left[\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2\right]\psi_n. \quad (2.1)$$

They are given in terms of Hermite polynomials,

$$\psi_n = \left(\frac{a_0}{\pi^{1/2} 2^n n!}\right)^{1/2} e^{-\frac{1}{2} \frac{a_0^2}{x^2}} H_n(a_0 x), \quad (2.2)$$

$$a_0 \equiv \left(\frac{m\omega}{\hbar}\right)^{1/2}, \quad (2.3)$$

and have eigenenergies

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \quad (2.4)$$

In particular, and this will be important later on, note that for the simple harmonic oscillator the difference in energy between adjacent eigenstates is independent of n . It is always the constant $\hbar\omega$ no matter what n .

As you all probably also know, the position and momentum can be defined as the sum and difference of the lowering and raising operators, a^- and a^+ ,

$$x = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a^- + a^+), \quad (2.5)$$

$$p = \left(\frac{m\hbar\omega}{2} \right)^{\frac{1}{2}} \frac{1}{i} (a^- - a^+) . \quad (2.6)$$

The lowering and raising operators satisfy the standard commutation relation,

$$[a^-, a^+] = 1 , \quad (2.7)$$

and have the property

$$a^\mp |n\rangle = \left(n + \frac{1}{2} \mp \frac{1}{2} \right)^{\frac{1}{2}} |n \mp 1\rangle . \quad (2.8)$$

Eq. (2.8) tells us that a^- and a^+ are also independent of n .

Given the above, the "coherent states" for the simple harmonic oscillator can be defined in many equivalent ways. Although these ways are equivalent for the harmonic oscillator it is not clear that they will be equivalent in general. [Indeed, they are not.]

a) Minimum Uncertainty Coherent States (MUCS)

One definition of the coherent states comes from the uncertainty relation between x and p . Since

$$[x, p] = i\hbar , \quad (2.9)$$

one has

$$\Delta x \Delta p = \hbar/2 . \quad (2.10)$$

The coherent states are those states which satisfy the equality in (2.10), SUBJECT to the condition that

$$\Delta x / \Delta p = 1/m\omega . \quad (2.11)$$

The point in (2.11) is commonly fluffed over. All minimum uncertainty states are not coherent states for all oscillators. A subset of them, restricted by (2.11), are coherent states for a particular oscillator.

(We'll see that very clearly in a minute or two.) Note that with the restriction (2.11), the ground state is one of the coherent states.

Anyway, this is one definition of the coherent states. The definition yields that the coherent states are particular Gaussians. How do you get that? This comes from a general result that if you have three Hermitian operators A, B, and C satisfying

$$[A, B] = iC, \quad (2.12)$$

(in general C is a q-number not a c-number), this implies that

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \langle C \rangle^2. \quad (2.13)$$

The states which will satisfy the equality in this generalized uncertainty relation are given by the solutions to the eigenvalue equation

$$\left[A + \frac{i\langle C \rangle}{2(\Delta B)^2} B \right] \psi = \left[\langle A \rangle + \frac{i\langle C \rangle}{2(\Delta B)^2} \langle B \rangle \right] \psi. \quad (2.14)$$

For the harmonic oscillator with A and B being x and p, the solution to Eq. (2.14), subject to the restriction (2.11) are

$$\psi = [2\pi(\Delta x)^2]^{-1/4} \exp \left\{ - \left[\frac{x - \langle x \rangle}{2\Delta x} \right]^2 + \frac{i}{\hbar} \langle p \rangle x \right\}. \quad (2.15)$$

That gives the first definition of the coherent states.

b) Annihilation Operator Coherent States (AOCS)

Another definition of the coherent states is that they are the eigenstates $|\alpha\rangle$ of the annihilation operator a^- :

$$\begin{aligned} a|\alpha\rangle &= \frac{1}{2} \left(\frac{x}{\Delta x} + i \frac{p}{\Delta p} \right) |\alpha\rangle = \alpha |\alpha\rangle \\ &= \frac{1}{2} \left(\frac{\langle x \rangle}{\Delta x} + i \frac{\langle p \rangle}{\Delta p} \right) |\alpha\rangle. \end{aligned} \quad (2.16)$$

As is well known, these turn out to be the following linear combination of the number states:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \quad (2.17)$$

Further, by using the generating function of the Hermite polynomials, a little algebra allows one to show that, up to a phase, Eq. (2.17) is the same as Eq. (2.15).

It is from this point of view that one can see the property (2.11) that I emphasized before. Notice that the coherent states we are discussing are in principle defined in terms of four parameters: $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$. But α has only two parameters, $(\text{Re } \alpha)$ and $(\text{Im } \alpha)$. You've got to eliminate two. They are eliminated by the equality in (2.10) and the restriction (2.11).

If you were to put in a different value of $\Delta x / \Delta p$ and then decompose these other states into the number states, they would not be Hermite polynomials of $a_0 x$, where a_0 is the proper a_0 , but of a different $a_0' x$.

c) Displacement Operator Coherent States (DOCS)

Finally, there is a third standard definition. This is that one apply the unitary operator

$$A(\alpha) = \exp[\alpha a^+ - \alpha^* a^-] \quad (2.18)$$

onto the ground state,

$$|\alpha\rangle = A(\alpha) |0\rangle , \quad (2.19)$$

where α is the α we defined before.

All three of the above definitions are equivalent in the simple harmonic oscillator system. You can use any of them and you come up with the same set of states. These coherent states satisfy the following property, they will follow the classical motion no matter how small is $\langle \alpha | H | \alpha \rangle = \bar{E}$. What do I mean by that? If I take a coherent state whose energy is $\bar{E}$, then $\langle \alpha | x(t) | \alpha \rangle$ will have the same motion as $x(t)$ for a classical particle of energy $(\bar{E} - E_0)$, where E_0 is the ground state energy $\hbar\omega/2$. Further, there will be no dispersion of the wave-packet with time.

This concept, that these states follow the classical motion, is going to be the motivation for our generalization.

Before proceeding I wish to point out that Barut and Girardello³ and Perelomov⁴ have separately proposed generalizations of the AOCs and the DOCS which have been applicable, using group theory methods, to systems with equally spaced levels, such as spin and angular momentum, including with magnetic fields.

But our question was the following. How do you apply these techniques to systems which aren't equally spaced? Like really, you usually don't have equally spaced levels. You can have a continuum. What if the system is not exactly solvable -- you may not know a dynamical group or anything about it.

Finally, there was one more thing which really tickled us. Some people who are doing laser-molecular physics problems have a great interest in classical analyses of these problems, sometimes simply because that's the only game that one knows how to play. It turns out, at least according to the adherents of this school, that often these classical analyses work surprisingly well. This too made us interested in coherent states which would follow the motion of a classical particle.

So, the question we have been asking is, "Can one define states which follow the classical motion in arbitrary potentials; real honest-to-God ones?" That being said I'm not going to start with MUCS, AOCS, or DOCS. Rather, I am going to look at the classical motion as Schrödinger⁵ originally did, and then see if there are any connections to MUCS, AOCS, or DOCS. We will see that the most direct connection is to the MUCS.

III. CLASSICAL MOTION GENERALIZATION

OK, back to classical physics. These next equations you saw even before your first course in quantum mechanics.

$$T + V = E , \quad (3.1)$$

$$\frac{1}{2} m \dot{x}^2 + V(x) = E , \quad (3.2)$$

$$\dot{x} = \left(\frac{2}{m} \right)^{\frac{1}{2}} [E - V(x)]^{\frac{1}{2}} . \quad (3.3)$$

Recall that for the simple harmonic oscillator with

$$V(x) = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2 , \quad (3.4)$$

the solution for $x(t)$ is

$$x(t) = \left[\frac{2E}{m\omega^2} \right]^{\frac{1}{2}} \sin(\omega t) , \quad (3.5)$$

so that

$$p(t) = m\dot{x} = m \left[\frac{2E}{m} \right]^{\frac{1}{2}} \cos(\omega t) . \quad (3.6)$$

Note that ω is a constant, no matter how large the amplitude - no matter what the energy. That's a property only of the harmonic oscillator. In general ω is classically a function of the energy. This is the reason why people can put ωm^2 in place of k , the force constant, in the potential equation (3.4).

What we propose to do, and this is our first physical ansatz, is to take a classical potential and find a map which changes its Hamiltonian equation into variables $X_c(x)$ and $P_c(x, \dot{x})$ such that the Hamiltonian equation is "harmonic-oscillator-like" in these variables, for a given energy.

(The subscript "c" denotes classical.) That means we want to find $X_c(x)$ and $P_c(x, \dot{x})$ whose time variations are given by

$$X_c = A_c \sin \omega_c t , \quad (3.7)$$

$$P_c = B_c \cos \omega_c t . \quad (3.8)$$

Eq. (3.7) means that (prime denotes d/dx)

$$\dot{X}_c = \dot{x} X'_c = A_c \omega_c \cos \omega_c t . \quad (3.9)$$

Thus, our P_c can be given as

$$P_c = m\dot{X}_c = pX'_c . \quad (3.10)$$

By relating the cosine to the square-root of one-minus-sin-squared, one can combine (3.3), (3.7), and (3.9), to give X_c as the solution to the equation

$$X'_c(x) = \left(\frac{m}{2}\right)^{\frac{1}{2}} \omega_c \left[\frac{A_c^2 - X_c^2(x)}{E - V(x)} \right]^{\frac{1}{2}} . \quad (3.11)$$

Putting this all back in to the original Hamiltonian equation (3.1) you obtain

$$\frac{1}{2m} P_c^2 + \frac{1}{2} m \omega_c^2 X_c^2 = \left[\frac{1}{2} m \omega_c^2 A_c^2 \right] . \quad (3.12)$$

QUESTION: (Rohrlich) Can you map the Coulomb potential into the harmonic oscillator potential?

ANSWER: Are you talking classically or quantum mechanically?

REPLY: (Rohrlich) Classically.

ANSWER: Classically, yes. The "natural variables" turn out to be just the variables which you use in solving the classical

Kepler problem. $(1/r - 1/r_0)$ is one of them and p_r is the other one. It turns out, however, that in this multidimensional problem one does not discuss things in terms of ω_c , but rather in terms of the angular velocity between apsidal distances. (This is one of the reasons why we want to talk to Mostowsky⁶ after this talk, since this is our version of the problem he looked at.) But there is an X_c and there is a P_c , so the answer classically is, "Yes." The answer quantum mechanically is, "Almost." Although there is a mapping of the Coulomb potential eigenstates into the harmonic oscillator eigenstates, the mapping is not one-to-one. Almost, but not quite.

Anyway, if we have done the above we have solved the classical problem. Now what do we propose to do quantum mechanically? We say that one should consider the quantum operators which are the quantum analogues of these "natural" classical variables X_c and P_c . That is, consider

$$X_c \rightarrow X(x) , \quad (3.13)$$

$$P_c \rightarrow P = \frac{\hbar}{2i} \left[\frac{d}{dx} X'(x) + X'(x) \frac{d}{dx} \right] . \quad (3.14)$$

(Note that in (3.14) we have symmetrized $X'(x)$ and p .) Because X and P satisfy

$$[X, P] = i\hbar (X')^2 , \quad (3.15)$$

they have the associated uncertainty relation

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{\hbar^2}{4} \langle X'^2 \rangle^2 . \quad (3.16)$$

We claim that the appropriate coherent states are those states which satisfy the equality in (3.16); i.e., they satisfy

$$A^- \psi_\alpha = \frac{1}{2} \left(\frac{X}{\Delta X} + j \frac{P}{\Delta P} \right) \psi_\alpha = \alpha \psi_\alpha , \quad (3.17)$$

SUBJECT to the restriction that the ground state is one of the set. (This, of course, is an alternative to setting a particular value to $\Delta X/\Delta P$.) A^- will turn out to be the proportional to the ground state annihilation operator.

[A comment to be made here is that in all our special cases, with our choice of "natural" classical variables, it is always possible to set one of the three parameters of the states which satisfy the equality in (3.16) equal to a specific value such that the ground state is one of the set. This is not possible if one were to take $X_c(x)$ and the momentum which is classically conjugate to $X_c(x)$ as the variables. Thus, our choice of variables is appropriate.]

Therefore, we have a continuous, two parameter set of states. That is the whole physics of it, right there. It is very simple. First year quantum mechanics students should be able to understand it. Now lets see what happens with this formulation.

IV. THE SYMMETRIC ROSEN-MORSE POTENTIAL

We have examined in more or less great detail a number of specific potentials. Hopefully this number will continue to grow. What I am going to concentrate on here is the symmetric Rosen-Morse (RM) potential,^{7,8}

$$V(x) = U_0 \tanh^2 ax \quad (4.1)$$

$$\equiv \mathcal{E}_0 s(s+1) \tanh^2 z, \quad (4.2)$$

$$\mathcal{E}_0 = \frac{\hbar^2 a^2}{2m}, \quad (4.3)$$

because it has built into it many of the possible problems. There's a continuum, the discrete eigenstates are not equally spaced, and, depending on how deep the potential is, I can have many bound states or few bound states. However, despite this, we can handle this thing analytically, as you will see. (We were also interested in this potential because of its relation to the soliton problem.)

Rosen and Morse⁷ invented this potential in 1932, to discuss diatomic molecules. They actually had it in an asymmetric form, which is a slight generalization of (4.1), and with the symmetric part being proportional to $(\cosh z)^{-2}$. For convenience we have added "1" to this so that the minimum of the well is at the origin. That makes it $(\tanh z)^2$.

As an aside, it's very interesting that in doing this work we have gone through many of the old quantum mechanics potentials, and it turned out that, at least to our knowledge, some of the properties of these solutions have never been obtained, such as the exact normalization constants. For example, for the Rosen-Morse potential things were originally solved in terms of hypergeometric functions, and then dropped. So, partially we have been going along up the old QM papers.^{2,8}

Now, we will solve the classical problem, although, with foresight, in Eqs. (4.1)-(4.3), we have already defined U_0 in terms of the parameters which will be useful quantum mechanically. The classical solutions for X_c and P_c are

$$X_c = \sinh z = \left[\frac{E}{U_0 - E} \right]^{\frac{1}{2}} \sin \omega_c t , \quad (4.4)$$

$$P_c = p \cosh z = [2ma^2 E]^{\frac{1}{2}} \cos \omega_c t , \quad (4.5)$$

$$\omega_c = \left[\frac{2a^2(U_0 - E)}{m} \right]^{\frac{1}{2}} . \quad (4.6)$$

You put the above in Eqs. (3.11) and you find that it is satisfied. The classical equations of motion are

$$\dot{X}_c = P_c/m , \quad (4.7)$$

$$\dot{P}_c = -m\omega_c^2 X_c . \quad (4.8)$$

Now we are going to do our coherent states procedure. The "natural" quantum operators are

$$X = X_c = \sinh z , \quad (4.9)$$

$$P = \frac{\hbar}{2i} a^2 \left[\frac{d}{dz} \cosh z + \cosh z \frac{d}{dz} \right] , \quad (4.10)$$

so that we are looking for states which satisfy the equality in the uncertainty relation

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{\hbar^2}{4} a^4 \langle \cosh^2 z \rangle^2 . \quad (4.11)$$

These states, complete with normalization constants, are

$$\psi_{cs} = N(C,s) \phi_{cs}(x) , \quad (4.12a)$$

$$\psi_{cs} \equiv \left[\frac{a\Gamma(B+\frac{1}{2}) \Gamma(B+\frac{1}{2}-iu)}{\pi^{\frac{1}{2}}\Gamma(B)\Gamma(B+\frac{1}{2})} \right]^{\frac{1}{2}} \\ \times (\cosh z)^{-B} \exp[C \sin^{-1}(\tanh z)] , \quad (4.12b)$$

$$B \equiv \frac{1}{2} \frac{\langle \cosh^2 z \rangle}{(\Delta \sinh z)^2} + \frac{1}{2} , \quad (4.13)$$

$$C \equiv u + iv = B\langle \sinh z \rangle + \langle \cosh z \frac{d}{dz} \rangle . \quad (4.14)$$

Finally, if

$$B \equiv s , \quad (4.15)$$

this set of states includes the ground state. Therefore, imposing (4.15) gives us our coherent states.

Before continuing, I want to take a closer look at the quantum eigenvalue problem, and show how one can get our "natural quantum operators" from the raising and lowering operators. (In fact, this is how we first stumbled on our "natural operators.") The exact eigenstates and eigenvalues are

$$\psi_n = N(n,s) \phi_n(x) \quad (4.16a)$$

$$= \left[\frac{a(s-n)\Gamma(2s-n+1)}{\Gamma(n+1)} \right]^{\frac{1}{2}} p_s^{n-s} (\tanh z) , \quad (4.16b)$$

$$E_n = \epsilon_0 [2ns - n^2 + s] . \quad (4.17)$$

Now, because the eigenfunctions are in terms of associated Legendre functions, you can derive the raising and lowering operators. However, for those of you who may be familiar with the work of Infeld and Hull,⁹ our "n" raising and lowering operators are not theirs. They obtain a factorization which is correct mathematically, but is not the useful one physically. What their factorization does is to raise and lower "s." Thus, essentially they raise and lower you from one potential to a different potential, keeping the number state constant. To actually generate the number states for a specific potential from their operators would drive you out of your mind. Anyway, the raising and lowering operators are

$$A_n^\pm = (s-n) \sinh z \mp \cosh z \frac{d}{dz} . \quad (4.18)$$

It turns out that, in a generalization of the way x and p are defined in terms of a^- and a^+ for the harmonic oscillator (see Eqs. (2.5) and (2.6)), X and P can be defined in terms of Hermitian sums and differences of the raising and lowering operators. I say, "Hermitian," since here and in general A_n^- is not the adjoint of A_n^+ . Specifically,

$$X = \frac{1}{4(s-n)} [(A_n^- + (A_n^+)^\dagger) + (A_n^+ + (A_n^-)^\dagger)] , \quad (4.19)$$

$$P = \frac{a^2}{4i} [(A_n^- + (A_n^+)^\dagger) - (A_n^+ + (A_n^-)^\dagger)] . \quad (4.20)$$

The above shows in one more way how what we are doing is a generalization of the harmonic oscillator. This generalization is elucidated by showing how all our results go over to the harmonic oscillator in the specific limit

$$\lim_{\hbar \rightarrow 0} \equiv \lim_{(s \rightarrow \infty, a \rightarrow 0, s a^2 \rightarrow m\omega/\hbar)} . \quad (4.21)$$

It is straightforward, but somewhat involved, to show that in the limit (4.21) the eigenstates, eigenvalues, and coherent states for the RM potential go over to the same objects for the harmonic oscillator. For example, for the eigenstates you start with the associated Legendre functions and change them to $(\cosh z)^{n-s}$ times Gegenbauer polynomials. There is a relation between Gegenbauer polynomials and Hermite polynomials in the limit we are considering. With the above transformations you can show that the RM eigenfunctions become the harmonic oscillator eigenfunctions, complete with normalizations.

The quantum equations of motion are

$$\dot{X} = -i\hbar^{-1} [X, H] = P/m , \quad (4.22)$$

$$\dot{P} = -a\{U_0 - H - \frac{1}{4} \epsilon_0, X\} . \quad (4.23)$$

Believe it or not, the above allow one to exactly calculate the operators $X(t)$ and $X^2(t)$, and similarly $P(t)$ and $P^2(t)$. Specifically,

$$X(t) = e^{iHt/\hbar} X e^{-iHt/\hbar} \quad (4.24)$$

$$\begin{aligned} &= X e^{-i\omega_0 t} \left[\cos \omega_H t + i \left(\frac{\epsilon_0}{U_0 - H} \right)^{\frac{1}{2}} \sin \omega_H t \right] \\ &+ P e^{-i\omega_0 t} \left[\frac{1}{2ma^2(U_0 - H)} \right]^{\frac{1}{2}} \sin \omega_H t , \end{aligned} \quad (4.25)$$

$$\begin{aligned}
X^2(t) = & \left[X^2 - \frac{1}{2} \left(\frac{H+\mathcal{E}_0}{U_0-H-\mathcal{E}_0} \right) \right] \exp[-i4\omega_0 t] \cos(2\omega_H t) \\
& - i \left[XQ + \frac{1}{2} \left(\frac{U_0}{U_0-H-\mathcal{E}_0} \right) \right] \left(\frac{\mathcal{E}_0}{U_0-H} \right)^{\frac{1}{2}} \exp[-i4\omega_0 t] \sin(2\omega_H t) \\
& + \frac{1}{2} \left(\frac{H+\mathcal{E}_0}{U_0-H-\mathcal{E}_0} \right),
\end{aligned} \tag{4.26}$$

where

$$\omega_0 \equiv \mathcal{E}_0/\hbar, \tag{4.27}$$

$$\omega_H \equiv \left[\frac{2a^2}{m} (U_0-H) \right]^{\frac{1}{2}}, \tag{4.28}$$

$$Q \equiv \cosh z \frac{d}{dz}. \tag{4.29}$$

The calculation of (4.25), for example, is done by expanding $\exp(\pm iHt/\hbar)$ in the right hand side of (4.24) as series of nested commutators, and then using (4.22) and (4.23) to obtain two three-term recursion relations. They can be solved. [See Ref. 2 for a description.] I have written the results with X and P (or Q for convenience) on the left and all the functions of H on the right. That turns out to be useful in calculations. If you go through the same type of calculations for the harmonic oscillator you get simpler but similar looking messes. Observe that ω_H of Eq. (4.28) would be the classical frequency ω_c of Eq. (4.6) if E were substituted for H .

QUESTION: It's an operator?

ANSWER: It's an operator. Right.

Returning to our coherent states of Eqs. (4.12)-(4.15), we can calculate $\langle X \rangle$, $\langle X^2 \rangle$, $\langle P \rangle$, $\langle P^2 \rangle$, $\langle H \rangle$, and all the rest. They satisfy the equality in the uncertainty relation (3.16).

By decomposing the coherent states at $t = 0$ into the eigenstates, the time-evolution of the coherent states can be given by

$$\Psi_{cs}(x, t) = e^{-iHt/\hbar} \psi_{cs}(x) \quad (4.30)$$

$$= \frac{N(C, s)}{a} \sum_{n=0}^{[s]} N^2(n, s) \exp[-iE_n t/\hbar] \\ \times O(n, C, s) P_s^{n-s}(\tanh z)$$

$$+ \text{Continuum} , \quad (4.31)$$

where $O(n, C, s)$ is the overlap. Further, one can calculate the overlap analytically. The n -even overlap function is

$$O(n\text{-even}, C, s) = \frac{\pi^{\frac{1}{2}}}{\Gamma(s+1+n)2^{s-n}} \frac{\Gamma(s-\frac{n}{2})\Gamma(s+\frac{1}{2}-\frac{n}{2})}{|\Gamma(s+\frac{1}{2}-\frac{n}{2}-i\frac{C}{2})|^2} \\ \times {}_4F_3 \left[\begin{matrix} s+\frac{1}{2}-\frac{n}{2} , s+\frac{1}{2}-\frac{n}{2} , s-\frac{n}{2} , -n; \\ s+\frac{1}{2}-\frac{n}{2}-i\frac{C}{2} , s+\frac{1}{2}-\frac{n}{2}+i\frac{C}{2} , s+1-n \end{matrix} \right] , \quad (4.32)$$

with a similar relation for n -odd. The ${}_4F_3$ hypergeometric series terminates, since one of the upper indices is a negative integer, so the overlaps are finite sums. It is all very complicated, but in exact closed form.

V. DISCUSSION

In the above decomposition I have not written out the continuum contribution. To see why, let us first look at the bound states. The bound state superscript on the associated Legendre functions of (4.16) is $(n-s)$, $n < s$. But $(n-s)^2$ measures the energy between the top of the potential and the particle eigenenergy. That is to say, $|n-s|$ essentially is proportional to the magnitude of a wave vector. When you go into the continuum, n tries to get larger than s , and that amounts to introducing $\pm i$ in front of the quantity $|n-s|$ which is the superscript to the Legendre function. That is, the continuum wave functions go as $\exp(\pm ikz)$. So, they are plain waves at infinity which, as they come to the well, zoom through and go out again. So, with time these continuum contributions just go off to the sides, leaving the bound state contributions confined.

Strictly speaking, there is always some overlap with the continuum (except for the coherent state which is the ground state). As stated, this will go off the sides with time. However, if you are reasonably deep in the well the overlap of the CS with the continuum will be exponentially small. This is consistent with the numerical calculations of Preston and Walker¹⁰ for the Morse harmonic oscillator. Taking a harmonic oscillator coherent state with $\langle n \rangle = |\alpha|^2$, they only needed on the order of $|\alpha|^2$ states to describe the time evolution. It is strictly true that there are vanishingly small pieces of all the higher states, but to numerical accuracy, they don't contribute.

Because one knows what $\psi_{cs}(x)$ is in terms of the eigenfunctions (just set $t = 0$ in Eq. (4.31)), all the operators involving $f(H)$ in Eqs. (4.24)-(4.29) will just give you $f(E_n)$ when operating on each particular eigenstate contribution (ψ_n) to the decomposition of ψ_{cs} . That means that we can

exactly calculate the bound state portions of $\langle X(t) \rangle_{cs}$, $\langle X^2(t) \rangle_{cs}$, etc., and they can be evaluated numerically.

Given all our analyses, do our coherent states follow the classical motion? If you look at all these complicated formulas and make a first, rough, analytic approximation, you can show the states follow the classical motion in this approximation. But you have to drop all kinds of terms to state this. The real question is how good is this approximation under various circumstances, like if the average energy is high or low compared to the well depth. The exact answer depends on the parameters of the system.

We are now programming our results to get precise numerical answers. Notice that the formulas are exact analytically, so you don't have to do any solving of differential equations. What we've got are just functions of x and t . Plot them! It's complicated but exact. A friend of mine by the name of Vincent Gutschick, who is a theoretical chemist by trade, -- he has many trades but that's his real one -- is at this very moment crunching away on the time evolution of these coherent states.

Then we will complete this same program for other cases. The analytic calculations have been done completely for some potentials, and partially for others. [Now the one-dimensional analytic calculations are complete.] Finally, we will also numerically look at the time evolution of the coherent states for these different potentials.

So, the conclusion is that we have a physical ansatz from which we feel one can define coherent states which follow the classical motion, of a particle in generalized potentials. For the example we have given, we know this is true in first analytic approximation. How well does their exact motion follow the classical motion? Come back in about one month and we hopefully can give you some answers.

And, in 29 minutes and 1 second, that's it. Thank you.

[After this talk was presented, a series of detailed numerical calculations were done on the coherent states for this and other potentials. These states were found to follow the classical motion quite well. In general, the deeper the state is in a well and, given that, the more eigenstates which have significant overlap with the coherent states, the longer the state coheres in terms of the number of classical oscillations. Each time the packet hits the wall, a piece of it goes off to the background, but the main hump still follows the classical motion. A computer generated film is available showing the time evolution of these states.¹¹ A detailed discussion of our general coherent states will appear elsewhere.¹²]

VI. QUESTIONS

QUESTION: To the effect of, "Can you give the connections of your formalism to those of Barut and Perelomov?"

ANSWER : I can give a connection to Barut³ faster than I can to Perelomov.⁴

A connection to Barut's is that one uses the ground state lowering operator in the eigenvalue equation (3.17). The connection to Perelomov's we are now not so sure of, although we have a couple of ideas. Our big problem in making a connection there is that the raising and lowering operators, as we have defined them, are n -dependent. One can make an ad-hoc choice and use the ground state lowering operator to make a connection to Barut's formalism.

[And it is ad-hoc since one picks out the A_0^- over the general A_n^- .] But because in Perelomov's formulation you have an exponentiation of operators, we're not exactly sure what's a proper choice. It may turn out that a connection can only be made if you modify or change the AOCS or DOCS formulations for generalized systems. That is possible. That is something which we have looked at, but which we have not concluded something on yet.

[Since the conference we have found a way to generalize the AOCS and DOCS concepts to non-equally spaced eigenvalue systems. In general they are not equivalent to what can be called the generalized MUCS that we have espoused here. This will be discussed elsewhere.¹²]

QUESTION: (Bialynicki-Birula) To the effect of, "You can also produce coherent states based on higher- n raising and lowering operators?"

ANSWER : Yes, but you see the way I presented this talk is completely "bass-ackwards" from the way that we actually obtained our

results. It turns out, and this is true for all our examples. If you start by looking for those X_c and P_c which classically vary as the sine and cosine of $\omega_c t$, and then you change to the quantum mechanical operators, by magic every single time the X and P can be defined as the Hermitian symmetric sum and difference of the raising and lowering operators -- you know -- with the n -dependence normalized out.

QUESTION: Concerning the relation of our analytic work to the group theory point of view.

ANSWER : Miller, where's he at?

REPLY : (Simmons) Willard Miller - Minnesota.

ANSWER : He has done a tremendous amount of work on the connection between group symmetries and the special functions of mathematical physics. As you saw, here the point basically was to find out that the wave functions are Legendre functions of $(\tanh z)$. It turns out that in all our other examples you can do similar weird transformations. If you like the special functions of mathematical physics, it's a fun game.

QUESTION: (Jaynes) Concerning what was done about the eigenfunctions in the original Rosen-Morse paper.

ANSWER : I want to make sure I quoted it right, first. OK, '32 it is, Phys. Rev. The solutions were given by them in terms of unnormalized hydrogeometric functions.

QUESTION: (Zwanziger) "Can you state again the class of potentials that you've considered?"

ANSWER : So far we have considered the radial part of the Coulomb problem, the inverse cosh-squared (or the tanh-squared, as discussed here

today), the Pöschl-Teller potential, which limits into the infinite square well--most people consider them separately but there is a certain limit connecting them, of course we have the harmonic oscillator, and the Morse potential.

QUESTION: (Zwanziger) What's the generality of the method, then?

ANSWER : The generality of the method is that it is based on the relation between X' and X . In principle it can be solved numerically. So, people who know how to handle differential operators on computers can hopefully take an arbitrary potential, find the eigenvalues, find the raising and lowering operators, define the classical problem and therefore numerically go through the whole procedure that we have done analytically in this example. It so happens that what we have discussed is an analytic problem to see if in an analytic problem you can define states which are "coherent" in the sense that we have defined them; i.e., they follow the classical motion and don't disperse very fast. Clearly for few level systems, and so forth, they will quickly disperse. The question is how fast, for all situations.

QUESTION: "What about the number of dimensions? Can you do anything for higher dimensions?"

ANSWER : What we feel you do there is to look as we have in the Coulomb problem.¹ You look at the radial oscillation. There ω_c is replaced in the classical problem by $\theta(t)$. That's the circular variable that you work with. People¹³ have beaten us to the punch azimuthally for the angular part of the problem for spherically symmetric potential. These states are called "intelligent spin states." If you look at their solutions they

amount to using $L_{\pm}$ raising and lowering operators in the MUCS formalism.

QUESTION: But is that not special because there is no precession?

ANSWER : I agree, but in general you can solve all such problems numerically. You have to follow $\theta(t)$ between successive apsidal distances. The orbit will not be closed except for the Coulomb potential and the harmonic oscillator. Other systems will have to be done approximately or numerically, but in principle they can be studied this way.

QUESTION: (Rohrlich) "Have you looked at potentials with two minima?

ANSWER : Ahhh... You are a son of a bitch, Fritz. The answer is we don't know yet. Let me tell you what the problem is. Clearly, quantum mechanically you've got tunneling. Classically, if you are in one well you are going to stay in one well. What I think will end up happening is that one will have a set of states which originally will start in one well oscillating back and forth and slowly will spread out. That's what we think will happen. But we do not know yet.

QUESTION: (Rohrlich) But you think you can handle it?

ANSWER : Hope? How's that? Yeah, we have hopes, I think that is a fair way to put it. We have hopes that we can do it. Thank you very much.

[From the results of our numerical work, we think we can now give a more definitive answer to Rohrlich's question. It appears that as a matter of principle, the time evolution of a coherent wave packet in a potential with more than one minimum is no more complicated (in principle) than for a potential with one minimum. For definiteness, consider the potential

$U = 2(U_R)^{\frac{1}{2}} x^2 + R x^4$, with the particle confined to the right at $t = 0$.

Ordinarily, with one minimum, the coherent state starts at $t = 0$ as a minimum uncertainty wavepacket, and then slowly disperses. When one thinks of the CS as being a superposition of eigenstates, the superposition is such that the nodes of all the eigenstates cancel each other except for the one hump. As time evolves, this cancelation gradually is ruined.

But it is no different for our double-minimum potential. With the double minimum one has a set of almost degenerate eigenstates (even and odd), which cancel on the left and also cancel on the right, except for one hump. As time evolves, the "tunneling" can be simply viewed as the cancelation on the left gradually getting ruined, just as it is getting ruined on the right at places different from the vicinity of the classical position.]

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