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ENTROPY FOR MOTION ALONG A
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BY

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**PLASMA PHYSICS
LABORATORY**

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CALCULATION OF THE KOLMOGOROV
ENTROPY FOR MOTION ALONG A STOCHASTIC
MAGNETIC FIELD*

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ABSTRACT

An expression for the Kolmogorov entropy has been derived. Excellent agreement between a probability description and direct dynamical computations has been found.

A typical magnetic field of interest to magnetic confinement physics may be taken to be of the form:

$$\vec{B} = B_0 \hat{z} + B_y(x) \hat{y} + \delta \vec{B} , \quad (1)$$

where $|\delta \vec{B}| \ll B_y$, B_0 might result from any short wavelength plasma microinstabilities. When δB exceeds a very small value such that neighboring magnetic islands overlap, the field structure becomes stochastic, and can be characterized by two geometrical properties.¹ First, a given field line diffuses in x , and, second, two neighboring field lines diverge from each other - the mean distance between them increasing as $d \sim \exp(z/L_c)$. The diffusion of a field line has obvious implications for the confinement of particles orbiting along the field and has been calculated using quasilinear theory.² However, the divergence of field lines is also important in calculating transport since it prevents the reversible wandering of a particle back and forth along a single field line as velocity is reversed by collisions.¹ The value $1/L_c$ is called the Kolmogorov entropy and is defined by the formula:³

$$h = \lim_{z \rightarrow \infty} \lim_{d_0 \rightarrow 0} \left[\frac{1}{z} \ln \left(\frac{d(z)}{d_0} \right) \right] . \quad (2)$$

Since exponential divergence is a statistical property one expects that averaging along trajectories ($z \rightarrow \infty$) can be replaced by phase space averaging with a proper distribution function. It is our purpose in this note to introduce such a statistical description and to give a derivation of the dependence of h on the properties of the fields given by Eq. (1). Our main result, Eq. (19),

is similar to that already obtained by Krommes, Kleva and Oberman,⁴ but we have tried to make the basic assumptions more transparent and the derivation more quantitative. We have also developed a simplified model of a stochastic field for which we have computed h directly using Eq. (2). This model allows detailed numerical study which agrees very well with the theoretical predictions.

Consider a magnetic field given by Eq. (1). In order to model poloidal and toroidal periodicity we assume that the system is periodic in the y direction with period $2\pi a$ and in the z direction with period $2\pi R$. Then $\delta\vec{B}$ can be written in the form:

$$\delta\vec{B} = B_0 \sum_{m,n} \vec{b}_{mn}(x) \exp[i(my/a - nz/R)] + c.c. \quad (3)$$

The turbulent nature of $\vec{b}_{mn}$ is introduced by assuming that they have random phases but constant, saturated, amplitudes. Equations for the coordinates of the field lines are:

$$\frac{dx}{dz} = \frac{B_x}{B_0}; \quad \frac{dy}{dz} = \frac{B_y}{B_0}. \quad (4)$$

Assuming shear, $L_s^{-1} = (1/B_0)(dB_y/dx)$, being constant we can write (4) in the dimensionless form:

$$dx'/dz' = b; \quad dy'/dz' = x'. \quad (5)$$

Here $b = \delta B_x/B_0$ is a small parameter in our problem and $x' = x/L_s$, $y' = y/L_s$, $z' = z/L_s$. We will omit primes in the following. In order to calculate h we have to find the divergence rate of two arbitrarily close particle trajectories. Equations for the differences in position $X = x_2 - x_1$ and $Y = y_2 - y_1$ can be easily written using Eq. (5)

$$\frac{dX}{dz} = \frac{\partial b}{\partial x} X + \frac{\partial b}{\partial y} Y, \quad \frac{dY}{dz} = X. \quad (6)$$

It is convenient to put $X = \Lambda Y$ so that

$$\frac{d\Lambda}{dz} = -\Lambda^2 + \frac{\partial b}{\partial x} \Lambda + \frac{\partial b}{\partial y}; \quad \frac{dY}{dz} = \Lambda Y. \quad (7)$$

Since it will turn out later that $\langle \Lambda \rangle = h \sim b^{2/3}$, we may neglect the term $\Lambda(\partial b/\partial x)$ if $h \ll 1$. The same condition allows us to neglect the contribution from the term δB_y in comparison with the shear term in Eq. (6). The case of zero shear has been recently considered by Kadomtsev and Pogutse.⁵ Thus the basic limitation of our theory is determined by the condition $h \ll 1$. Note that we use here dimensionless h appropriate to Eq. (5).

Consider a number of test particles which are distributed initially with the probability $P(x_i, 0)$ (two point distribution function) in a cross section $z = 0$, x_i is a vector with the components (x, y, Λ, Y) . If these particles are moving freely along the field lines remaining in the same cross section z then the evolution of their distribution function $P(x_i, z)$ is described by the continuity equation

$$\frac{\partial P}{\partial z} + \sum_{i=1}^4 \frac{\partial}{\partial x_i} v_i P = 0, \quad (8)$$

where v_i is a generalized velocity with components $(b, x, -\Lambda^2 + \partial b/\partial y, \Lambda Y)$ and the z coordinate plays the role of a time. The Kolmogorov entropy is defined by the formula:

$$h = \frac{\partial}{\partial z} \frac{1}{2} \int \ln(X^2 + Y^2) P d\Lambda dY, \quad (9)$$

$\bar{P}(\lambda, Y, z)$ is a probability averaged over x and y which varies slowly in z . It satisfies the equation:

$$\frac{\partial \bar{P}}{\partial z} - \frac{\partial}{\partial \lambda} \lambda^2 \bar{P} + \frac{\partial b}{\partial Y} \frac{\partial \bar{P}}{\partial \lambda} + \frac{\partial}{\partial Y} \lambda Y \bar{P} = 0. \quad (10)$$

We derive this equation by averaging Eq. (8). The fluctuating part $\tilde{P} = P - \bar{P}$ is small and satisfies a first order equation

$$\frac{\partial \tilde{P}}{\partial z} + \frac{\partial}{\partial Y} x \tilde{P} - \frac{\partial}{\partial \lambda} \lambda^2 \tilde{P} + \frac{\partial}{\partial Y} \lambda Y \tilde{P} = - \frac{\partial b}{\partial x} \bar{P} - \frac{\partial b}{\partial Y} \frac{\partial \bar{P}}{\partial \lambda}. \quad (11)$$

We may neglect the 4th and 5th terms in this equation because $\lambda \ll 1$. We can now solve Eq. (11) taking $\partial \bar{P} / \partial \lambda$ to be slowly varying due to the smallness of b and substitute $\tilde{P}$ into (10) to get

$$\frac{\partial \bar{P}}{\partial z} - \frac{\partial}{\partial \lambda} \lambda^2 \bar{P} + \frac{\partial}{\partial Y} \lambda Y \bar{P} - D_{ef} \frac{\partial^2 \bar{P}}{\partial \lambda^2} = 0, \quad (12)$$

$$D_{ef} = \pi \frac{RL}{a^2} \sum_{m,n} \frac{|mb_{mnx}(x)|^2 \delta(mxR/a - n)}{m^2}. \quad (13)$$

These calculations are very similar to the derivation of the magnetic diffusion coefficient in quasilinear theory.² Notice that the first term on the right hand side of Eq. (10) averages to zero. We will deal later with $\tilde{P}$ and suppress the bar.

A general initial value solution for $P(\lambda, Y, z)$ is impractical to obtain. It is sufficient to look for moments. Let us define

$$P_0(\lambda, z) = \int_{-\infty}^{\infty} P dY \quad \text{and} \quad P_e(\lambda, z) = \frac{1}{2} \int_{-\infty}^{\infty} (\ln Y^2) P dY$$

then

$$\frac{\partial P_0}{\partial z} - \frac{\partial}{\partial \Lambda} \Lambda^2 P_0 - D_{ef} \frac{\partial^2 P_0}{\partial \Lambda^2} = 0, \quad (14)$$

and

$$\frac{\partial P_e}{\partial z} - \frac{\partial}{\partial \Lambda} \Lambda^2 P_e - D_{ef} \frac{\partial^2 P_e}{\partial \Lambda^2} = \Lambda P_0. \quad (15)$$

The steady state solution of Eq. (14) valid for large z , and well behaved at $\Lambda = \pm\infty$ is

$$P_0(\Lambda) = C \exp(-\Lambda^3/3D_{ef}) \int_{-\infty}^{\Lambda} \exp(\Lambda'^3/3D_{ef}) d\Lambda', \quad (16)$$

C is a normalization constant determined from the condition

$\int P_0 d\Lambda = 1$. We may now solve for

$$P_e = h z P_0(\Lambda) + Q(\Lambda), \quad (17)$$

where Q obeys

$$\frac{d}{d\Lambda} \Lambda^2 Q + D_{ef} \frac{d^2 Q}{d\Lambda^2} = (h - \Lambda) P_0. \quad (18)$$

We note that the constant h is just the Kolmogorov entropy as defined by Eq. (9). It can be determined by the solubility condition for Eq. (18). This may be obtained by noting that $\Lambda = \pm\infty$ corresponds to $Y = 0$ [as seen from Eq. (6)]. Since the probability function at $Y = +0$ must be equal to that at $Y = -0$ it follows that $\Lambda^2 Q(+\infty) = \Lambda^2 Q(-\infty)$. Hence integrating Eq. (18) over Λ we deduce that h is given by

$$h = \langle \Lambda \rangle = \frac{1}{2} \frac{(2/3)!}{(1/3)!} (3D_{ef})^{1/3}. \quad (19)$$

Going back to dimensional units we can write $L_c = L_s/h$.

Let us turn now to the results of our numerical computations. We used the following dimensionless units: $2\pi a = 1$,

$2\pi R = 1$, $L_s = 1$ and took $b_{mnx} = \epsilon \exp(i2\pi m\phi_{mp})/2i$, $m = 1, \dots, M$,
 $p = n \text{ modulo } N$, $n = 0 \pm 1, \pm 2, \dots$, where phases ϕ_{mp} are randomly
 choosen numbers between 0 and 1. Summing over n with fixed p in
 Eq. (3) gives us δ functions, allowing differential Eq. (5) to
 be reduced to a simple mapping:

$$\begin{aligned} Y_{i+1} &= Y_i + x_i/N \\ x_{i+1} &= x_i + f(Y_{i+1})/N \end{aligned} \quad (20)$$

Here $f(Y_{i+1}) = \epsilon \sum_{m=1}^M \sum_{p=0}^{N-1} \sin 2\pi [m(Y_{i+1} + \phi_{mp}) - (i+1)p/N]$

and one step corresponds to $\Delta z = 1/N$. Equations (13) and (19)
 applied to this specific model give

$$D_{ef} = \pi^2 \epsilon^2 \sum_{m=1}^M m^2 \approx \pi^2 \epsilon^2 M^3/3, \quad M \gg 1,$$

and

$$h_{th} = 0.54 M \epsilon^{2/3}. \quad (21)$$

On the other hand h has been calculated directly using Eqs. (20)
 and the numerical method described in Ref. 6. To obtain reasonable
 accuracy approximately 10^4 steps were necessary. The results of
 these computations are presented in Fig. 1. One can see a
 remarkable agreement with the theoretical formula (21). We have
 found that h is equal to zero below the stochastic transition
 (where there exist good magnetic surfaces) and has a well defined
 positive value above the stochasticity threshold. This value
 is independent of initial conditions, that is, it appears numerically
 that h has the same value for almost all trajectories. Stochastic
 transition takes place at $\epsilon_M \approx 2/M^3$, in good agreement with the

resonance overlapping criterion.^{1,3} The results do not depend on the step size $\Delta z = 1/N$ in the region $h < N$, but h does depend on N when $h > N$, and scales in a completely different way with ϵ and M in this region:⁷ $h \sim \ln(M^3 \epsilon^2)$. In order to derive a formula for h in this case let us consider eigenvalues of tangential transformation for the mapping (20).³

$$\lambda^{\pm} = 1 + K/2 \pm (K^2/4 + K)^{1/2}, \quad (22)$$

where $K = f'(y_{i+1})/N^2$. If $|K| \gg 1$ then $\lambda^+ \approx K$ and divergence takes place preferentially in the x direction. The Kolmogorov entropy can be easily written in this case³

$$h = \frac{1}{2} N \overline{\ln K^2}. \quad (23)$$

The bar here means averaging along the trajectory. Because $M \gg 1$ and ϕ_{mp} are random we can make use of a central limit theorem and perform averaging in Eq. (23) with the Gaussian distribution $P(K) = \exp(-K^2/2\overline{K^2})/(2\pi\overline{K^2})^{1/2}$ where $\overline{K^2} = (2/3)\pi^2\epsilon^2(M/N)^3$. After a simple calculation we can get from (23):

$$h = \frac{1}{2} N \ln \left[\frac{\pi^2}{3} e^{-c} \left(\frac{M}{N} \right)^3 \epsilon^2 \right]. \quad (24)$$

Here $c = 0.577$ - the Euler constant. This formula is also in very good agreement with computer calculations.⁷ Using Eq. (21) we can rewrite Eq. (24) as $h = (3/2)N \ln(2.3h_{th}/N)$, applicable when $h_{th} > h > N$. But in the region $h_{th} < N$ we have to use Eq. (21).

We believe that our results have some general interest: they illustrate how a statistical description can be introduced in a deterministic but stochastically unstable system.

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⁷More detailed results of these computations will be published
elsewhere.

FIGURE CAPTION

Fig. 1 The numerically obtained Kolmogorov entropy. The solid curve is the theoretical result. Also shown are the stochastic transition points ε_M .

Fig. 1. 792023

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