CONF 900687--1

LA-UR -90-3376

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36

**TITLE: QUASIPERIODICITY, MODE-LOCKING, AND UNIVERSAL SCALING IN
RAYLEIGH-BENARD CONVECTION**

AUTHOR(S): R. E. ECKE

LA-UR--90-3376

DE91 002312

**SUBMITTED TO: NATO ASI WORKSHOP "CHAOS, ORDER AND PATTERNS"
COMO, ITALY. JUNE 25-JULY 6, 1990**

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**QUASIPERIODICITY, MODE-LOCKING, AND UNIVERSAL SCALING IN
RAYLEIGH-BENARD CONVECTION**

Robert E. Ecke, P-10, LANL

Full paper submitted to NATO ASI Workshop "Chaos, Order and Patterns", Como, Italy, June 25-July 6, 1990.

LA-UR-90-

This major review paper describes research on a model nonlinear dynamical system of small-aspect-ratio Rayleigh-Benard convection in ^3He - ^4He mixtures. The nonlinear effects of mode locking and quasiperiodic behavior are described. Analysis techniques for characterizing the state of the dynamical system include Fourier transforms, Poincare sections, phase differences, transients, multifractal $f(\alpha)$ spectra and scaling function dynamics. Theoretical results such as the fractal staircase of mode-locked intervals and the Arnold tongues are reproduced in experimental data. New techniques for analyzing scaling dynamics are developed and discussed. This is a tutorial article that introduces the major important concepts in nonlinear dynamics and focuses on experimental problems and techniques.

Quasiperiodicity, Mode-Locking, and Universal Scaling in Rayleigh-Bénard Convection

Robert E. Ecke

Physics Division and Center for Nonlinear Studies

Los Alamos National Laboratory, Los Alamos, NM 87545

Abstract

Quasiperiodicity, mode-locking and universal scaling dynamics are described for Rayleigh-Bénard convection in a dilute solution of ^3He in superfluid ^4He . Examples from experimental data are used to illustrate analysis techniques of nonlinear dynamics: power spectra, phase space reconstruction, Poincaré sections, transients and stability eigenvalues, return maps, multifractal $f(\alpha)$ analysis, and scaling function dynamics. Using these tools we show that the route to chaos in this system of two intrinsic oscillatory modes has the same universality as the sine circle map.

INTRODUCTION

These lectures on quasiperiodicity and mode-locking in a model fluid system, Rayleigh-Bénard convection, will review the theory of quasiperiodicity derived from 1-D circle maps, discuss how this perspective fits with the classical view of mode-locking, and illustrate many techniques of dynamical systems, applied in a real experiment to obtain a detailed understanding of experimental data. In particular, Section deals with simple physical models of mode-locking and with the theory of the generic 1-D map model of mode locking, the sine circle map. An historical perspective is presented on experiments on quasiperiodicity, mode-locking and chaos with an emphasis on fluid experiments. In section , the experimental work and basic analysis techniques of convection in a small-aspect-ratio cell using a ^3He -superfluid- ^4He mixture

are discussed. Comparisons of fundamental ideas of mode locking with experiment will then be presented: phase angles, convective amplitudes, 1D return maps and demonstration of the tangent bifurcation to mode locking. Another interesting and promising tool in experimental dynamical systems, the study of transients, will be illustrated. Section will concentrate on analysis tools for making comparisons between the theoretical predictions of the universal theory of quasiperiodicity and experimental data. This will focus on understanding the dynamics of the attractor in the language of multifractal thermodynamics and the scaling function formalism. This "statistical mechanical" approach yields effective algorithms for analyzing data.

Quasiperiodicity and Mode Locking

Frequency entrainment of two nonlinear oscillators was first observed by Christian Huyghens in 1665 when he noticed that clocks set on a shelf would synchronize their motion [1]. The simple example of pendulum-style clocks is an intuitive way to understand the basic elements of frequency entrainment [2]. Consider two uncoupled clocks, each composed of a pendulum motion and an energy input to overcome dissipation. In Fig. 1, if the upper plate is rigidly supported then the motions of the two clocks are uncoupled. The crucial element of the oscillation is that it be nonlinear, i.e., that the oscillation frequency depends on the amplitude of the motion. In the case of a clock the nonlinearity comes primarily from the impulses of the ratchet mechanism which sustains the motion. The individual oscillators are in limit cycle states. The phase space of the two oscillators is composed of the amplitude and phase of each oscillator and is thus four dimensional. When the amplitudes are only weakly perturbed by the coupling (i.e. slaved to the phase) it is adequate to consider the motions of the system as being confined to the surface of a torus imbedded in a 3D phase space, Fig. 2. In the absence of coupling one motion winds around the large amplitude (major) axis with amplitude A and phase $\phi_A = \omega_A t$. The other motion winds around the minor axis with amplitude $B < A$ and with $\phi_B = \omega_B t$. We denote Ω as the ratio of uncoupled oscillators $\Omega \equiv \omega_B/\omega_A$. For almost all (measure one) values of Ω the motion on the torus is ergodic, i.e. the trajectory of the motion eventually passes through all points on the torus. An equivalent statement is that the frequencies are incommensurate (not the ratio of two integers) and

Figure 1: Schematic illustration of coupled pendulum clocks.

Figure 2: Schematic illustration of the phase space of weakly coupled oscillators. Trajectories lie on a 2-D torus.

Figure 3: Schematic illustration of a driven pendulum clock. Phase portraits of the drive (A) and the clock with no coupling (B) are shown.

that the motion is not periodic but quasiperiodic.

The two oscillators can now be coupled together by, for example, a platform that is allowed to move, Fig. 1. The theorem of Peixoto [3] guarantees that for any type and strength of coupling the two oscillators will mode lock, the locking will consist of q pairs of stable and unstable fixed points for a period q cycle, and the mode-locked interval is structurally stable. Mode locking (entrainment, resonance) means that a particular periodic trajectory with frequency ratio p/q exists over a finite interval in Ω whereas in the uncoupled case the periodic state only exists for the precise value $\Omega = p/q$. The widths of mode-locked intervals are very small for small coupling, however, and Peixoto's theorem does not give much in the way of physical intuition. In order to develop some physical intuition we consider a simpler model of mode-locking consisting of one clock driven by a periodic forcing, Fig. 3. For simplicity we confine the discussion to entrainment at the drive frequency, i.e., the 1:1 resonance. When the drive is precisely tuned to the natural frequency ω_0 of the clock there are three periodic points to this dynamical system. The clock without forcing has an unstable fixed point corresponding to no motion (you have to give it a little kick to get it going). This state becomes a periodic repeller with small oscillation amplitude as forcing is applied. There are two other periodic states associated with entrainment of the

Figure 4: Schematic frequency response curves for the a) amplitude deviation $A - A_0$ (A_0 is the unperturbed amplitude) and b) phase difference (normalized by π) of the driven clock plotted vs. detuning parameter $\omega - \omega_0$. The solid curve is for the attracting cycle and the dashed curve for the unstable cycle.

limit cycle. The attracting periodic point corresponds to in-phase motion of the drive and the pendulum, while the unstable point is associated with out-of-phase motion. For weak forcing the in-phase drive slightly increases the amplitude of the pendulum while the out-of-phase motion decreases the amplitude a little. The stability boundary of the locking can be understood by considering the frequency response curve of the oscillator at fixed drive amplitude, Fig. 4. As the drive is detuned away from ω_0 , less amplitude is gained by the in-phase oscillation and less is lost in the out-of-phase mode. At some detuning boundary the oscillator gains (or loses) no net amplitude relative to the undriven oscillation. This marks the end of the mode-locked interval. Another thing to note is that the phase difference between the oscillators changes as the system is detuned. The phase of the attracting cycle goes from zero at resonance to $\pm\pi/2$ at the locking boundaries. Similarly the phase for the unstable cycle changes from π at resonance to $\pm\pi/2$ at the boundaries. In other words the amplitude difference and the phase difference between the stable and unstable cycles goes to zero at the locking boundary. We will come back to this point later in section .

We now have some idea about the behavior of a driven oscillator at low

amplitude. Similar ideas apply for arbitrary rational ratios of drive frequency to natural frequency so that resonances exist for all rational p/q values of Ω . For very small coupling, the width of a q -cycle resonance horn (the region over which the frequencies are entrained) varies as K^q where K is the coupling strength [4,5]. At the other extreme of very strong forcing everything responds at the drive frequency. In between there can be complex dynamics and complicated structure of resonance horns. The whole range of such behavior has been demonstrated by numerical studies of forced ODEs [6,7] and in experiments on forced magnetic oscillators [8]. In the case of two intrinsic oscillators the situation is more complex. Recent numerical work on coupled chemical reactions [9] show that for large coupling one or the other oscillators becomes dominant while in certain parameter regimes one can actually drive the system to a fixed point, that is no oscillations at all. Again in between these extremes quasiperiodic, mode-locked and chaotic dynamics can occur. In this intermediate regime, where the coupling is not too great, circle-map-like dynamics occurs and certain universal scaling behavior is expected.

In general mode-locking and quasiperiodic behavior are adequately modeled, from a dynamics perspective, by annulus maps of the plane. As illustrated in Fig. 2 we can describe weakly coupled oscillators as the phase space motion on a torus. To convert continuous dynamics to an iterated mapping in discrete time, one studies the intersections of the system trajectories with a plane (a Poincaré section). The resultant points form a mapping of the plane onto itself or, in the case of small radial perturbations, of an annulus onto itself. A family of such maps [10] is $P_{\Omega,K,\lambda}(r, \theta)$ given by

$$r_{n+1} = 1 + \lambda(r_n - 1) - \frac{K}{2\pi} \sin(2\pi\theta_n),$$

$$\theta_{n+1} = \theta_n + \Omega + \lambda(r_n - 1) - \frac{K}{2\pi} \sin(2\pi\theta_n).$$

In the case that the radial contraction is infinite (or very large) then $\lambda = 0$ and the equations reduce to the extensively studied sine circle map:

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi} \sin(2\pi\theta_n), \text{ mod } 1. \quad (1)$$

This mapping has two control variables: Ω the bare winding number and K which represents the nonlinearity/coupling. Many excellent papers de-

Figure 5: Discrete mappings of the circle from the sine circle map with $K=0.95$ and $p/q=3/5$ at the tongue boundary: a) θ_{n+1} vs. θ_n and b) θ_{n+5} vs. θ_n .

scribe the fractal structure of the mode-lockings [11,12,13] the renormalization group theory of universal scalings, [10,14,15] and other features of circle map behavior [16]. The basic features are presented here. One can graphically represent the circle map (1) by plotting θ_{n+1} versus θ_n as in Fig. 5. For $K < 1$ this is an invertible mapping; the mapping is single valued for forward and backward iterations.

As K is increased from zero at fixed Ω the ratio of the two motions, the rotation number, defined as

$$\rho \equiv \lim_{n \rightarrow \infty} \left[\frac{\theta_n - \theta_0}{n} \right].$$

deviates from Ω . In the case of mode locking, ρ takes on the value of some rational ratio p/q , where p, q are integer, over some finite interval in Ω . The condition for mode locking to a p/q resonance in some general 1D circle map, $\theta_{n+1} = f(\theta_n)$, is that $f^q(\Omega_q^p, x_q) = p + x_q$, where x_q is a periodic point of a q -cycle, Ω_q^p is a value of Ω for which $\rho = p/q$, and f^q is the q th iterate of the mapping function f . One can see graphically what this condition means by plotting the q th iterate map, θ_{n+q} versus θ_n , see Fig. 5. At either locking boundary the mapping becomes tangent to the $\theta_{n+q} = \theta_n$ line at q points. This is called a tangent bifurcation and is the transition

Figure 6: Regions of mode-locking for the sine circle map in the K, Ω space. Only large lockings are shown here.

between quasiperiodic and mode-locked states. For values of Ω between the boundaries there are q stable periodic points and q unstable periodic points, the stability depending on the slope of the curve through the $\theta_{n+q} = \theta_n$ line.

There are two main features of the circle map in the K, Ω parameter space. One is the tongue structure of resonance horns as K is varied, Fig. 6. At $K = 0$ the irrationals have measure one on the unit interval and the rationals, which make up the resonance intervals, have zero width. The measure of irrational holes, μ decreases, Fig. 7, as the resonance widths increase until at $K = 1$ the mode-locked intervals form a devil's staircase, Fig. 8, with fractal dimension $d_F = 0.870$ [11,12]. In the subcritical interval $0 < K < 1$ the set of $p : q$ resonances are not a "thin" fractal but instead obeys so-called fat fractal scaling [17,18]. A fat fractal has finite measure but still has scaling at all sizes given by $\mu(\epsilon) = \mu(0) + A\epsilon^\beta$ where β is the fat fractal exponent and ϵ the coarse graining size. From numerical experiments it is conjectured that $\beta = 2/3$ for all subcritical staircases [5]. The transition to fractal scaling at $K=1$ comes from the development of a cubic inflection point in the circle map. This inflection point also produces universal scaling dynamics at special irrational winding numbers. These special irrationals can be written as an infinite periodic continued fraction expansion of the

Figure 7: Irrational measure of holes between mode-locked intervals.

Figure 8: A theoretical devil staircase of mode-locked intervals. Large lockings are labeled.

form

$$\rho = \frac{1}{n + \frac{1}{m + \frac{1}{r + \dots}}}, \quad (2)$$

where $n, m, r, \dots$ are integers. This series can be represented by a sequence of numbers written as $[n, m, r, \dots]$. The simplest such expansion is for all the terms to be one, the sequence $[1, 1, 1, \dots]$. One can sum this series and obtain the so called "golden mean" winding number $\rho_g \equiv (\sqrt{5} - 1)/2$. The theoretical results for this winding number are the ones most often compared with experiment. Any winding number which has the same asymptotic expansion of ones should also obey the predicted scalings but convergence will be slower.

Experiment θ_n on nonlinear dynamical systems prompted the development of many of the theoretical models described here. We briefly review those original experiments and discuss the more recent experiments on forced nonlinear oscillators. Many of these results are reviewed in reference [19]. Mode-locking was first observed quantitatively in fluid experiments where a sequence of Hopf bifurcations resulted in states with two incommensurate frequencies [20,21,22]. Subsequent theoretical work attempting to understand elements of mode-locking in these experiments resulted in the universal theory of golden-mean scalings and the fractal "devil's" staircase. Many of the universal predictions, however, require the simultaneous tuning of two experimental parameters to obtain both criticality and a rotation number at the golden mean. The systems originally studied did not have multiple control parameters which were readily accessible. As a result researchers adopted the approach of producing a stable limit cycle in some nonlinear system and forcing it externally with a periodic signal of adjustable amplitude and frequency. Such a system is more likely to be a close approximation to the circle map model because the radial contraction to the stable limit cycle can be tuned by varying the control parameter of the nonlinear system. A multitude of such experiments ensued with systems of fluid convection [23,24,25], electronic materials [26,27,28,29,30,31], open fluid flows [32], laser oscillations [33], and recently thermoacoustic oscillations [34,35]. Most studies compared the fractal dimension of the locking intervals and the power spectra of the dynamics [10,36]. The scaling exponent δ [14] is somewhat harder to obtain experimentally but was measured quite accurately in the convection work of

Stavans et. al. [24,25]. Another characterization of the universal circle map scaling which focuses on the dynamics instead of the resonance widths is the multifractal $f(\alpha)$ analysis [37]. The $f(\alpha)$ analysis was first applied to circle map data from Rayleigh-Bénard experiments with great success [38]. All forcing experiments [39] report good agreement with the predictions of circle map scaling. From these experiments one can conclude that the critical behavior of many externally-forced nonlinear oscillators is the universality class of the sine circle map.

The originally studied experiments of two intrinsic oscillators received much less attention owing to the experimental difficulties. The case for two independent oscillations is more complex, in principle, because each oscillator can adjust to the dynamics of the other; an additional degree of freedom is available. In addition, The size of the radial dissipation is constrained by the physical system and there is no guarantee that the dissipation is large. Thus the study of two independent oscillators offers a more severe test for not only the universality of the theoretical model but also for the genericity of the results. In addition to the work reviewed here on two frequency Rayleigh-Bénard convection [41,42,43] there have recently been several other experimental studies of systems with two internal modes with independent frequencies [44,45].

Small-Aspect-Ratio ^3He - ^4He Convection: Data and Analysis

We have made a detailed study of the dynamics and mode-locking of two intrinsic modes of thermal convection in small-aspect-ratio Rayleigh-Bénard convection in a dilute solution of ^3He in superfluid ^4He [41,42,43,47,46,48,49, 50]. In the parameter space of Rayleigh number R , a dimensionless control parameter proportional to the temperature difference across the fluid layer, and Prandtl number Pr , another dimensionless parameter which is a strong function of the mean cell temperature many regions of different convective behavior exist, Fig. 9. The superfluid nature of the mixture does not play an important role here and we can consider it to be a classical single component fluid. Details and references to this assertion can be found in references [51,52]. For small R , the fluid conducts heat diffusively. At $R=2000$ there is a

Figure 9: Phase diagram of convective states in parameter space of Rayleigh number and Prandtl number. Regions are described in the text.

supercritical (continuous) transition to steady convection [51] where the fluid motion is believed to be two convection rolls oriented parallel to the short side of the rectangular convection cell (height 0.8 cm, length 2.1 cm and width 1.6 cm). The actual flow is inferred from room temperature experiments since visualization of helium fluid convection has not yet been realized. The state of the convective flow is partially determined from measurements of the spatially-averaged convective heat transport. Another probe of the state of the system, more sensitive to time dependence, consists of a local temperature transducer. In small aspect ratio convection the side walls severely constrain the spatial structure of the fluid flow and measurement of the dynamics of the system at a single point is adequate to characterize the dynamical state. The local probe is a differential thermocouple with high temperature sensitivity, $0.3 \mu K/\sqrt{Hz}$. Details of the experimental techniques can be found elsewhere [51]. The output of the probe, a voltage proportional to local temperature differences $\delta T(t)$, is digitized to produce a time series. Standard techniques of phase space reconstruction [53,54,55] are used to produce phase space trajectories of the dynamical system. Frequency spectra are also calculated from the time series using an FFT algorithm and standard windowing techniques. The peak frequencies for periodic states are obtained by interpolation of the windowing function, thereby increasing the accuracy of the frequency deter-

Figure 11: Experimental devil's staircase of mode-locked intervals for $1/Pr=14.9$, slightly above criticality. Prominent lockings are indicated.

Figure 12: Experimental regions of mode locking in the $1/Pr$ and R/R_c parameter space. The dashed lines represent the hysteresis discontinuity at low R/R_c , increasing R/R_c (---) and decreasing (---), and the transition to high dimensional chaos at large R/R_c . For $1/Pr > 15$ there is structure in the tongues not shown on this global plot.

dimensional and that are well described by circle map models. An important characteristic of the theory is the critical line where certain critical scaling behavior is expected. In the more complex experimental parameter space of R and Pr one needs a way to define and locate such a line if it exists. Below we discuss how we accomplish this experimentally. First we describe simple quasiperiodic and mode-locked dynamics that occur in a region of parameter space in our fluid system that is far below criticality.

In our simple picture of mode locking and frequency entrainment developed in section , there are several ideas that can be demonstrated with experimental data. One is the concept that there is energy pumping between modes as a mechanism for frequency locking. In figure 13 are three plots that illustrate this nicely. On the top is the winding number as a function of R/R_c showing locking to a $W=2/13$ resonance. In the second and third figures the amplitudes A_2 and A_1 of the fundamental frequencies f_2 and f_1 are shown. In the locking interval A_2 is boosted slightly relative to a baseline trend (shown as a solid line in the figure) while A_1 is sharply reduced; the motion of mode 2 is being increased at the expense of mode 1. The other idea is the importance of the phase of one oscillator with respect to the other. Recall that in-phase motion right on resonance is associated with the attracting state whereas out of phase motion is characteristic of the unstable cycle. It is straightforward to obtain the absolute phase at a particular frequency from a Fourier transform of the time series. When the resonance is not one to one, however, the phase difference is a little difficult to define. In order to understand this in the general case, we will first consider two oscillators with motions $\cos(\omega_2 t + \phi_2)$ and $\cos(\omega_1 t + \phi_1)$ where the absolute phases of the oscillations represent the uncertainty in the time origin. If the frequencies are equal then the phase difference is fixed, does not change with time and is a physically relevant quantity. This is true only so long as the frequencies are exactly equal since any small frequency difference will cause the phases to change according to $\Delta\phi = (\omega_2 - \omega_1)t$. We want to define a phase difference in the general case of a $p:q$ mode locking that has the same properties of being fixed inside the resonance interval and time dependent outside. If we define the phase difference $\Delta\Psi \equiv (\omega_1 - \omega_2)t + (\phi_1 - \phi_2)$ it is still time dependent even if the state is mode-locked. But by defining $\Delta\Psi \equiv (p\omega_1 - q\omega_2)t + (\phi_1 - \phi_2)$ the coefficient of t is zero for a $p:q$ locking. This definition works nicely and is quite useful in analyzing the interactions between two interacting modes. In the simple model described in section

Figure 13: Convective mode amplitudes A_2 and A_1 and winding number f_2/f_1 vs. R/R_c for experimental data around the $W=2/13$ locking. Data inside (•) and outside (o) the locking region are labeled and the boundaries of the locking indicated (- - - -).

Figure 14: Phase angle difference normalized by 2π for experimental data with points inside (●) and outside (○) the locking interval labeled.

, the phase difference evolves over an interval of π as the locking region is scanned. In experiment this phase difference is well defined and measurable but it is not always equal to π . In [41] several examples are shown and the analysis is described in more detail. Here we illustrate the phase angle dependence of the data of Fig. 13. In Fig. 14 the phase is shown to be fixed over the mode-locked region and spans a total range of about $\pi/2$. No theory is presently available that explains quantitative features of this phase angle difference. Phase differences constructed for the sine circle map also show deviations from π [41].

We now consider quasiperiodicity and mode locking from a dynamical systems perspective. One can construct the dynamical state of a system composed of two nonlinear, interacting modes of oscillation in a three dimensional phase space consisting of delay coordinates of the local probe temperature oscillations. The trajectories in this phase space lie on a manifold which is topologically equivalent to a 2-D torus (in practice the torus is often quite distorted). To better understand the dynamics one constructs a 2-D mapping (Poincaré section) of the plane by plotting the interpolated intersections of system trajectories with a 2-D surface. We use a planar surface oriented in such a way as to produce the “best” mapping, i.e. one that avoids crossings, kinks, etc. In this way we can avoid distortions due to the imperfect

Figure 15: Experimental Poincaré section in delay coordinate phase space (units normalized to size of attractor). Angle θ indicates parameter for generation of return map.

imbedding of delay coordinate schemes [53,55]. Note that our method of generating a Poincaré section is more general than the strobing technique used in forced oscillator experiments. In the subcritical regime, this Poincaré map will form a smooth 1D curve diffeomorphic to a circle. In Fig. 15 an example is shown for experimental data in which the dynamical state is very close to a period-2 resonance. A 1-D return map is then constructed by parametrizing the iteration of points on the circle by an angular coordinate θ , producing a circle map, θ_{n+1} as a function of θ_n , Fig. 16(a). The rotation number of this Poincaré section is different from W obtained from the frequency spectrum because of the ambiguity in choosing the plane of the Poincaré section that cuts the torus. It is operationally easier to produce a section by cutting through the minor axis of the torus thereby producing a rotation number ρ which is related to W by $\rho = 1/W, \text{ mod } 1$. The state represented in Figs. 15 and 16 has $W \approx 2/13$ corresponding to $\rho \approx 1/2$. In this and all future Poincaré sections we shall refer to ρ as the appropriate winding number. To illustrate the tangent bifurcation structure of mode locking we plot the second iterate map for the data in Figs. 15 and 16 which are close to a period-2 locking. (For a period- q locking a tangent bifurcation will occur in the q th iterate map). This is illustrated in Fig. 16(b) for the experimental data.

Figure 16: Return maps for a) first iterate θ_{n+1} vs. θ_n and b) second iterate θ_{n+2} vs. θ_n for experimental data near locking to a period-2 resonance.

Having demonstrated aspects of the circle map from experimental data one would like to proceed to develop a more quantitative comparison based on the critical properties of the mapping. The first problem is how to define the critical line in the real experimental parameter space. In the circle map criticality comes about when the map becomes noninvertible at a cubic inflection point. In the more general case of a 2-D map one can think of criticality as resulting from a loss of smoothness or differentiability of the invariant circle as in general the return map will not have a cubic inflection point on this critical line [60,61]. A theoretical approach [62,63] was based on the structure of the stable and unstable manifolds of the periodic points in the interior of a resonance horn [64,10]. At the edges of the locking the periodic points arise from a saddle-node bifurcation (the 2D phase-space version of the tangent bifurcation) and thus the stable manifold must be characterized by two real eigenvalues (Floquet multipliers), see Fig. 17(a). If somewhere in the interior of the horn the periodic points take on spiral stability, i.e., complex conjugate eigenvalues, then the stable manifold is no longer differentiable at those points, see Fig. 17(b). Other mechanisms can also lead to loss of smoothness of the circle as discussed in references [64,10,62] but for simplicity we define our critical line to consist of the minima of the spiral stability regions. Other criteria based on the qualities of the time series at

Figure 17: Stable and unstable manifolds of a) saddle-node and b) spiral periodic states. Saddles are denoted by (o) and attracting periodic points by (•)

the golden mean are most often used but this technique provides an independent measure of criticality. To realize this method the linear stability of the periodic points has to be determined. This requires the analysis of transient relaxation of the system to the attracting periodic points. Therefore we turn our attention to the production and analysis of transients in real physical systems with many potential degrees of freedom.

In simple systems such as electronic oscillators it is easy to produce transients because one can specify the initial conditions with a small number of parameters. On the other hand systems described by partial differential equations have spatial degrees of freedom and require specifying initial conditions at every point in space. This is impossible experimentally for arbitrary initial conditions and so one typically uses another stable state of the system as the initial state and produces transients by making sudden changes in a control parameter. An important point to note here is that even if the long time dynamics of the system is low dimensional as in our experiments on quasiperiodic convection, a sudden change in control parameter is likely to induce dynamics in a much larger dimensional phase space. These higher dimensional motions should damp out more quickly than the low dimensional transients of interest but there is often a mixing of time scales which complicates interpretation of the dynamics. With that caveat we describe cases

Figure 18: Transient Poincaré section showing spiral approach to a periodic point in the 2/13 resonance horn; $R/R_c=12.025$ and $1/Pr=14.77$. Solid curves are a guide to the eye and not system trajectories.

where the interpretation is straightforward and quantitative results obtainable. Consider transient relaxation to a single periodic point in a period-2 cycle. This means that we take every other iterate point from a Poincaré section of the mode-locked state. (The axes of the section are proportional to the temperature oscillations of the local probe but are scaled so that the entire range of the attractor is ± 1 .) The data in Fig. 18 are for a state with spiral stability for which the real and imaginary parts of the complex conjugate eigenvalues λ_R and λ_I can be defined by considering the position of the points relative to the asymptotic stable state: $\vec{r}(r, \theta, n) = \vec{r}_0 e^{-\lambda_R n + i \lambda_I n}$ where n is the discrete time iterate variable. λ_R is obtained by plotting $\ln r$ versus n as in Fig. 19(a). Likewise the angular rotation is just the slope of θ versus n , Fig. 19(b). We get $\lambda_R = 0.18/\text{iterate}$ and $\lambda_I = 1.7$ radians/iterate. In the region of the resonance horn where two real eigenvalues govern the stability of the periodic points one eigenvalue is typically much larger than the other and only the smaller one is accessible to experimental measurement. An example of pure radial relaxation is shown in Fig. 20 where we get $\lambda_R = 0.3/\text{iterate}$. By making a series of such measurements across the 2:13 resonance we can specify the stability of the periodic point inside that tongue. In Fig. 21 λ_R and λ_I are shown for $Pr^{-1} = 14.77$ where spirals ex-

Figure 19: Plots of a) $\log_{10} r$ and b) θ vs. iterate number n for data from Fig. 18. Solid lines are straight line least squares fits to the data where the slope determines the eigenvalues λ_R and λ_I .

Figure 20: $\log_{10} r$ vs. n for experimental data with pure real eigenvalues. Only the smaller eigenvalue is measured by the fit to the data (—).

ist over most of the horn and for $Pr^{-1} = 14.56$ where there are no spirals. Notice how the radial eigenvalue decreases towards the middle of the locking for the spiral case but is peaked for the non-spiral data. From this information the spiral stability boundaries of the 2:13 resonance horn are identified. Similarly the approximate threshold for spirals is determined for the 3/19 and 4/25 resonances. The accuracy with which this lower boundary can be determined decreases as the cycle length increases. Another limitation is that as the two real eigenvalues become close in magnitude, as they must before the transition to spiral stability, it becomes difficult to resolve the spiral component before the transient has decayed to the noise threshold. The critical line, determined by this constraint and by the condition that the critical line is below any tongue hysteresis, is placed slightly below the locus of apparent spirals. The tongue diagram, the spiral region of the 2:13 resonance and the critical line are shown in Fig. 22. The general shape of the critical line follows reference [62]. Also illustrated are lines of constant winding number with asymptotic golden mean expansions. These are defined as $\rho_g^{(n)} = [n, 1, 1, 1, \dots]$ according to (2). It is at the intersections of these constant golden mean lines and the critical line that universal predictions of the sine circle map model should apply. In the next section we describe those predictions and the analysis tools for making quantitative comparisons between theory and experiment.

Universal Scaling Dynamics: Experimental Analysis

There are a number of quantitative comparisons that can be made between experimental data and predictions of universal scaling theory. Calculation of the fractal dimension of the staircase, evaluation of the contraction exponent δ for interval widths around the golden mean, the self similar structure of the power spectrum of the data at criticality are all useful and well described analysis techniques. These are reviewed in detail elsewhere [19] and are difficult to apply for our system. Instead this section will focus on analysis of the dynamics of the system represented by a Poincaré section of the phase space trajectories. As described earlier the phase space motion, for states below criticality, is constrained to a 2-D torus. The Poincaré section of a

Figure 21: Eigenvalues for periodic points (Floquet multipliers) across $W=2/13$ locking interval: a) λ_R vs. R/R_c , $1/\text{Pr}=14.77$; b) λ_R vs. R/R_c for $1/\text{Pr}=14.6$; c) λ_I vs. R/R_c for $1/\text{Pr}=14.77$. Lines are guide to eye and error bars reflect statistical uncertainty in the fits.

Figure 22: Resonance horns in parameter space of inverse Prandtl number $1/Pr$ and normalized Rayleigh number R/R_c . Dashed vertical lines trace golden-mean lines with winding numbers $\rho_g^{(1)}$, $\rho_g^{(2)}$, $\rho_g^{(3)}$, and $\rho_g^{(4)}$. Lightly shaded regions indicate hysteresis; the interior structure of the $2/13$ resonance consists of stable spiral periodic cycles and secondary Hopf bifurcations (SHB) of the periodic cycles. Also shown are estimated location of critical line (—) from analysis of spiral periodic cycles (o) and of $f(\alpha)$ spectra.

Figure 23: Poincaré section for experimental data with $R/R_c = 12.22$, $1/Pr=14.57$, and ρ within 50 ppm of $\rho_g^{(4)}$. Inset: connected segments for $p/q=28/129$ approximation to data points.

torus is a circle, Fig. 12, and even at criticality the section is very line like, i.e., a 1-D curve. In Fig. 23 the Poincaré section is shown for a state with $\rho = \rho_g^{(4)}$ to within 0.005%. The inset shows a high order approximation to the data with a 28:129 cycle. This curve is used to compute the arc lengths that define the segments discussed below. Figure 24 is the corresponding 1D return map. Before discussing this data in detail the general analysis techniques of multifractal $f(\alpha)$ spectra and scaling functions [65] need to be presented.

A simple fractal structure is one that can be generated by successive application of a single scale factor. The center-third Cantor set is such a set. Many fractal sets, however, have multiple scalings and cannot be adequately described by a single number, the fractal dimension of the set. The concepts of generalized dimension D_q and $f(\alpha)$ spectrum were created to provide a framework for analyzing such fractals. Many aspects of multifractals are reviewed in reference [66]. To understand the $f(\alpha)$ and scaling function techniques it is useful to consider the dynamics of a model system which generates a Cantor set by successively dividing up the unit interval: the zeroth level consists of the unit interval, the second level has two segments, the third has four, etc. In general there are Q_n segments labeled $\Delta_k^{(n)}$, $k = 0, \dots, Q_n - 1$ at

Figure 24: Return map θ_{n+1} vs. θ_n for data in Fig. 23.

the n th level. Consider a Cantor set where the two intervals at the first level are divided into four different intervals on the second level by application of four scale factors, $\{\sigma_i\}$. Successive application of these scalings produces a fractal set. In the case of the center-third Cantor set there is only a single scale so $\sigma = \sigma_i = 1/3$ so that on the n th level all intervals have size $(1/3)^n$. For multiple scales, however, an interval at level n has a size that depends on the history of application of the scale factors, e.g., $\Delta^{(n)} = \sigma_1 \sigma_2 \sigma_1 \sigma_3 \dots$. The sizes are bounded from above by successive application of the largest scale factor and similarly from below by the smallest scale. This range of scales is an important element in characterizing a multifractal and will be one of the things extracted in an $f(\alpha)$ analysis. The other important parameter is the Hausdorff dimension of the set which corresponds in a rough sense to the average fractal dimension. The $f(\alpha)$ formalism is really just a technique for extracting this information from a set of intervals. A better characterization of this fractal is the set of scales which generate the intervals. The scaling function approach is just the identification of these scale factors.

A useful analogy exists between some of the averaging over scales in a multifractal and the averaging of energy states in statistical mechanics [67]. In this picture the statistical averaging over intervals gives a kind of thermodynamics whereas elucidation of the individual scale factors is a more microscopic description of the fractal as derived from statistical mechanics. A

“thermodynamic” description of these intervals can be defined by a “partition function”

$$Q_n^{-P(\beta)} = \sum_{k=0}^{Q_n-1} |\Delta_k^{(n)}|^\beta,$$

from which follows the definition of the thermodynamic “pressure”:

$$P(\beta) = - \lim_{n \rightarrow \infty} \frac{\ln \sum_{k=1}^{Q_n} |\Delta_k^{(n)}|^\beta}{\ln Q_n}. \quad (3)$$

(Here we follow Feigenbaum [68] in defining the canonical formulation of multifractal thermodynamics.) The “internal energy” is defined as

$$u(\beta) = \frac{\partial P(\beta)}{\partial \beta} = - \lim_{n \rightarrow \infty} \frac{\sum_k |\Delta_k^{(n)}|^\beta \ln \Delta_k^{(n)}}{\ln Q_n \sum_k |\Delta_k^{(n)}|^\beta}, \quad (4)$$

from which follow relationships for α , $f(\alpha)$, D_q , etc., from the microcanonical formulation [37] in terms of $P(\beta)$ and $u(\beta)$:

$$\begin{aligned} \alpha &= 1/u(\beta), \quad \tau = -\beta, \quad f(\alpha) = \beta - \frac{P(\beta)}{u(\beta)}, \\ D_q &= \frac{\beta}{1+P(\beta)}, \quad q = -P(\beta). \end{aligned} \quad (5)$$

Applying these definitions to the 1/3 Cantor set with $Q_n = 2^n$ and $\Delta_k^{(n)} = 3^{-n}$ one gets $\alpha = f(\alpha) = \ln 2 / \ln 3$ as expected. Another way to obtain this result uses a statistical mechanical transfer matrix formalism that reveals more about the scaling dynamics than the averages computed from the thermodynamic approach [69,70]. We define the transfer matrix as operating on two segments Δ_0 and Δ_1 with scale factors $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Consider a two-by-two matrix:

$$\mathbf{T}^\beta = \begin{pmatrix} \sigma_1^\beta & \sigma_2^\beta \\ \sigma_3^\beta & \sigma_4^\beta \end{pmatrix} \quad (6)$$

where the scale factors $\sigma_i < 1$ and we used $\mathbf{T}^\beta$ to indicate each element to the power β and not the matrix. $\mathbf{T}$ operates on two intervals Δ_0 and Δ_1 to produce sums of intervals composed with the dynamical scale factors. From the eigenvalues of $\mathbf{T}^\beta$ one can get the thermodynamics directly. The two eigenvalues λ_1 and λ_2 of $\mathbf{T}^\beta$ are obtained from the secular equation $|\mathbf{T}^\beta - \mathbf{I}\lambda| = 0$.

Figure 25: Four-scale Cantor set construction.

The pressure $P(\beta)$ can be written in terms of these eigenvalues:

$$P(\beta) = - \lim_{n \rightarrow \infty} \frac{\ln (c_1 \lambda_1^{n-1} + c_2 \lambda_2^{n-1})}{n \ln 2}$$

which reduces in the limit to $P(\beta) = -\ln \lambda_1 / \ln 2$ where $\lambda_1 > \lambda_2$. The convergence to this limit is given in this expression by the ratio λ_1 / λ_2 and by the nonuniversal constants c_1 and c_2 . For the $1/3$ Cantor set $\sigma_i = 1/3$, $\lambda_1 = 2 [3]^\beta$, and $\lambda_2 = 0$. Results for α and $f(\alpha)$ follow from (4) and (5).

The more interesting example for generalized Cantor sets is one where the dynamic scale factors are not the same. Solving the secular equation for arbitrary σ_i yields an equation for the largest eigenvalue

$$\lambda_{max}(\beta) = \frac{\sigma_1^\beta + \sigma_4^\beta}{2} + \left[\left(\frac{\sigma_1^\beta - \sigma_4^\beta}{2} \right)^2 + (\sigma_2 \sigma_3)^\beta \right]^{1/2}$$

from which follows $P(\beta)$, α , and $f(\alpha)$. One thing to notice is that the scale factors σ_2 and σ_3 only appear as a product and thus the thermodynamics of the four scale Cantor set is infinitely degenerate over internal scales in that there is a 1-D family of sets with the same product $\sigma_2 \sigma_3$. A more direct approach in determining the scale factors is to measure ratios of intervals.

In the four scale example, Fig. 25, the relationships are $\sigma_1 = |\Delta_1^{(2)}|/|\Delta_1^{(1)}|$, $\sigma_2 = |\Delta_2^{(2)}|/|\Delta_1^{(1)}|$, $\sigma_3 = |\Delta_3^{(2)}|/|\Delta_2^{(1)}|$, and $\sigma_4 = |\Delta_4^{(2)}|/|\Delta_2^{(1)}|$. With suitable relabeling these relationships hold at every level although for this example the results are the same. In real dynamical systems, however, scalings usually apply asymptotically for small intervals and so the smallest size compatible with noise considerations should be used. The advantage of this direct technique, an unambiguous determination of the scaling dynamics, is balanced by the practical disadvantage of sensitivity to experimental (or numerical) errors. The thermodynamic approach on the other hand should be more robust since it averages over many intervals. These problems are considered next in the analysis of experimental data.

The dynamics of the circle map produces an ordering of points on the circle which divides up the interval 0 to 2π into segments, each segment consisting of the arc length between points on the circle. For a winding number that is a rational approximant to the golden mean there is a generation of segments on different levels similar to the Cantor set example. The rational approximants for the sequence $[1,1,1,1,\dots]$ are $1/1, 1/2, 2/3, 3/5, 5/8, 8/13, \dots$ with cycles that are the Fibonacci numbers $F_i = [1,2,3,5,8,13,\dots]$. Successively higher approximants correspond to higher cycles, a closer approximation to the golden mean, and more levels in the scaling tree. In Fig. 26 this tree is illustrated for an $8/13$ cycle which generates 3 levels and either 3 or 5 scale lengths depending on whether one takes the transfer matrix $\mathbf{T}$ as operating on segments $\Delta_0^{(1)}$ and $\Delta_1^{(1)}$ or on the three segments $\Delta_i^{(2)}$ at level 2. The dynamics of the circle map excludes certain intervals and thus constrains certain elements in $\mathbf{T}$ to be zero, thereby reducing the possible four scales for a 2×2 matrix to three and the possible nine scales for a 3×3 matrix to five. The 2×2 transfer matrix is

$$\mathbf{T} = \begin{pmatrix} \sigma_0 & \sigma_1 \\ \sigma_2 & 0 \end{pmatrix} \quad (7)$$

and the 3×3 transfer matrix is

$$\mathbf{T} = \begin{pmatrix} \sigma_0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \\ \sigma_3 & \sigma_4 & 0 \end{pmatrix}. \quad (8)$$

In the limit as the golden mean is approached the number of levels becomes large and an $N \times N$ matrix is needed to describe all the scalings. This limit

Figure 26: Universal scaling tree for circle map segments with $p/q=8/13$ and scale factors $\{\sigma_i\}$.

can be thought of as describing a function $T(x)$ whose argument is just the index of the scale factor normalized by the total number of scales. The magnitude of $T(x)$ is the value of the scaling factor at a particular value of x . Approximations to this function consist of increasingly bigger matrices starting with 2×2 , then 3×3 , etc. As in (7) and (8) this corresponds to 3-scale and 5-scale approximations of the continuous function. The scale factors are only asymptotically convergent to universal values. In Fig. 27 we show a 5-scale approximation to the scaling function derived from the data shown in Fig. 23. Three of the scales show good agreement with the predictions of the theory whereas the higher order scalings are not at all close to the theory. Some of the discrepancies can be understood to arise from the sensitivity of the segments to noise and drift in the data: the higher order scales σ_3 and σ_4 involve smaller segments mapped more times around the circle and thus are more sensitive to experimental imperfections. This dependence can be avoided by performing thermodynamic averages which produce the $f(\alpha)$ spectrum. Before discussing the relative advantages and disadvantages of the techniques we describe our methods for constructing the

Figure 27: Universal scaling function in the 5-scale approximation of experimental data (---) of Fig. 23 and for a many-scale theoretical approximation (—)

thermodynamic averages. Other results on the determination of a scaling function from experimental data are reported in reference [71].

Standard techniques for calculating $f(\alpha)$ spectra suffer from an arbitrariness in picking scaling regions. Using the canonical approach of Feigenbaum [68] these problems do not arise and the statistics required to obtain reasonable results is much reduced [72,42]. Consider the Poincaré section in Fig. 23. The arc length between points defines a set of segments which can be inserted into the definitions for $P(\beta)$ and $u(\beta)$ (3), and (4), which are combined to yield α and $f(\alpha)$ from (5). The essential difference between this technique compared to standard methods is the explicit evaluation of the derivative sum rather than performing a finite difference derivative. To justify the use of the canonical equations the limit must be shown to exist from the experimental data. The straight line slopes for different values of β in a plot, Fig. 28, of $\ln \sum_k |\Delta_k^{(n)}|^\beta$ versus $\ln Q_n$ where Q_n is an approximant cycle length, indicate that the limit does exist. We now describe problems associated with the evaluation of $P(\beta)$ as they affect the $f(\alpha)$ spectrum.

A straightforward evaluation of the $f(\alpha)$ curve for the small number of data points that we have here, of order 100, gives reasonable spectra but one quickly observes that the procedure is not robust and is strongly affected by a few segments. In particular the largest and smallest segments dominate the determination of the range of scales, i.e., the values of α_{max} and α_{min} . This is because these quantities depend on the interval sizes raised to the

Figure 28: $\ln \sum_k |\Delta_k^{(n)}|^\beta$ vs. $\ln Q_n$ where Q_n is an approximant cycle length with a) $\beta = 10$ (■), b) $\beta = 0$ (+), and c) $\beta = -10$ (×).

power β and thus the largest segment dominates the sum for large positive β whereas the smallest segment is the important one for large negative β . In Fig. 29 we illustrate this for large segments by plotting $P(\beta)$ for all the data and comparing it with $P(\beta)$ for only the largest segment; for $\beta > 8$ there is no difference between the two. There is the additional problem of slow convergence of $P(\beta)$ for large β , see Fig. 30. On the basis of these two difficulties we adopt a somewhat different strategy based on the transfer matrix formalism. An equivalent approach using a different method was used by Feigenbaum et. al. [73] to analyze forced Rayleigh-Bénard convection data. First we verify that the maximum of the $f(\alpha)$ spectrum is equal to one which corresponds to the condition $P(\beta = 1) = 0$. (If all the segments are used then this condition is met automatically so it is necessary to compute $f(\alpha)$ using the subset of segments shown in Fig. 26.) This is found to be true to within several percent. Given that condition Feigenbaum has shown [69] that the $f(\alpha)$ curve is determined to high precision by two parameters denoted here as s_1 and s_2 . These parameters are combinations of scale factors that come about from solving for the largest eigenvalue of the 5-scale transfer matrix, (7). Defining $z = 1/\lambda$, we get the secular equation $1 - z\sigma_0^\beta - z^2(\sigma_2\sigma_4)^\beta - z^3((\sigma_1\sigma_2\sigma_3)^\beta - (\sigma_0\sigma_2\sigma_4)^\beta) = 0$. With the condition $P(\beta = 1) = 0$ we have $z=1$

Figure 29: $P(\beta)$ vs. β for experimental data of Fig. 23. For (•) all the data was used and for (—) only the largest segment was used.

Figure 30: $P(\beta)$ vs. β for experimental data (•) of Fig. 23 and fitting curve (—) parameterized by s_1 and s_2 . Notice poor convergence of experimental data for positive β .

Figure 31: $f(\alpha)$ curves for the experimental data in Fig. 23 ($\bullet$), for theoretical data from a sine circle map at criticality (—) and for a subcritical experimental data set with $R/R_c=12.375$ and $1/Pr=14.17$ (- - - -). The error bars for α that are smaller than 1.1 are of the order of the data points and are not plotted.

for $\beta = 1$ and we get the simplified expression:

$$(1 - zs_1^\beta)(1 - z^2s_2^\beta) - z^3(1 - s_1)^\beta(1 - s_2)^\beta = 0, \quad (9)$$

where $s_1 = \sigma_0$ and $s_2 = \sigma_2\sigma_4$. To find s_1 and s_2 from experimental data we solve (9) using a nonlinear least-squares fitting procedure with z_{min} obtained from the experimental $P(\beta)$ by inverting the relation $P(\beta) = \ln z_{min} / \ln \rho_g$. The range of β is small and centered around zero where there are many segments and the averaging is effective. The values of $s_1 = 0.47 \pm 0.01$ and $s_2 = 0.61 \pm 0.2$ do not depend sensitively on the range of β chosen. The $f(\alpha)$ spectrum obtained using these parameters is shown in Fig. 31. Also shown for comparison is a subcritical $f(\alpha)$ curve. In principle a subcritical data set is not multifractal and should collapse to a point. Finite size effects, however, associated with finite cycle approximations to the golden mean cause some residual multifractal scaling [74] and a narrow but finite width curve [75]. From the quantities s_1 and s_2 we can calculate the value of the inflection point

exponent which characterizes the universality of a particular map. For inflection points of the type $\theta |2\theta|^{\nu-1}$ we have that $\nu = 2 \ln s_1 / \ln s_2 = \alpha_{\max} / \alpha_{\min}$. From the experimental data for $\rho_g^{(4)}$, Fig. 23, we get $\nu = 3.1 \pm 0.3$, putting the state in the universality class of the sine circle map 1. Other golden mean lines in our parameter space, $\rho_g^{(2)}$ and $\rho_g^{(3)}$ yield $\nu^{(2)} = 2.9 \pm 0.3$ and $\nu^{(2)} = 2.8 \pm 0.4$. Details of the $f(\alpha)$ and scaling function analyses and how it is applied to winding number sequences of the form $\rho = [n, 1, 1, 1, \dots]$ are described in reference [43].

The thermodynamic approach which generates the $f(\alpha)$ spectra provides a robust technique that takes advantage of averaging over the complete fractal structure. On the other hand direct evaluation of the scaling function gives a more complete description of the fractal dynamics. In our experimental scaling function three scale factors σ_0 , σ_1 , and σ_2 are resolved to within about 10% whereas the thermodynamic approach determines s_1 and s_2 which gives σ_0 and the combination $\sigma_2\sigma_4$. It is interesting that while the direct determinations of σ_3 and σ_4 are quite poor, averaging gives $\sigma_2\sigma_4$ quite accurately. It would be worthwhile to understand the details of the averaging process and to average in a reasonable manner over the structure of the fractal to get a more robust scaling function.

The methods described here work quite well for circle map data and presumably for any dynamical system for which the scaling dynamics is understood. Periodic orbit decomposition methods [76] should also be applicable in such systems but we have not done this analysis. The techniques fail when there is no model for the dynamics but for some cases one can construct the dynamics directly [77]. Finally, I hope that the discussions provided here will help in understanding the pitfalls and limitations of the multifractal analysis; some of the analysis methods are not robust and can lead to poor comparisons with theory. It is vitally important to establish that the point ordering of the experimental circle map is the same as for the theory and that a scaling function can be constructed with the correct properties. Without such consistency checks the nonrobust features of the $f(\alpha)$ methods suggest reason for concern in comparisons between theory and experiment.

Acknowledgements

I would like to acknowledge all the people who have contributed to this work: John Wheatley for the original work on ^3He -superfluid- ^4He mixtures, Hans Haucke and Yoshi Maeno who built the experimental apparatus and did most of the early convection work, Ioannis Kevrekidis who taught me about bifurcation theory and helped in understanding the Arnold tongue structure, Doyne Farmer and Dave Umberger for their work on fat fractals, Tim Sullivan for his contributions to the transient analysis and the critical dynamics, and a special thanks to Ronnie Mainieri for teaching me everything I know about $f(\alpha)$ analysis and scaling function dynamics and for helping prepare parts of these notes. This research was sponsored by the U.S. Department of Energy, Basic Energy Sciences, Division of Materials Science.

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<i>combination</i>	<i>f measured</i>	<i>f calculated</i>
$f_1 - 6f_2$	0.06257355	0.06257358
f_2	0.10124070	—
$8f_2 - f_1$	0.13990820	0.13990782
$2f_2$	0.20248148	0.20248140
$9f_2 - f_1$	0.24114892	0.24114852
$3f_2$	0.30372200	0.30372210
$10f_2 - f_1$	0.34239028	0.34238922
$4f_2$	0.40496285	0.40496280
$f_1 - 2f_2$	0.46753636	0.46753638
$5f_2$	0.50620357	0.50620350
$f_1 - f_2$	0.56877589	0.56877708
$6f_2$	0.60744363	0.60744420
f_1	0.67001778	—

Table 1: Quasiperiodic frequency decomposition showing the deduced combination of the fundamental frequencies f_1 and f_2 , the measured frequency and the calculated frequency for a given combination.

combination frequencies. The mode-locking structure known as the “devil’s” staircase is constructed from a sequence of such spectra as a function of R at fixed Pr . This structure is illustrated in Fig. 11 for experimental data and is qualitatively similar to the circle map version shown in Fig. 8. Note that the range of experimentally accessible winding numbers, $1/8 < W < 2/11$, is substantially less than one. By making a series of such measurements at different values of Pr the locking regions in the $\{R, Pr\}$ parameter space are determined, Fig. 12. The tongues are seen to broaden as Pr decreases (this is why we plot $1/Pr$ in Fig. 12) while the value of the winding number is controlled primarily by changing R . A simple mapping of the control variables R and Pr onto the circle map variables of K and Ω assigns R the role of Ω and Pr the part of $1/K$. In general one is not guaranteed such a direct relationship but it works well here.

The boundaries of the regions where quasiperiodic and mode-locked states are observed are higher dimensional attractors for large R/R_c and the simple quasiperiodic state with no mode-locking at smaller R/R_c [41,46,47,48,50,49]. In this review we will concentrate on the dynamical states that are not higher

Figure 10: Quasiperiodic a) time series of temperature oscillations and b) corresponding power spectrum. Combination frequencies are labeled in the spectrum.

mination over just taking the peak value by as much as 3 orders of magnitude [56]. Using these techniques we can characterize the evolution of dynamical states of the system as R is increased.

The first time dependence begins at the transition to periodic oscillations of frequency f_1 . This transition is a forward Hopf bifurcation [57,58,59] and the onset value depends strongly on Pr . Another Hopf bifurcation at higher R gives rise to a second frequency f_2 , incommensurate with the first. This second mode is only weakly interacting with the initial limit cycle mode and not until there is a discontinuous transition to a different second mode does measurable mode-locking occur. The physical mechanism for this transition is not known. Within a region of parameter space above the discontinuous transition, quasiperiodic (incommensurate frequencies), mode-locked and chaotic states exist. As in the theoretical discussions the state of the dynamics is often well represented by the winding number $W \equiv f_2/f_1$, where f_1 and f_2 are the fundamental frequencies determined from spectral analysis of time series data. (W is related to ρ in the theory as discussed below). In Fig. 10 is a typical quasiperiodic time series and power spectrum in which many distinct peaks are observed. Each peak can be written as a sum, difference, or harmonic of the two fundamental peaks. In table 1 the fundamental frequencies and some of the other peaks are compared with the decomposition into

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