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Vortex Collapse from a Viewpoint of Complex-time Singularity

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When collisions of point vortices occur under certain initial and parametrical conditions, a singularity appears on the real time axis. If some of the conditions are deviated from the collapse ones, the singularity splits to a (or some) complex-conjugate pair(s). Recently a scaling law between a deviation of an initial condition and a resulted shift of the position of singularity from the real point was proposed for a system of point vortices. This scaling law is verified using the collisions of three and four point vortices.

key words: collapse, complex-time singularity, vortex motion, scaling law

§1 Introduction

In various collapse phenomena (phenomena in which a quantity or its derivative becomes infinite), we can associate the concept of a real singularity, i.e. a singularity on the real restriction in the complex domain. For examples, we can raise collisions of point masses in celestial mechanics [1] or phase transitions in statistical mechanics [2]. Mathematically speaking, these singularities are called "movable" owing to the fact that their complex positions and orders are subject to initial or parametric conditions for the nonlinear equations. Under special conditions, a real singularity can occur and for other cases it takes complex positions, in general. If we assume that small amount of perturbations to those collapse conditions only cause small changes of the property of singularities (i.e. position and order), which will be validated by a structural study of solutions of differential equations, a relation between an amount of perturbation and a change of the properties stimulates our interests. Here, we call this type of changes of position of singularity "parametric motion".

The objective of this paper is to seek a relation between a perturbation to a condition (in particular, initial one) and a shift of position of complex-time singularity using a system of interacting point vortices. The reasons of this choice are,

- (1) their motion is governed by a set of nonlinear ordinary differential equations, which can be written in polynomial couplings [3]. This fact is suitable for detecting complex-time singularities by successive calculations of the radius of convergence of the Taylor series of a solution.
- (2) There are several particular solutions for point vortices such as the rigid rotation solution with a constant angular velocity when three vortices are in the equilateral triangle configuration (it is irrespective of the strengths of three vortices!). As the solution is a sinusoidal function of time, there is no singularity in a finite domain of complex time plane,
- (3) Self-similar collapse solutions are known under certain initial and parametrical conditions. When a collapse occurs, a real time singularity appears in a finite time.

As a numerical method to detect complex-time singularities, a ratio-like test named "three terms test" by Chang and Corliss [4] was adopted. It is a numerical technique to count the radius of convergence of the Taylor series as a complex number (not only its absolute value but also its argument) which substantially provides a complex distance from an expansion point to the nearest singularity. It is almost equivalent in concept with other well-know methods for detecting complex singularities such as the Padé approximants method [5] or the Domb-Syke's plotting method [6].

As a numerical procedure, first the equations of motion modified in polynomial couplings is integrated to a certain point t_0 in the complex-time plane along a certain complex integration path. The integration should be as accurate as possible and we have used the fourth-order Runge-Kutta scheme. (Of-course, every variable (dependent and independent) should be defined as complex.) Then taking advantage of "the polynomiality", coefficients of the Taylor expansion are calculated recursively, and by computing a certain ratio of the coefficients and checking its convergence, the complex radius of convergence is estimated and a guess of a position of the nearest singularity from t_0 is obtained. If t_0 is closer to the singularity, the convergence is faster and we get better estimation, and thus a good choice of the integration path is very essential for the accuracy.

In the following sections, by perturbing the rigid rotation solution, a parametric motion towards infinity is shown in §2. Before proceeding to parametric motions towards a real point corresponding to a collapse of vortices (§4), the general theory of the similarity solution (the homographic solution) is briefly reviewed and the particular solutions for three- and four-vortex cases, which have been discussed by Novikov [7] and Aref [8] independently, are presented in §3. Section 5 is devoted to discussion of the importance of the present and the future research.

§2 Parametric motion towards infinity

Motion of N -point vortices is governed by

$$\frac{dz_j}{dt} = \frac{-1}{2\pi i} \sum_{m \neq j}^N \frac{\kappa_m}{z_j^* - z_m^*} \quad (j = 1, 2, \dots, N) \quad (2.1)$$

where $z_j (= x_j + i y_j)$ and κ_j denote complex position and strength of j -th vortex, respectively, and the asterisk indicates complex conjugation. By regarding the $1/\text{denominator's}$ in (2.1) as other dependent variables and coupling the time evolution of those variables, we have an expanded system which has polynomial couplings [3].

As mentioned in the introduction, (2.1) has the equilateral triangle solution for which the initial triangle rotates rigidly about the center of vorticity with angular velocity $(\kappa_1 + \kappa_2 + \kappa_3)/2\pi\rho$ where ρ is a length of a side [8]. In [8], the result of linear stability analysis of the solution is also presented and the neutrally stable mode has angular velocity

$$(3\kappa_1\kappa_2 + 3\kappa_2\kappa_3 + 3\kappa_3\kappa_1)^{1/2} / 2\pi\rho. \quad (2.2)$$

It is one of the eigenvalues of the linearized equation of (2.1) around the equilateral triangle solution in the rotating frame with the same angular velocity of the system. (See §4.)

As the rigid rotation solutions are entire-function solutions of time, no singularities exist in the finite complex-time domain. As an initial configuration is approached to an equilateral triangle the singularities go to infinity (Fig. 1). We show this fact using a specific example in Fig. 2. For this example, as the third vortex is approached to the point $(0, \sqrt{3})$ vertically from above, the nearest singularity to the origin goes to infinity along the real-time axis while the imaginary component becomes constant asymptotically. To see the detail, we define ϵ as deviations of the y-component of the third vortex from $\sqrt{3}$ and plot $-\log \epsilon$ and R_c , the real component of the nearest singularity in Fig. 3. The line in Fig. 3 is one with a slope of $4/3$. Hence we observe an asymptotic scaling law $\epsilon \sim e^{-3/4 R_c}$ for this particular case. The scaling law can be generalized for other equilateral triangle cases with different combinations of strengths and sizes as

$$\epsilon \sim e^{-\lambda t_0}, \quad (2.3)$$

where t_0 is the complex position of the nearest singularity and λ is a scaling exponent. We have verified numerically that λ is equivalent to the eigenvalue of the linear stability problem for the equilateral solution (2.2). (A heuristic explanation is provided in [3] and [9].) The above relation tells us that when λ is either real or pure imaginary t_0 goes to infinity asymptotically along the real or the imaginary axis of time for the real perturbation ϵ , and we called these motions lateral and vertical sweepings, respectively.

§3 Similarity solution for the motion of point vortices

In §4 we extend the result in the preceding section to parametric motion towards a point on the real axis by applying perturbations to the collapse solutions. Before that we briefly review the theory of similarity solutions for the motion of point vortices in this section. The reader is referred to [10],[11] and references therein for the details.

To get a similarity solution, we assume the following decomposition into space and time variables:

$$z_j = k_j f(t). \quad (3.1)$$

where k_j is the complex-position of the j -th vortex at $t = 0$, and $f(t)$ is the time dependence of the whole system. Substituting (3.1) into (2.1), we obtain equations for $f(t)$ and k_j , respectively,

$$\dot{f} f^* \equiv C = A + i B \quad (3.2)$$

$$C = \frac{i}{k_j} \sum_{m (\neq j)}^N \frac{\Gamma_m}{k_j^* - k_m^*} \quad (j = 1, 2, \dots, N) \quad (3.3)$$

where the dot denotes differentiation with respect to time, A and B are real constants, and $\Gamma_m = \kappa_m/2\pi$.

Equation (3.2) can be easily integrated for the initial condition $f(0) = 1$,

$$f(t) = \begin{cases} \sqrt{2At+1} \exp \left[i \frac{B}{2A} \log(2At+1) \right] & (A \neq 0) \\ \exp(i Bt) & (A = 0) \end{cases} \quad (3.4)$$

$$(A = 0) \quad (3.5)$$

and thus the system collapse at the time $t^* = -1/2A$ unless A vanishes. From the consideration of integrals of the motion, the Γ_j 's should satisfy, when $A \neq 0$,

$$\sum_{i > j} \Gamma_i \Gamma_j = 0, \quad (3.6)$$

as is explained in previous studies.

We present here only some of the particular solutions of (3.2) and (3.3) for three- and four-vortex cases separately.

Three-vortex case [7], [8], [10]

combination of the strength: $(\Gamma_1, \Gamma_2, \Gamma_3) = (2, 2, -1)$,

initial positions: $(x_1, y_1) = ((3 + \sqrt{3} \cos \theta) / 6, \sqrt{3} \sin \theta / 6)$,

$(x_2, y_2) = ((-3 + \sqrt{3} \cos \theta) / 6, \sqrt{3} \sin \theta / 6)$,

$(x_3, y_3) = (2\sqrt{3} \cos \theta / 3, 2\sqrt{3} \sin \theta / 3)$,

$$\text{collapse time:} \quad t_0 \approx 1.90512... \quad \text{for } \theta = 4\pi/9. \quad (3.7)$$

Four-vortex case [7]

$$\text{combination of the strength: } (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) = (1, -(2 + \sqrt{3}), 1, -(2 + \sqrt{3})),$$

$$\text{initial positions: } (x_1, y_1) = ((\sqrt{3} + 1)/\sqrt{2}, 0),$$

$$(x_2, y_2) = (\cos \theta, \sin \theta),$$

$$(x_3, y_3) = (-(\sqrt{3} + 1)/\sqrt{2}, 0),$$

$$(x_4, y_4) = (-\cos \theta, -\sin \theta),$$

$$\text{collapse time:} \quad t_0 \approx 1.44337... \quad \text{for } \theta = \pi/3. \quad (3.8)$$

Collapse solutions always appear as one-parameter family of θ , and a different θ gives a different collapse time. Those families include the rigid rotation solutions for which t_0 becomes infinite. In Fig. 4, we present trajectories of three and four vortices for the above conditions.

Though for $N \geq 4$, (2.1) is nonintegrable, in general [12], the collapse solution seems well-behaved as long as the numerical calculation is concerned.

§ 4 Parametric motion towards real collapse

We perturb the initial configurations of the collapse solutions mentioned in the previous section as Fig. 5 shows. It is expected that as the perturbation ϵ goes to 0, singularities approach real points denoted as the collapse times in (3.7) and (3.8). In Fig. 6, we show the result of the numerical hunting of singularities in the upper half of the complex time-plane according to changes of ϵ . (We have a symmetric result in the lower half plane because of reality of the solutions on the real time axis.) Singularities move on a line which is asymptotically tangent to the real time axis. We can obtain a quantitative information on this parametric motion analogously with the previous discussion by applying a modified linear stability theory to the collapse solutions. This argument is based on a fact that self-similar motions remain fixed points if we observe them in moving coordinates with the same time evolution with the system.

To derive a linearized equation for a perturbation on the moving coordinates $f(t)$ (3.4), we substitute

$$z_j = f(t) (k_j + \epsilon w_j(t)) \quad (4.1)$$

into (2.1) where k_j is the solution for the unperturbed equation, and ϵ is a small parameter.

Expanding the right hand side at order ϵ , substituting (3.2) and (3.4) we obtain

$$(2At+1) \frac{dw_j}{dt} = -i \sum_{m (\neq j)} \frac{\Gamma_m (w_j^* - w_m^*)}{(k_j^* - k_m^*)^2} - C w_j \quad (j = 1, 2, \dots, N) \quad (4.2)$$

(note that zeroth order terms cancel in virtue of (3.2) and (3.3)). Introducing a scaled time τ :

$$\tau = \frac{1}{2A} \log (2At + 1) \quad (A \neq 0), \quad (4.3)$$

the coefficient of the left hand side of (4.2) is absorbed and we have a linearized equation for w_j with τ as the time variable,

$$\frac{dw_j}{d\tau} = -i \sum_{m (\neq j)} \frac{\Gamma_m (w_j^* - w_m^*)}{(k_j^* - k_m^*)^2} - C w_j \quad (j = 1, 2, \dots, N). \quad (4.4)$$

Coupled with its complex conjugate, (4.4) is expressed as a $(2N) \times (2N)$ off-diagonal block matrix (N : number of vortices) equation for a column vector of w_j 's and w_j^* 's.

If we assume that the same scaling law is applicable to this singular case, the relation (2.3) is expected in the regularized coordinates (τ -frame), which is pulled back to the original frame by substituting (4.3),

$$\begin{aligned} \epsilon &\approx \exp \left[-\frac{\lambda}{2A} \log (2At_0 + 1) \right], \\ &= (2At_0 + 1)^{-\frac{\lambda}{2A}}, \end{aligned} \quad (4.3)$$

where λ is one of the eigenvalues of the $(2N \times 2N)$ matrix.

Here we show results of calculation of eigenvalues for the three- and four-vortex cases (3.7) and (3.8). (For the three-vortex cases, eigenvalues are obtainable analytically [9].)

Three-vortex case:

$$\begin{aligned} 0.26245... + i \ 3.02314... & \quad (= - C) \\ 0.26245... - i \ 3.02314... & \quad (= - C^*) \\ 0.52490... \text{ (double)} & \quad (= - (C+C^*)) \\ 0 & \quad \text{(double)} \end{aligned}$$

Four-vortex case:

$$\begin{aligned} 0.34641... + i \ 1.55884... & \quad (= - C) \\ 0.34641... - i \ 1.55884... & \quad (= - C^*) \\ 0.69282... \text{ (double)} & \quad (= - (C+C^*)) \\ 0 & \quad \text{(double)} \\ 0.34641... + i \ 1.21244... & \\ 0.34641... - i \ 1.21244... & \end{aligned}$$

where C is a complex constant determined by (3.3) and "double" denotes the multiplicity. It can be proven that for any number of vortex cases, 0 , $-C$, $-C^*$ and $-(C+C^*)$ become eigenvalues of the $(2N) \times (2N)$ matrix [13].

Taking the fact that the approaching routes of singularities are tangent to the real time axis for the three- and four-vortex cases into account, we predict that the real eigenvalues should play roles of the scaling exponent λ . Then the scaling (4.3) should hold for the real displacements ϵ and the real parts t_r of the position of the singularity asymptotically. In Fig. 7, we show the plots with $-\log \epsilon$ and $(\log(2At_r + 1))/2A$ as abscissa and ordinate, respectively for the both cases. Comparing with a straight lines of slope of $1/\lambda$, we observe that the scaling law (4.3) is satisfied asymptotically.

§ 5 Conclusion

In the preceding sections we have presented an asymptotic scaling law between a departure of initial conditions (from a non-singular or a collapse configuration) and the position of the nearest singularity to the origin in the complex time plane, where the scaling exponent is one of the eigenvalues of the linearized stability problems with or without some regularization process. In other words, the route of parametric motion of complex singularity towards the real collapse has an intimate relation between the stability of the singular solution and the type of the singularity.

It is surprising that the scaling holds for a case in nonintegrable regime. The reason is entirely an open problem. It is reported, however, that the pathological logarithmic accumulation of complex time singularities is an evidence of chaos [14], and according to this criterion, the above case has only a few singularities, and chaotic behaviors might be avoided.

For the cases we treated in this paper, only real eigenvalues take charge in the scaling property. If there are cases in which complex eigenvalues play a role of the scaling exponent, we can expect that asymptotes of singularities make certain angles with the real axis and that the angle is a function of the ratio of the real and the imaginary parts of the eigenvalues.

Seeking families of equations which can be analyzed with the same methods discussed in the present paper would be a future problem. Is Hamiltonian structure necessary? Do zero eigenvalues have a relation with the scaling property? These are some of the interesting questions to be addressed. [13]

Even though exact collapse conditions are point-like (measure zero), one can somehow define a domain around those points in the initial- or parametrical-condition space. For conditions in that domain, solutions become nearly singular and collapse eventually occurs. We predict the size of the domain is related with the stability problem of the real singularity, and thus the parametric motion towards the real point.

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References

- [1] See for example R. McGehee, Dyn. Syst., Theor. Appl., Battelle Seattle 1974 Renc., Lect. Notes Phys. **38**, 550, and references therein.
- [2] C.N. Yang and T.D. Lee, Phys. Rev., **87** (1952) 404, 410.
- [3] Y. Kimura, I. Zawadzki, and H. Aref, Phys. Fluids **A2** (1990) 214.
- [4] Y. F. Chang and G. Corliss, J. Inst. Math. Appl. **25** (1980) 349.
- [5] G.A. Baker, Jr., Essentials of Padé Approximants (Academic Press, New York, 1975).
- [6] D.S. Gaunt and A.J. Guttmann, in Phase Transitions and Critical Phenomena, eds. C. Domb, M.S. Green, (Academic Press, London) Vol. 3, 181
- [7] E.A. Novikov, Sov. Phys. JETP **50** (1979) 297.
- [8] H. Aref, Phys. Fluids **22** (1979) 393.
- [9] Y. Kimura Physica D (in press)
- [10] Y. Kimura, J. Phys. Soc. Jpn. **56** (1987) 2024.,
- [11] K.A. O'neil, Trans. Am. Math. Soc. **302**(1987) 383.
- [12] H. Aref, Ann. Rev. Fluid Mech. **15** (1983) 345
- [13] H. Ohmoto and Y. Shikata, private communication
- [14] J.D. Fournier, G. Levine, and M. Tabor, J. Phys. A: Math. Gen. **21** (1988) 33.;
T. Dombre, U. Frisch, J.M. Greene, M. Hénon, A. Mehr, and A. M. Soward, J. Fluid Mech. **167** (1986) 353; T. Bountis, V. Papageorgiou, and M. Bier, Physica D **24**, (1987) 209, A. Ramani, B. Grammaticos and T. Bountis, PHYSICS REPORTS **180** , No. 3 (1989) 159.

Figure captions

Fig. 1: As the initial configurations approach the equilateral triangle, the pair of complex time singularities move to infinity along either the real or the imaginary time axis.

Fig. 2: Plot of the y-component of the third vortex and R_c ; the real component of the position of complex time singularities. As the y-component approach $\sqrt{3}$ (the equilateral configuration) R_c becomes infinite (lateral sweeping)

Fig. 3: Log-linear plot of the result for the upper part of the perturbations ($\epsilon = y - \sqrt{3}$) and R_c in Fig. 2. The slope of the line is $4/3$.

Fig. 4: Trajectories of (a) three and (b) four vortices with initial conditions (a) (3.7) and (b) (3.8).

Fig. 5: The perturbations ϵ to the initial positions of (3.7) and (3.8).

Fig. 6: Parametric motion of singularities in the upper half of the complex time plane according to changes of the perturbation ϵ for (a) the three-vortex case and (b) the four-vortex case. As ϵ goes to zero singularities approach the collapse times (a) 1.9512 and (b) 1.44337. They are on lines asymptotically tangent to the real time axis.

Fig. 7: Plot of the real part of singularities and the disturbance ϵ for (a) the three-vortex case and (b) the four-vortex case. The abscissa is $-\log \epsilon$ and the ordinate is $\log(2At+1)/2A$. They approach lines of slopes $1/\lambda$, (a) 1/0.5249 and (b) 1/0.69282.

Fig. 2

Fig. 3

Fig 4(a)

Fig. 4(b)

Fig. 5

Fig - 6 (a)

Fig. 6(b)

Fig. 7 (a)

Fig. 7 (b)

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