Establishment of Magnetic Coordinates for a Given Magnetic Field

by

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A method is given for expressing the magnetic field strength in magnetic coordinates for a given field. This expression is central to the study of equilibrium, stability, and transport in asymmetric plasmas.

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I. Introduction

The fundamental importance of coordinate systems based on the magnetic field lines, so-called magnetic coordinate systems, was first appreciated in plasma equilibrium^{1,2} and stability studies.³ More recently it was shown that the transport due to particle drifts and collisions could be evaluated provided the field strength in magnetic coordinates was known.^{4,5} The field strength in magnetic coordinates also permits the calculation of the Pfirsch-Schluter currents⁵ and the v^{**} stability criterion.^{5,6} However, the gyromotion or ordinary classical transport⁵ and the shear stabilization for flutes⁷ require a knowledge of $|\vec{\nabla}\psi|^2$ in magnetic coordinates with $2\pi\psi$ being the toroidal flux inside a pressure surface.

Although the benefits of magnetic coordinates have long been known, the difficulty of solving for the magnetic coordinates, even when the magnetic field is known, has limited the use of these systems. Fortunately, one does not really need expressions for the magnetic coordinates as functions of the ordinary cartesian coordinates. Expressions are required for only a few scalar functions like the field strength and $|\vec{\nabla}\psi|^2$. In this paper a method is given for finding the Fourier decomposition of scalar functions in magnetic coordinates for toroidal, scalar pressure, plasmas. This Fourier decomposition method gives expressions which are especially useful for both analytic and computational work. The simplest scalar function to evaluate in magnetic coordinates is the field strength. Since it is also the most important function, the emphasis of the paper will be in this scalar function. The other scalar function of significant importance, $|\vec{\nabla}\psi|^2$, can in principle be calculated by the same procedure as the field strength. Its calculation, however, is complicated by the numerical problem of

evaluating $|\vec{\nabla}\psi|$ or equivalently the distance between adjacent ψ surfaces as a function of position.

II. Evaluation of the Field Strength

To evaluate the magnetic field strength, a number of properties of the ψ , θ_0 , χ magnetic coordinate system will be used. For scalar pressure equilibria, these coordinates can be defined by⁴

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta_0 \quad (1)$$

$$= \vec{\nabla}\chi + \beta \vec{\nabla}\psi \quad (2)$$

and $2\pi\psi$ equal to the toroidal magnetic flux inside a pressure surface. The definition of the ψ , θ_0 , χ coordinates implies $\theta_0 \rightarrow \theta_0 + \theta_*(\psi)$, $\chi \rightarrow \chi + \chi_*(\psi)$, and $\beta \rightarrow \beta - d\chi_*/d\psi$ gives the same magnetic field $\vec{B}$. Except for the arbitrary functions $\theta_*(\psi)$ and $\chi_*(\psi)$ the ψ , θ_0 , χ coordinates are uniquely defined. The limited arbitrariness of the ψ , θ_0 , χ coordinates implies that in a torus, a poloidal and a toroidal angle θ and ϕ can be defined such that⁵

$$\chi = g(\psi)\phi + I(\psi)\theta \quad (3)$$

$$\theta = \theta_0 + \kappa(\psi)\phi. \quad (4)$$

The two periodicities of the torus are given by θ and ϕ advancing by 2π . One can show $cg/2$ is the total poloidal current lying outside a pressure surface and $cI/2$ is the total toroidal current lying inside a pressure surface. The function $\kappa(\psi)$ is the rotational transform.

The periodicities of the torus imply that any scalar function, in particular $1/B^2$, can be written as

$$\frac{1}{B^2} = \sum_{n,m} [a_{nm} \cos(n\phi - m\theta) + b_{nm} \sin(n\phi - m\theta)] . \quad (5)$$

The convenience of expanding $1/B^2$ rather than the field strength B will become clear. Expressing $n\phi - m\theta$ in terms of χ and θ_0 using Eqs.(3) and (4) one finds

$$n\phi - m\theta = \frac{n - m\chi}{g + \chi I} \chi - \frac{mg + nI}{g + \chi I} \theta_0 . \quad (6)$$

To evaluate the Fourier coefficients, a_{nm} and b_{nm} , a field line integration is started at an arbitrary point. The arbitrariness in θ_0 and χ allows us to assume they are zero at the starting point. Since a field line is defined by ψ and θ_0 constant, θ_0 remains zero as the field line integration maps out the flux surface. Rational magnetic surfaces, which have χ equal to the ratio of two integers, present a problem which will be discussed later. As the integration proceeds, one can evaluate χ , for Eqs. (1) and (2) imply

$$\chi = \int B d\ell \quad (7)$$

with $d\ell$ the differential distance along a line. From the field line integration we obtain $B(\chi)$ with

$$B(\chi) \equiv B(\psi, \theta_0 = 0, \chi) \quad (8)$$

Eq. (5) implies that the known function $B(\chi)$ can be written as

$$\frac{1}{B^2}(\chi) = \sum_{n,m} [a_{nm} \cos(\omega_{nm}\chi) + b_{nm} \sin(\omega_{nm}\chi)] \quad (9)$$

with

$$\omega_{nm} = \frac{n - m\chi}{g + \chi I} \quad (10)$$

A Fourier decomposition $1/B^2(\chi)$ will give the coefficients and the associated discrete frequencies ω_{nm} .

Expressions for $g(\psi)$, $I(\psi)$, and $r(\psi)$ together with the decomposition of $1/B^2(\chi)$ would give the Fourier coefficients and hence $B(\psi, \theta_0, \chi)$ on a ψ surface. In practice $g(\psi)$ and $I(\psi)$ are known due to their relation to the total poloidal and toroidal current and the equations given by Bonzer⁵ for g and I

$$\frac{dg}{d\psi} = -\nu - \left(\frac{4\pi}{B_0} \frac{dP}{d\psi}\right) g \quad (11)$$

$$\frac{dI}{d\psi} = \nu(\psi) - \left(\frac{4\pi}{B_0} \frac{dP}{d\psi}\right) I \quad (12)$$

with $\nu(\psi)$ the force free or $n = 0$, $m = 0$ Fourier component of $j_{\parallel}/B$, $1/B_0^2$ the $n = 0$, $m = 0$ Fourier component of $1/B^2$, and $P(\psi)$ the pressure. In the case of vacuum stellarator fields, $I = 0$ and g is a constant. The rotational transform $r(\psi)$ can be estimated from the field line tracing, which is the way the transform is traditionally computed. A more accurate value for r can be found by adjusting its value so the ω_{nm} have the discrete values given by the Fourier transform of $1/B^2(\chi)$.

Once the Fourier decomposition of $1/B^2$ has been made on one magnetic surface, new starting points for field line integrations must be chosen to evaluate the Fourier coefficients for different values of ψ . One can always make a_0 zero at the new starting point, because other choices differ by an additive function of ψ , which does not change the representation of $\vec{B}$, Eqs. (1) and (2). Unfortunately, one must be more careful with the initial value of χ . Equation (2) implies that the change in χ along an outward normal to the flux surface, $\vec{\nabla}\psi$, is

$$\chi_{,\psi} = -\rho/a_0. \quad (12)$$

Given ψ on the original flux surface, the starting value of χ can be evaluated using this equation. It can be shown⁵ that

$$\chi = \frac{4\pi}{B_0^2} \frac{dP}{d\psi} \quad \chi = a_0 + \rho_* \quad (13)$$

$$\rho_* = \frac{4\pi}{B_0^2} \frac{dP}{d\psi} \sum_{n,m} \frac{g_n + \chi I}{n - \chi m} [a_{nm} \cos(n\psi - m\phi) + b_{nm} \sin(n\psi - m\phi)] \quad (14)$$

with the prime on the sum implying the $n = 0, m = 0$ term is eliminated. For vacuum or force-free magnetic fields, one can set $\chi = 0, a_0 = 0$ all along an outward normal to the magnetic surfaces and choose the starting points for the field line integrations along this line.

On rational surfaces, which have χ the ratio of integers, a field line integration does not cover the magnetic surface. Unless the integers in the ratio are small, a field line integration nonetheless covers the surface adequately to define the Fourier coefficients. For low order rational surface, which have $\chi = N/M$ with N and M less than or of order the n and m of

significant terms in the Fourier decomposition, a field line integration does not adequately cover the surface. However, the evaluation of the Fourier coefficients on neighboring irrational surfaces will specify the coefficients on the low order rational surfaces.

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