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MOTION OF A MATERIAL WHOSE PARTICLE VELOCITY
IS A LINEAR FUNCTION OF THE SPATIAL POSITION

J. MICHAEL MCGLAUN

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MOTION OF A MATERIAL WHOSE PARTICLE VELOCITY
IS A LINEAR FUNCTION OF THE SPATIAL POSITION

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ABSTRACT

The equations of motion are derived for plane one-dimensional motion of a material whose particle velocity is a linear function of the spatial position. It is proved the mass density is uniform at all times if it is initially uniform. A closed form solution to the equations of motion is obtained for a special case. Sensitivity of the solution to various parameters is investigated. An equation for the efficiency of a power history is derived and discussed.

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1. Introduction

It is often desired to have a quick reasonably accurate method of determining the plane motion of two masses confining a high pressure propellant. Several investigations [1]-[6] have obtained approximate solutions by assuming the velocity is a linear function of the distance between the propellant's two moving boundaries. The resulting equations of motion can be written as nonlinear ordinary differential equations. These ordinary differential equations are solved in closed form or numerically.

In each of the reports [1]-[5] the material between the two moving boundaries is assumed to be a perfect gas at uniform pressure. In this derivation we do not assume a form for the specific internal energy or pressure, nor do we assume the pressure is uniform. A relationship between the velocity of the gas and the motion of the boundary is also assumed in the aforementioned reports, e.g., rigid body motion, fixed, growing tamper, etc. In this derivation the differential form of the equations of motion are independent of any relationship between the velocity of the propellant and the motion of the boundaries. Our two primary assumptions in this paper are that the material velocity of the propellant is a linear function of the distance between the propellant's two boundaries and that there is no heat transfer across the boundaries of the propellant.

From the initial assumptions, independent of the constitutive equations for the specific internal energy or pressure, we show the velocity profiles can be expressed as a linear function of the Eulerian coordinate system or the material particle's initial position. We also show the mass density is uniform at all times if and only if it is initially uniform and the velocity

is a linear function of the distance between the propellant's two moving boundaries.

The equations of motion are derived for plane motion. The pressure is proved to be a quadratic function of the material particle's initial position. We also discuss various relationships between the motion of the propellant and the motion of the boundary.

We introduce additional assumptions which enable us to obtain a closed form solution to the equations of motion. Using the closed form solution we investigate the sensitivity of the system to changes of flyer mass. We also develop an expression for the efficiency of an infinitesimal power pulse and the efficiency of a complete power history.

2. Description of Problem and Nomenclature

The physical components of the object we wish to investigate are displayed in figure 1. We desire to accelerate the flyer to a high velocity in a short period of time. This is done by rapidly depositing energy in the propellant. Two possible sources of energy are chemical (high explosives) and electrical (ohmic heating). The propellant lies between the flyer and tamper. The tamper's role is to confine the high pressure propellant. The propellant accelerates the flyer down the barrel. The barrel is the region of space above the flyer. Our problem begins at time t_0 . The origin of our coordinate system coincides with the propellant-tamper interface at t_0 .

We would like to know the velocity of the flyer-propellant and the tamper-propellant interfaces as a function of time. We wish to determine the sensitivity of the system to changes in flyer mass, flyer material or tamper material. We would also like to be able to compare, in a quantitative manner, the efficiency of different energy histories deposited in the propellant.

The problem stated is one-dimensional but is often used as an approximation to a two-dimensional problem, see [2] and [4].

Figure 1, Physical Components

3. Initial Assumptions, Governing Equations and General Results

Assumptions

- 3.1. The particle velocity of the propellant varies linearly with distance between the propellant-tamper and propellant-flyer interfaces.
- 3.2. The mass density of the propellant is initially uniform.
- 3.3. There is no transfer of heat between the propellant and the flyer or tamper.

Assumption 3.1 is often referred to as the Gurney assumption, see [1].
The following equations govern one-dimensional plane motion.

Conservation of momentum (zero body forces)

$$-\frac{d}{dt} \int_{r(Y,t)}^{r(X,t)} \rho(z,t) V(z,t) dz = P(r(X,t),t) - P(r(Y,t),t) \quad (3.1)$$

$$-\rho(x,t) \dot{V}(x,t) = \frac{\partial}{\partial x} P(x,t)$$

x, y each denote spatial position.

$V(x,t)$ denotes the velocity of the material particle occupying the spatial position x at time t .

P denotes the pressure and is positive in compression.

ρ is the mass density.

X, Y each denote the initial position of material particles and are used as material coordinates (Lagrangian coordinates).

$r(X,t)$ maps the material particle X onto its spatial position at time t .

A superimposed dot denotes material time differentiation.

Conservation of mass

$$\int_Y^X \rho(z, t_o) dz = \int_{r(Y, t)}^{r(X, t)} \rho(z, t) dz \quad (3.2)$$

$$\rho(r(X, t), t) \frac{\partial r(X, t)}{\partial X} = \rho(X, t_o)$$

Equation (3.3) relates the velocity of the material particle to the mapping $r(X, t)$.

$$V(r(X, t), t) = \dot{r}(X, t) = \frac{\partial}{\partial t} r(X, t) \quad (3.3)$$

Conservation of energy

Using assumption 3.3, the equation expressing the balance of energy in the propellant is

$$\begin{aligned} \frac{1}{2} \int_{r_T(t)}^{r_F(t)} \rho(x, t) V^2(x, t) dx + W_T(t) + W_F(t) \\ + \int_{r_T(t)}^{r_F(t)} \rho(x, t) \mathcal{E}(x, t) dx = E_p(t) + E_o. \end{aligned} \quad (3.4)$$

$r_F(t)$ is the position of the propellant-flyer interface at time t .

$r_T(t)$ is the position of the propellant-tamper interface at time t .

$W_T(t)$ is the work done on the tamper up to time t .

$W_F(t)$ is the work done on the flyer up to time t .

$\mathcal{E}(x, t)$ is the specific internal energy of the propellant at (x, t) .

$E_p(t)$ is the energy deposited in the propellant up to time t .

E_o is the initial energy of the propellant.

$$E_o = \int_{r_T(t_o)}^{r_F(t_o)} \{ \rho(x, t_o) \mathcal{E}(x, t_o) + \frac{1}{2} \rho(x, t_o) v^2(x, t_o) \} dx \quad (3.5)$$

Assumption 3.1 may be expressed mathematically as

$$V(r(X, t), t) - V_T(t) = \frac{r(X, t) - r_T(t)}{r_F(t) - r_T(t)} (V_F(t) - V_T(t)) \quad (3.6)$$

$$t \geq t_o, \quad X \in [X_T, X_F] .$$

$V_F(t)$ is the velocity of the propellant-flyer interface at time t .

$V_T(t)$ is the velocity of the propellant-tamper interface at time t .

X_F is the initial position of the propellant-flyer interface.

$X_T(=0)$ is the initial position of the propellant-tamper interface.

Theorem 3.1. Let $r_F(t)$ and $r_T(t)$ be given. Assumption 3.1 is satisfied if and only if

$$r(X, t) - r_T(t) = \frac{X - X_F}{X_F - X_T} (r_F(t) - r_T(t)). \quad (3.7)$$

Proof. Assume assumption 3.1 is valid. Fixing X and substituting (3.3) into (3.6) we see the resulting equation is an ordinary differential equation. The unique solution satisfying the initial condition

$$r(X, t_o) = X$$

is (3.7).

Assume (3.7) is valid.

$$\dot{r} - \dot{r}_T = \frac{X - X_T}{X_F - X_T} (\dot{r}_F - \dot{r}_T) = \frac{r - r_T}{r_F - r_T} (\dot{r}_F - \dot{r}_T) \quad \blacksquare (3.8)$$

Theorem 3.1 proves the velocity is a linear function of the spatial coordinates if and only if it is a linear function of the material particle's initial position.

Theorem 3.2. Assumption 3.1 is satisfied if and only if the mass density of a material particle, at any given time, divided by the initial mass density of the same material particle is independent of the material particle.

Proof. Assume assumption 3.1 is valid. Using $(3.2)_2$ and (3.7)

$$\rho(r(X,t),t) = \frac{\partial X}{\partial r} \rho(X,t_0) = \frac{X_F - X_T}{r_F - r_T} \rho(X,t_0) . \quad (3.9)$$

Assume the mass density of a material particle, at any given time, divided by the initial mass density of the same material particle is independent of the material particle.

$$\frac{\rho(r(X,t),t)}{\rho(X,t_0)} = c(t)$$

By equation $(3.2)_2$

$$\frac{\partial r}{\partial X} = \frac{1}{c(t)} \quad (3.10)$$

Integrate (3.10) from X_T to X .

$$r(X,t) - r_T(t) = \frac{1}{c(t)} (X - X_T) \quad (3.11)$$

Let $X = X_F$ in (3.11). Solve the resulting eqn. for $c(t)$ and substitute into (3.11)

$$r(X,t) - r_T(t) = \frac{X - X_T}{X_F - X_T} (r_F(t) - r_T(t)) \quad \blacksquare$$

Theorem 3.2 proves the mass density at any time is equal to the initial mass density multiplied by a quantity which is dependent only on the time.

Corollary 3.2.1. Assumptions 3.1 and 3.2 are satisfied if and only if the mass density is uniform across the propellant at all times.

Proof. This follows from theorem 3.2. ■

In this section we stated our initial assumptions and governing equations. We were able to derive kinematic relationships based upon the first two initial assumptions and the conservation of mass. We did not introduce any constitutive equations for the specific internal energy or pressure so these results are valid for all materials. Theorem 3.1 proved the velocity is a linear function of the spatial coordinates if and only if it is a linear function of the material particle's initial position. Theorem 3.2 proved the mass density at any time is equal to the initial mass density multiplied by a quantity which is independent of the spatial position. The corollary to theorem 3.2 proves the mass density is uniform across the propellant at all times if and only if it is initially uniform and assumption 3.1 is satisfied.

4. Derivation of Equations of Motion

Assumptions:

- 4.1. The energy deposited in the propellant, $E_p(t)$, is given.
- 4.2. The specific internal energy, $\mathcal{E}$, and pressure, P , are functions of mass density, ρ , and temperature, T .

Using (3.7) we can evaluate the first integral appearing in (3.4).

Using the change of variables theorem, (3.2)₂, (3.3), and (3.7)

$$\frac{1}{2} \int_{r_T(t)}^{r_F(t)} \rho(x,t) V^2(x,t) dx = m_p [V_F^2(t) + V_F(t)V_T(t) + V_T^2(t)]/6 \quad (4.1)$$

m_p denotes the mass of the propellant.

$X_T = 0$ Recall the origin of our coordinate system coincides with the initial position of the propellant-tamper interface.

Substituting (4.1) into (3.4) yields

$$\begin{aligned} \int_{r_T(t)}^{r_F(t)} \rho(z,t) \mathcal{E}(z,t) dz &= E_p(t) + E_o - W_T(t) - W_F(t) \\ &- m_p [V_F^2(t) + V_F(t)V_T(t) + V_T^2(t)]/6 \end{aligned} \quad (4.2)$$

Thus far we have not used equation (3.1). Using assumption 3.1 we find,

Theorem 4.1. The pressure is a quadratic function of the initial position of the material particle.

$$P(r(X,t),t) = m_p \left\{ (\dot{V}_F(t) - \dot{V}_T(t)) \left(\frac{X_F^2 - X^2}{2X_F^2} \right) + \dot{V}_T(t) \left(\frac{X_F - X}{X_F} \right) \right\} + P(r_F,t) \quad (4.3)$$

Proof. Using the change of variables theorem, (3.2), (3.3), and (3.7)

$$\int_{r(X,t)}^{r_F(t)} \rho(z,t) \dot{V}(x,t) dz = m_p \left\{ (\dot{V}_F(t) - \dot{V}_T(t)) \left(\frac{X_F^2 - X^2}{2X_F^2} \right) + \dot{V}_T(t) \left(\frac{X_F - X}{X_F} \right) \right\} \quad (4.4)$$

Substituting (4.4) into (3.1) proves (4.3) ■

In many studies the pressure is assumed to be uniform. If this assumption is made and the propellant is accelerating, momentum cannot be conserved. This is evident from equation (3.1)₂. If the pressure is uniform its spatial derivative is zero and the acceleration must be zero.

Corollary. $P(r_T,t)$ and $P(r_F,t)$ are related by

$$P(r_T,t) - P(r_F,t) = m_p (\dot{V}_F(t) + \dot{V}_T(t))/2 \quad (4.5)$$

Proof. Set $X = X_T = 0$ in equation (4.3). ■

The work and pressure at the boundary are related by

$$\frac{d}{dt} W_T(t) = V_T(t) P(r_T(t),t) \quad W_T(t_0) = 0 \quad (4.6)$$

$$\frac{d}{dt} W_F(t) = V_F(t) P(r_F(t),t) \quad W_F(t_0) = 0$$

Using assumptions 4.1 and 4.2, we have ten equations in twelve unknowns. The ten equations are (3.12), (4.2), (4.3), (4.5), (4.6), and four ordinary differential equations relating $r_T(t)$, $V_T(t)$, $\dot{V}_T(t)$, $r_F(t)$, $V_F(t)$, and $\dot{V}_F(t)$. The twelve unknowns are $W_T(t)$, $W_F(t)$, $P(r_T, t)$, $P(r_F, t)$, $T(x, t)$, $\rho(x, t)$, $r_T(t)$, $V_T(t)$, $\dot{V}_T(t)$, $r_F(t)$, $V_F(t)$ and $\dot{V}_F(t)$. All the unknowns are functions of time except the temperature which is a function of both time and spatial position.

If we introduce boundary conditions specifying $P(r_T, t)$ and $P(r_F, t)$ in terms of the twelve unknowns, we may substitute these equations into (4.2), (4.3), (4.5) and (4.6). We then have twelve equations in twelve unknowns.

5. Boundary Conditions

There are numerous choices for the boundary conditions. Below we present some of them.

Rigid Body Motion of Flyer or Tamper

We may assume the flyer or tamper moves as a rigid body. If m_F (respectively m_T) is the mass of the flyer (respectively tamper) then

$$P(r_F, t) = m_F \dot{V}_F(t)$$

$$P(r_T, t) = m_T \dot{V}_T(t)$$

$$W_F(t) = \frac{1}{2} m_F [V_F^2(t) - V_F^2(t_0)]$$

$$W_T(t) = \frac{1}{2} m_T [V_T^2(t) - V_T^2(t_0)] \quad . \quad (5.1)$$

Stress Waves

We may assume stress waves propagate into the flyer and tamper. We assume these stress waves are functions of $t, r_F, r_T, V_F, V_T, \dot{V}_F, \dot{V}_T, \rho$ and T .

$$P(r_F, t) = S_F(t, r_F, r_T, V_F, V_T, \dot{V}_F, \dot{V}_T, \rho, T)$$

$$P(r_T, t) = S_T(t, r_F, r_T, V_F, V_T, \dot{V}_F, \dot{V}_T, \rho, T) \quad (5.2)$$

Fixed Tamper

By setting $V_T(t) = 0$ we have a **fixed** tamper. As a result of (4.6) we have $W_T(t) = 0$.

Combination of Rigid Body and Stress Waves

We may assume a portion of the flyer, $X_F \leq X \leq X_R$, moves as a rigid body and stress waves propagate into another portion $X_R < X \leq X_S$.

For this case

$$P(r_F, t) = m_R \dot{V}_F(t) + S_F(t, r_F, r_T, V_F, V_T, \dot{V}_F, \dot{V}_T, \rho, T) \quad (5.3)$$

Where m_R is the mass of the flyer which moves as a rigid body.

Growing Tamper

We assume the velocity is uniform across the tamper, but the mass increases as a function of time. The mass of the tamper is taken to be the product of the initial mass density multiplied by the time integral of the tamper material's speed of sound.

$$P(r_T, t) = \dot{V}_T(t) \rho_T(t) C(t - t_0) \quad (5.4)$$

Where C is the tamper material's speed of sound.

6. Closed Form Solution for a Special Case

The equations of motion derived in section 4 are quite general. They are so general, it would be extremely difficult to derive a closed form solution to the equations of motion. A closed form solution can be quite useful for determining the sensitivity of the solution to system parameters. In this section we impose additional assumptions enabling us to derive a closed form solution to the resulting equations of motion.

Assumptions:

6.1. Assumptions 3.1, 3.2, 3.3, and 4.1 are valid. We do not assume the validity of the remaining assumptions made in the preceding sections.

6.2. The specific internal energy of the propellant may be written as

$$\mathcal{E}(x,t) = P(x,t)/(\rho(x,t)(\gamma - 1)) + H(\rho(x,t)) \quad (6.1)$$

where γ is the adiabatic gas constant, $\gamma > 1$. $H(\cdot)$ is a general function.

6.3. The tamper is fixed.

$$r_T(t) = 0 \quad (6.2)$$

6.4. The flyer moves as a rigid body.

$$P(r_F(t), t) = m_F \dot{V}_F(t) \quad (6.3)$$

$$W_F(t) = \frac{1}{2} m_F \left(V_F^2(t) - V_F^2(t_0) \right)$$

6.5. $r_F(t)$ is a monotone increasing function of time which is bounded by $R < \infty$. The flyer velocity as a function of spatial position, $V_F(x)$, uniformly satisfies a Lipschitz condition.

Using (6.2), equations (4.1) and (4.3) reduce to

$$\frac{1}{2} \int_{r_T(t)}^{r_F(t)} \rho(x,t) V^2(x,t) dx = m_p V_F^2(t)/6$$

$$P(r(X,t),t) = m_p \dot{V}_F(t) \left(\frac{X_F^2 - X^2}{2X_F^2} \right) + P(r_F,t) . \quad (6.4)$$

Using Corollary 3.2.1, (6.1), (6.3) and (6.4) we find

$$\int_{r_T(t)}^{r_F(t)} \rho(x,t) \mathcal{E}(x,t) dx = \frac{1}{\gamma - 1} \left\{ \frac{m_p}{3} + m_F \right\} \dot{V}_F(t) r_F(t)$$

$$+ m_p H \left(\frac{m_p}{r_F(t)} \right) . \quad (6.5)$$

Substituting (6.3), (6.4), and (6.5) into (3.4) yields

$$\frac{2}{\gamma - 1} \dot{V}_F(t) r_F(t) + V_F^2(t) = 2 \left[E_p(t) + E_o - m_p H \left(\frac{m_p}{r_F(t)} \right) \right] / m_p \left(\frac{1}{3} + \frac{m_F}{m_p} \right) . \quad (6.6)$$

$$E_o = \frac{1}{\gamma - 1} \int_0^{X_F} P(s,t_o) ds + m_p H \left(\frac{m_p}{X_F} \right) + \frac{1}{2} (m_F + m_p/3) V_F^2(t_o)$$

In the following we use two descriptions of flyer motion, flyer velocity vs. position, and flyer position (and velocity) vs. time. The following lemma proves the equivalence of the two descriptions.

Lemma 6.1 Given flyer velocity as a function of position, there exists a unique function specifying flyer position vs. time whose derivative is equal to the flyer velocity. Given flyer position as a function of time there exists a function specifying flyer velocity vs. position.

Proof: Given $\hat{V}_F(x)$, $x \in (X_F, R)$ we wish to find a function $r_F(t)$, $t \in (t_0, \tau)$ such that

$$\hat{V}_F(r_F(t)) = \frac{d}{dt} r_F(t) .$$

The existence and uniqueness of $r_F(t)$ follow from the existence and uniqueness of the solution of an ordinary differential equation and assumption 6.5.

Given $r_F(t)$, $t \in (t_0, \infty)$ we wish to find a function $\hat{V}_F(x)$, $x \in (X_F, R)$ such that

$$\hat{V}_F(r_F(t)) = \frac{d}{dt} r_F(t) .$$

Using assumption 6.5 we may invert $r_F(t)$ yielding

$$t = g(r_F(t)) . \quad (6.7)$$

The desired function is

$$\hat{V}_F(x) = V_F(g(x)) = \frac{d}{dt} r_F(g(x)) . \quad \blacksquare (6.8)$$

Using (6.8) and the chain rule

$$\begin{aligned} \frac{d}{dt} V_F(t) &= \frac{d}{dx} (\hat{V}_F(x)) \bigg|_{x = r_F(t)} \frac{dr_F(t)}{dt} \\ &= \frac{d}{dx} (\hat{V}_F(x)) \hat{V}_F(x) , \quad x = r_F(t) \\ &= \frac{1}{2} \frac{d}{dx} (\hat{V}_F^2(x)) , \quad x = r_F(t) \end{aligned} \quad (6.9)$$

Substitute (6.7), (6.8), and (6.9) into (6.6). The resulting equation in a nonlinear differential equation with independent variable $x = r_F(t)$ and dependent variable $\hat{V}_F^2$.

$$\begin{aligned} \frac{1}{\gamma - 1} x \frac{d}{dx} (\hat{V}_F^2(x)) + \hat{V}_F^2(x) &= 2[E_p(g(x)) + E_o \\ &- m_p H(\frac{m_p}{x})] / m_p (\frac{1}{3} + \frac{m_F}{m_p}) , \\ x &= r_F(t) \end{aligned} \quad (6.10)$$

A solution of (6.10) with initial condition $\hat{V}_F^2(r_F(t_o)) = V_o^2$ is

$$\begin{aligned} \hat{V}_F^2(r_F(t)) &= \left(\frac{1}{r_F(t)} \right)^{\gamma-1} \left[V_o^2 X_F^{\gamma-1} \right. \\ &\quad \left. + \frac{2(\gamma - 1)}{m_p \left(\frac{1}{3} + \frac{m_F}{m_p} \right)} \int_{X_F}^{r_F(t)} \left\{ E_p(g(s)) + E_o - m_p H \left(\frac{m_p}{s} \right) \right\} s^{\gamma-2} ds \right] \end{aligned} \quad (6.11)$$

The solution given by (6.11) is unique if $E_p(g(s)) - m_p H \left(\frac{m_p}{s} \right)$ satisfies a Lipschitz condition uniformly for $s \in [X_F, R]$. In the following we consider only solutions of the form (6.11). When the energy is instantaneously deposited, i.e. E_p is a constant, equation (6.11) simplifies to

$$\begin{aligned} \hat{V}_F^2(r_F(t)) &= \left(\frac{1}{r_F(t)} \right)^{\gamma-1} \left[\frac{2}{\left(\frac{1}{3} + \frac{m_F}{m_p} \right) m_p} \left\{ (E_p + E_o)(r_F(t))^{\gamma-1} - X_F^{\gamma-1} \right. \right. \\ &\quad \left. \left. - (\gamma - 1) \int_{X_F}^{r_F(t)} m_p H \left(\frac{m_p}{s} \right) s^{\gamma-2} ds \right\} + V_o^2 X_F^{\gamma-1} \right] . \end{aligned} \quad (6.12)$$

In the following we do not distinguish between $V_F(t)$ and $\hat{V}_F(r_F(t))$.

It should be clear from the context which function we mean.

Sensitivity of the Solution to the Energy History

Using (6.11) we investigate the relationship between the energy deposited in the propellant and the velocity of the flyer.

Proposition 6.1. Consider two systems which are identical except for the flyer masses, m_F and m_F^* , and energy histories, $E_p(t)$ and $E_p^*(t)$. The energy histories result in identical flyer velocity histories if and only if

$$\frac{E_p(t) + E_o - m_p H\left(\frac{m_p}{r_F(t)}\right)}{E_p^*(t) + E_o^* - m_p H\left(\frac{m_p}{r_F^*(t)}\right)} = \frac{\frac{1}{3} + \frac{m_F}{m_p}}{\frac{1}{3} + \frac{m_F^*}{m_p}}, \quad X_F \leq x \leq R. \quad (6.13)$$

Proof. Subtract (6.11) with flyer mass m_F from (6.11) with flyer mass m_F^* . Use x rather than $r_F(t)$ and $r_F^*(t)$ as the independent variable. Multiply the resulting equation by $x^{\gamma-1}/2(\gamma-1)$ yielding

$$\begin{aligned} & \left(\frac{1}{3} + \frac{m_F^*}{m_p}\right)^{-1} \int_{X_F}^x \left\{ E_p^*(g^*(s)) + E_o^* - m_p H\left(\frac{m_p}{s}\right) \right\} s^{\gamma-1} ds - \\ & - \left(\frac{1}{3} + \frac{m_F}{m_p}\right)^{-1} \int_{X_F}^x \left\{ E_p(g(s)) + E_o - m_p H\left(\frac{m_p}{s}\right) \right\} s^{\gamma-1} ds = 0. \end{aligned} \quad (6.14)$$

Differentiate (6.14) with respect to x and multiply the resulting equation by $x^{1-\gamma}$. Using the result of lemma 6.1 the proposition is proved. ■

It is interesting that the left hand side of (6.13) is a function of the time while the right hand side is a constant.

Using the results of proposition 6.1 we can define an equivalence class of systems which are identical except for their flyer masses and energy histories. Two systems, which are identical except for their flyer masses and energy histories are equivalent if and only if they satisfy (6.13). By proposition 6.1, the two systems are equivalent in the sense that their velocity histories are identical.

Proposition 6.2. Consider two energy histories $E_p(t)$ and $E_p^*(t)$. Assume, for a given system and set of initial conditions,

$$E_p(g(x)) \geq E_p^*(g(x)) \quad , \quad X_F \leq x < R . \quad (6.15)$$

Where $g(x)$ and $g^*(x)$ are defined as in equation (6.7). Then

$$V_F^2(x) \geq V_F^{*2}(x) \quad , \quad X_F \leq x < R . \quad (6.16)$$

where $V_F^2(r_F(t))$ ($V_F^{*2}(r_F^*(t))$, respectively) is the solution of (6.10) corresponding to $E_p(t)$ ($E_p^*(t)$, respectively).

Proof. Using (6.11)

$$V_F^2(x) - V_F^{*2}(x) = \left\{ \frac{2 \left(\frac{1}{x} \right)^{\gamma-1} (\gamma-1)}{\left(\frac{1}{3} + \frac{m_F}{m_p} \right) m_p} \int_{X_F}^x \left(E_p(g(s)) - E_p^*(g^*(s)) \right) s^{\gamma-2} ds \right\} . \quad (6.17)$$

Using (6.15) in (6.17) proves (6.16). ■

Corollary 6.2.1. For identical systems, identical initial velocities and energy histories that deposit the same amount of energy, the flyer velocity is a maximum if and only if the energy is deposited before the flyer begins to move.

Proof: If. With notation as in proposition 6.2, we have

$$E_P - E_P^* (g(x)) \geq 0 . \quad (6.18)$$

for all energy histories that deposit the same amount of energy. Substituting (6.18) in (6.17) proves the flyer velocity associated with the constant energy history is a maximum.

Only if. Using the notation of proposition 6.2., assume the starred and unstarred velocities are equal.

$$0 = \int_{X_F}^x \{ E_P(g(s)) - E_P^*(g^*(s)) \} s^{\gamma-2} ds \quad (6.19)$$

Equation (6.19) is true for all x . Differentiating (6.19) with respect to x proves

$$E_P(g(s)) = E_P^*(g^*(s)) . \quad (6.20)$$

Using the fact that the maximum flyer velocity is achieved when E_P is a constant, the corollary is proved. ■

Corollary 6.2.2. If we assume

$$\lim_{t \rightarrow \infty} r_F(t) \rightarrow \infty$$

$$\lim_{t \rightarrow \infty} \frac{(\gamma-1)}{r_F(t)} \int_{X_F}^{r_F(t)} m_P^H \left(\frac{m_P}{s} \right) \left(\frac{s}{r_F(t)} \right)^{\gamma-2} ds \rightarrow C \quad (6.21)$$

then for any energy history $E(t)$, $t_0 \leq t \leq \infty$

$$V_F^2(\infty) \leq \frac{2}{\left(\frac{1}{3} + \frac{m_F}{m_p}\right) m_p} \left\{ E_p(\infty) + E_0 - C \right\} \quad (6.22)$$

Proof. This follows from corollary 6.2.1 and (6.12). ■

Equation (6.22) places an upper bound on the velocity of a flyer. If $\frac{m_F}{m_p} \gg 1/3$ the maximum terminal velocity is controlled by the ratio of m_F to m_p . (In many practical situations this is the case.) If $\frac{m_F}{m_p} \ll 1/3$ the maximum terminal velocity is controlled by the $1/3$ and we have

$$V_F^2(\infty) \leq 6(E_p(\infty) + E_0 - C)/m_p .$$

In many cases $|E_0 - C| \ll E_p(\infty)$ and can be neglected in (6.22). Note that E_0 and C depend only on the initial conditions, the material and $\lim_{t \rightarrow \infty} r_F(t)$. E_0 and C depend on $E_p(t)$ only through the limiting value of $r_F(t)$.

Local and Global Energy Efficiency

Equation (6.11) shows the flyer's velocity depends on the energy history rather than just the total energy deposited. We show in corollary 6.2.1 the maximum velocity attainable for a fixed amount of energy occurs when the energy is deposited before the flyer begins to move. This is a limiting case and we would like to have a quantitative measure of the efficiency of energy deposition after the flyer begins to move.

In the following we compare energy histories that deposit the same amount of energy by the time the flyers reach a specified position down the barrel. Given two energy histories $E(t)$ and $E^*(t)$, and a specified position down the barrel l , we say $E(t)$ is more efficient than $E^*(t)$ if and only if the flyer associated with $E(t)$ is moving faster than the flyer associated with $E^*(t)$ at l . Therefore, corollary 6.2.1 describes the most efficient way of depositing energy.

In the following we define two efficiencies, a local and global efficiency. The local efficiency compares infinitesimal power pulses deposited when the flyer is at different positions down the barrel. The global efficiency compares entire power histories.

Consider a power pulse of arbitrarily short duration, $\delta t > 0$,

$$P(\tau) = \begin{cases} P^* & , \quad t^* \leq \tau \leq t^* + \delta t \\ 0 & , \quad \tau < t^*, \tau > t^* + \delta t \end{cases} \quad (6.23)$$

From corollary 6.2.1 we know the velocity of the flyer is maximized when $t^* = t_0$.

Proposition 6.3. Consider two vanishingly short power pulses of equal magnitude and of the form (6.23). Let one power pulse be deposited when the

propellant-flyer interface is at x_1 and the other when the propellant-flyer interface is at x_2 . The flyer receiving the energy deposited at x_1 will have the greater velocity at any point $x_3 \geq \max(x_1, x_2)$ if and only if

$$x_1 < x_2 . \quad (6.24)$$

Proof: Let $V_F^2(r_F(t))$ and $\bar{V}_F^2(\bar{r}_F(t))$ be the solutions of (6.10) corresponding to the power pulses

$$P(\tau) = \begin{cases} P & , \quad t^* \leq \tau \leq t^* + \delta t \\ 0 & , \quad \tau < t^*, \tau > t^* + \delta t . \end{cases} \quad (6.25)$$

$$\bar{P}(t) = \begin{cases} P & , \quad \bar{t}^* \leq \tau \leq \bar{t}^* + \delta t \\ 0 & , \quad \tau < \bar{t}^*, \tau > \bar{t}^* + \delta t , \end{cases}$$

respectively.

Let t^* ($\bar{t}^*$) be the time when the unbarred (barred) flyer reaches x_1 (x_2).

Consider the following function

$$e(x_1, x_2, x_3) = \lim_{\delta t \rightarrow 0} \frac{\int_{X_F}^{x_3} \left(\int_{t_o}^{\bar{g}(s)} \bar{P} d\tau \right) s^{\gamma-2} ds}{\int_{X_F}^{x_3} \left(\int_{t_o}^{g(s)} P d\tau \right) s^{\gamma-2} ds} . \quad (6.26)$$

From (6.11) we see $V_F^2(x_3) > \bar{V}_F^2(x_3)$ if and only if $e(x_1, x_2, x_3) < 1$. Using (6.25) and (6.27) we can show (6.28).

$$\int_{X_F}^{x_3} \int_{t_0}^{g(s)} (\bullet) d\tau ds = \int_{t_0}^{g(x_3)} \int_{r(\tau)}^{x_3} (\bullet) ds d\tau \quad (6.27)$$

$$e(x_1, x_2, x_3) = \frac{x_3^{\gamma-1} - x_2^{\gamma-1}}{x_3^{\gamma-1} - x_1^{\gamma-1}} \quad (6.28)$$

$e(x_1, x_2, x_3)$ is less than one if and only if $x_1 < x_2$. ■

The proof of proposition 6.3 may be constructed in the same manner as the proof of proposition 6.2.

Combining the results of propositions 6.2 and 6.3 we may quantitatively evaluate the efficiency of an infinitesimal power pulse by comparing it with an identical power pulse input at the start of the problem. We introduce one additional parameter in our definition, the barrel length. Often we are interested in the velocity at only one position, b , the barrel length.

Local Efficiency of a Power Pulse. Using the result of proposition 6.3 we define the local efficiency of an infinitesimal power pulse deposited at $x \geq X_F$ as

$$E(x, b) = \frac{\left(\frac{b}{X_F}\right)^{\gamma-1} - \left(\frac{x}{X_F}\right)^{\gamma-1}}{\left(\frac{b}{X_F}\right)^{\gamma-1} - 1} \quad (6.29)$$

where b is the barrel length. Note that $E(x, b)$ is a monotone decreasing function of x between X_F and b with value one at X_F and zero at b . From proposition 6.3 and the definition of the local efficiency we see an

infinitesimal power pulse input at x_1 will result in a higher velocity than an identical pulse input at x_2 if and only if

$$E(x_1, b) > E(x_2, b) \quad , \quad b > \max(x_1, x_2) \quad . \quad (6.30)$$

Using the results of proposition 6.3 we know that the earlier we deposit a power pulse the more efficiently the energy will accelerate the flyer. However, the local efficiency compares only infinitesimal power pulses. We would like to compare complete power histories. Below we derive a global efficiency which enables us to quantitatively compare the efficiency of entire power histories.

Proposition 6.4. Let two power histories $P(t)$ and $P^*(t)$ be given. Let $V_F(t)$ and $V_F^*(t)$ be the associated flyer velocity histories. For identical initial conditions and identical systems

$$V_F(b) > V_F^*(b) \quad (6.31)$$

if and only if

$$\int_{t_0}^{g(b)} P(\tau) E(r_F(\tau), b) d\tau > \int_{t_0}^{g^*(b)} P^*(\tau) E(r_F^*(\tau), b) d\tau \quad . \quad (6.32)$$

Where $E(x, b)$ is the local efficiency for the finite barrel length b , refer (6.29).

Proof. From (6.11) we see (6.31) is satisfied if and only if

$$\int_{X_F}^b \left(\int_{t_o}^{g(s)} P(\tau) d\tau \right) s^{\gamma-2} ds > \int_{X_F}^b \left(\int_{t_o}^{g^*(s)} P^*(\tau) d\tau \right) s^{\gamma-2} ds. \quad (6.33)$$

Using (6.7), (6.27) and (6.29) we have

$$\begin{aligned} & \int_{t_o}^{g(b)} P(\tau) E(r_F(\tau), b) d\tau = \\ & = \frac{\gamma - 1}{b^{\gamma-1} - X_F^{\gamma-1}} \int_{X_F}^b \left(\int_{t_o}^{g(s)} P(\tau) d\tau \right) s^{\gamma-2} ds. \end{aligned} \quad (6.34)$$

Inequality (6.31) is satisfied if and only if (6.33) is satisfied. And by (6.34) inequality (6.33) is satisfied if and only if inequality (6.32) is satisfied. ■

Proposition 6.5, below, establishes a correspondence between velocities associated with an arbitrary power history and an instantaneously deposited power history. This connection enables us to define a global energy efficiency.

Proposition 6.5. Let a power history, $P(t)$, initial flyer velocity, V_o , and barrel length, b , be given. If we deposit an amount of energy equal to

$$F(P, b) = \int_{t_o}^{g(b)} P(\tau) d\tau \quad (6.35)$$

at the start of the problem, the flyer velocity at b is identical to the flyer velocity at b associated with the power history $P(t)$, $t \in [t_o, g(b)]$.

Where

$$F(P,b) = \frac{\int_{t_o}^{g(b)} P(\tau) E(r(\tau),b) d\tau}{\int_{t_o}^{g(b)} P(\tau) d\tau} \quad (6.36)$$

$E(x,b)$ is the local efficiency of a power pulse, see (6.29).

$g(x)$, see Eqn. (6.7), is associated with the temporarily distributed power history rather than the instantaneously deposited energy.

Proof: Using (6.36), (6.27) and (6.29)

$$\left(\frac{b^{\gamma-1} - X_F^{\gamma-1}}{\gamma - 1} \right) F(P,b) \int_{t_o}^{g(b)} P(\tau) d\tau = \quad (6.37)$$

$$= \int_{X_F}^b \left(\int_{t_o}^{g(s)} P(\tau) d\tau \right) s^{\gamma-2} ds .$$

Substituting (6.37) into (6.11) yields (6.12) with E_p replaced by

$$F(P,b) \int_{t_o}^{g(b)} P(\tau) d\tau \quad \blacksquare \quad (6.38)$$

Global Efficiency of a Power History. For a given system, barrel length, b , and power history $P(t)$, we define $F(P,b)$, Eqn. (6.36), as the global efficiency of the power history. Note that the value of $F(P,b)$ lies between zero and one.

It assumes the value one in the limiting case of energy deposition before the flyer begins to move. From corollary 6.2.1. we know this is the only case when the efficiency is equal to one. From proposition 6.4, $V_F(b) > V_F^*(b)$ if and only if $F(P,b) > F(P^*,b)$ when

$$\int_{t_0}^{g(b)} P(\tau) d\tau = \int_{t_0}^{g^*(b)} P^*(\tau) d\tau .$$

The following corollary is closely related to proposition 6.2.

Corollary 6.5.1. For any given power history $P(t)$ and barrel length, b , we have

$$F(P,b) \int_{t_0}^{g(b)} P(\tau) d\tau \leq \int_{t_0}^{g(b)} P(\tau) d\tau . \quad (6.39)$$

Proof. Use (6.36) and the mean value theorem to show

$$F(P,b) \int_{t_0}^{g(b)} P(\tau) d\tau = E(x,b) \int_{t_0}^{g(b)} P(\tau) d\tau \quad (6.40)$$

for some $x \in [X_F, b]$. Recalling the value of $E(x,b)$ lies between zero and one, the corollary is proved. ■

Using proposition 6.5 we can establish an equivalence relation between power histories. In particular, for a given system, barrel length and initial condition, two power histories are equivalent if the flyers have the same velocity at the end of the barrel. From proposition 6.5 two power histories $P(t)$ and $P^*(t)$ are equivalent if and only if

$$\int_{t_0}^{g(b)} P(\tau) E(r_F(\tau), b) d\tau = \int_{t_0}^{g^*(b)} P^*(\tau) E(r_F^*(\tau), b) d\tau . \quad (6.41)$$

Quantities associated with $P^*(t)$ are starred.

Equivalence classes of power histories would be useful to know. However, to calculate the integral in (6.36) requires knowing the solution, which necessitates integrating (6.10) or evaluating (6.11).

The global efficiency of a power history may be easily computed during the integration of (6.10). This efficiency gives us a quantitative method of comparing power histories which deposit the same amount of energy by the time the flyer reaches the end of the barrel.

In this section we introduce additional assumptions which enable us to obtain a closed form solution to the equations of motion. We use this solution to investigate the sensitivity of the motion. Proposition 6.1 shows how the energy histories change as a function of the flyer mass for a fixed velocity history. Proposition 6.2 proves the quicker we deposit the energy in the propellant the more efficiently it accelerates the flyer. Corollary 6.2.1 proves instantaneous energy deposition is the most efficient way of depositing energy. Corollary 6.2.2 established an upper bound on the velocity of a flyer. The upper bound is a function of m_F, m_P and the total energy deposited in the propellant. Proposition 6.3 leads to the definition of a local efficiency of a power pulse, and propositions 6.4 and 6.5 lead to the definition of a global efficiency of a power history.

7. Summary

In this report we investigate plane one-dimensional motion of a material whose velocity is a linear function of the spatial position. The object studied is composed of a tamper, propellant, flyer and barrel. Energy is rapidly deposited in the propellant driving the flyer down the barrel.

We assume the particle velocity of the propellant varies linearly with the distance between the propellant-tamper and propellant-flyer interfaces. In Section 3 we show this assumption is identical to assuming the particle velocity of the propellant varies linearly with the material particle's initial position. Using this assumption and conservation of mass we show the ratio of the mass density at any time to the initial mass density is independent of the material particle. If we assume the mass density is initially spatially uniform, it remains spatially uniform through all time.

Using conservation of energy and momentum we derive the equations of motion in Section 4. To solve these equations we need to know the pressures at the propellant-flyer and propellant-tamper interfaces. Knowing these boundary pressures we have an equal number of equations and unknowns.

We prove the pressure in the propellant is a quadratic polynomial in the initial position of the material particle. In Section 5 we discuss various types of equations specifying boundary pressures, i.e., rigid body, fixed, growing tamper, etc.

We introduce additional assumptions in Section 6 which enable us to obtain a closed form solution to the equations of motion. We use this solution to investigate the sensitivity of the solution to system parameters.

We show how the energy histories change as a function of the flyer mass for a fixed velocity history. We prove the quicker we deposit the energy in the propellant the more effectively it accelerates the flyer. We prove the limiting case of instantaneous energy deposition is the most efficient way of depositing energy. We prove the flyer velocity is bounded by

$$V_F^2 \leq \frac{2}{\left(\frac{1}{3} + \frac{m_F}{m_P}\right)m_P} \{E_P(\infty) + E_O - C\} \quad (7.1)$$

where $E_P(\infty)$ is the total energy deposited in the propellant and $E_O - C$ depends on the material, the initial conditions, and the limiting value of $r_F(t)$, see (6.21).

We define a local and global power efficiency. The local power efficiency measures the efficiency of an infinitesimal power pulse deposited when the flyer is at various positions down the barrel. The global power efficiency assigns an efficiency to a complete power history.

List of Symbols

t_0	beginning of problem
x, y	spatial position, (3.1)
$V(x, t)$	velocity of the material particle occupying the spatial position x at time t , (3.1)
P	pressure which is taken positive in compression, (3.1)
ρ	mass density, (3.1)
X, Y	initial position of a material particle, used as a material (Lagrangian) coordinate, (3.1)
$r(X, t)$	maps the material particle X into its spatial position at time t , (3.1)
$r_F(t)$	position of the propellant-flyer interface at time t , (3.4)
$r_T(t)$	position of the propellant-tamper interface at time t , (3.4)
$W_T(t)$	work done on the tamper up to time t , (3.4)
$W_F(t)$	work done on the flyer up to time t , (3.4)
$\mathcal{E}$	specific internal energy, (3.4)
$E_p(t)$	energy deposited in the propellant up to time t , (3.4)
E_0	initial energy of the propellant, (3.4)
$X_T(=0)$	initial position of propellant-tamper interface, (3.6)
X_F	initial position of propellant-flyer interface, (3.6)
$V_F(t)$	velocity of the propellant-flyer interface at time t , (3.6)
$V_T(t)$	velocity of the propellant-tamper interface at time t , (3.6)
T	temperature
m_P	mass of propellant, (4.1)
m_F	mass of flyer, (5.1)
m_T	mass of tamper, (5.1)

R	upper bound on $r_F(t)$, $t > t_0$
γ	adiabatic gas constant, (6.1)
H	general function, (6.1)
g	inverse of $r_F(t)$, (6.7)
V_0	initial velocity, (6.11)
$P(t)$	power deposited in propellant at time t , (6.23)
b	barrel length (6.29)
$E(x,b)$	local efficiency of a power pulse (6.29)
$F(P,b)$	global efficiency of a power history, (6.36)

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