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Maxine J. Clover
Contract Specialist
Acquisition and Assistance Unit

AA:MJC

Enclosures:

1. Renewal Proposal w/Budget & Statement of Costs (4 copies)
2. Progress Report (4 copies)

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Sodium Boiling in LMFBF Fuel Assemblies

Progress Report

April 30, 1981

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Formal reporting for this project is annually. Since the next annual report is due at the end of FY 81, this progress report has been compiled in support of the renewal proposal.

MASTER1.0 Objective of Program

The objective of this program is to contribute to the development of an improved understanding of sodium voiding behavior under postulated LMFBF accident conditions.

2.0 Background and Summary of Program

The program consists of development of multi-dimensional computer models under low flow and low power conditions. Two codes are being developed in parallel. The first will use a two-fluid (6 equation) model which is more difficult to develop but has the potential for providing a code with the utmost in flexibility and physical consistency for use in the long term. The other will use a "mixture" (<6 equation) model which is less general but may be more amenable to interpretation and use with available experimental data. Therefore, it may be easier to develop for application in the near term. Both codes are being developed using the existing transient thermal-hydraulic analysis code, THERMIT, as a basis. A major effort has been the modification of the numerical scheme originally developed for water application to one that is suitable for

sodium application. To assure that the codes being developed are not design dependent, geometries and transient conditions typical of both foreign and U.S. designs are being considered for the process of code testing and application.

An effort has been made to maintain close communication with DOE contractors and laboratories in order to: (1) assure that maximum use is made of data and information available from related programs and, (2) facilitate their eventual acquisition and use of the codes.

Activities on the mixture model code was initiated about a year ago. A four equation drift flux approach will be utilized. All necessary two phase correlations as well as the specific numerical technique which will be utilized are now being identified. The two-fluid model code has been under development for two years. The objective of this effort is to develop calculational models for sodium boiling in fast reactor assemblies based on the two-fluid approach. In particular, two codes will be pursued: two dimensional model (NATOF-2D) and a three dimensional code based on the LWR code THERMIT-TF. The models will be verified by comparison to existing experimental results on sodium boiling. Where needed, additional experimental results will be defined.

The NATOF-2D and THERMIT-TF codes are now at an advanced state of development. The importance of the 2D effects under some transient conditions imply that development of these models should be pursued at the same time so that comparison of the 1D and 2D and 3D models will be helpful in verification of the appropriate scaling laws of small bundles usually employed in experiments.

The codes are in a relatively primitive state as regards physical realism. Thus, implementation of the correlations necessary to bring realism into the code is pursued. This means examining all sodium boiling data and creating models and correlations appropriate to the fluid formalism, including:

- 1) description of relative motion of the phases via interphase and wall friction correlations;
- 2) flow boiling heat transfer models;
- 3) flow regime description accounting for both heat transfer and fluid dynamics;
- 4) optionally (if super heat is important), interphase heat and mass transfer rate models.

The development of these multi-dimensional models will allow the study of the sensitivity of sodium boiling consequences to the geometrical effects of different designs. Assessment of such effects will define the design conditions that need to be factored in fast reactor assemblies in order to intensify the inherent safety features.

3.0 Major Program Elements

From the above summary it is apparent that the effort is now split among four subtasks. These are

- Development of NATOF-2D
- Assessment and Improvements to THERMIT-S-6E (the 6 equation two-fluid formulation)
- Development of THERMIT-S-4E (4 equation mixture formulation)
- Assessment of THERMIT-S by Comparison with Single Channel Test Results

In the next section each of these subtasks is discussed in a consistent format.

4.0 Detailed Discussion of Subtasks

4.1 Development of NATOF-2D (R. Zielinski, M. Kazimi)

Statement of Task:

The objective of this work is to improve the constitutive models of NATOF-2D and to develop models for NATOF-2D which were considered necessary to enhance its performance in the simulation of sodium experiments.

Accomplishments to Date:

1) The Direct Solution

As an alternative to using an iterative process to solve the pressure field, NATOF-2D now has the capability to solve this by a direct method. Results show a computational time savings of up to 50% during a transient.

2) Interfacial Heat Transfer

A more advanced model for interfacial heat transfer has been implemented into the code.

3) Interfacial Momentum Exchange

The effects of mass exchange have been added to the interfacial momentum exchange term by use of a modified interfacial shear coefficient.

Work in Progress:

1) Interfacial Mass Exchange

A more exact formulation based on the kinetic theory for interfacial mass exchange is presently being tested.

2) Radial Heat Transfer

Both an explicit and a semi-implicit model for radial heat transfer have been incorporated into the model. Presently, comparisons between

steady state sodium experiments and NATOF-2D simulations are being made in order to estimate a heat transfer coefficient between cells.

3) Specification of the Inlet Flow Rate

The ability to impose an inlet flow rate at the bottom boundary of the test section is being implemented.

Work to be Completed by September 30, 1981:

1) Simulation of Sodium Boiling Experiments

Loss of flow transients, and overpower transients of the W-1 Sodium Boiling Test runs, and the P3A Test runs will be simulated with NATOF-2D in order to test and compare modifications made to the code.

4.2 Assessment and Improvements to THERMIT-S-6E (the 6 equation two-fluid formulation) (H. C. No, M. S. Kazimi)

Statement of Task:

The principal task is to improve the THERMIT-S code for the better experimental simulation of the transient sodium boiling, especially caused by the loss of flow accident, or the low power and low flow.

Accomplishments to Date:

1) Investigation of Mathematical and Numerical Characteristics:

The virtual mass term, neglected terms in the THERMIT-S code, were found to have a profound effect upon the mathematical characteristics and numerical stability. It was concluded that the finite difference scheme with Lahey's model is restricted only by the convective stability conditions in the case that the coefficient of virtual mass, C_v is greater than $\sqrt{4-\alpha(1-\alpha)^3} \rho_v/\rho_\ell$.

2) Solution Scheme Improvement:

Sudden pressure build-up by violent vapor formulation and collapse was found to be likely to lead the divergence of iterated solutions. Several methods were investigated including steepest descent method, hybrid method, and underrelaxation Newton method. So far it was proven that underrelaxation Newton method may be a powerful tool to solve the above difficulties.

3) Physical Model Improvement:

(1) Mass Exchange Model

It was found that our mass exchange model was not exactly derived by kinetic theory. New model strictly based on Schrage's kinetic theory was derived.

(2) Wall Heat Transfer Model

Unlike the present model based on Chen's concept, a new approach based on the analogy between the phenomenological equations for momentum flux and heat flux was tried. Nu from the new model is in good agreement with the approximate predicted Nu .

4) Properties

The properties used in THERMIT-S were reviewed (for detail see Appendix A).

Work in Progress:

1) Solution Scheme Improvement

A more refined solution scheme of Underrelaxation Newton method is being investigated so that the convergence may be guaranteed to the broad extent.

2) Comparison between Old and New Mass Exchange Model

They are being compared with the experimental data by THORS 19 pin bundle single channel simulation.

Work to be Completed by September 30, 1981:

- 1) Putting the virtual mass term and the new wall heat transfer correlation in the THERMIT-S and evaluating their numerical and physical effects.
- 2) THORS 19 pin bundle multichannel simulation will be done with the modified version.

4.3 Development of THERMIT-S-4E (4 equation mixture formulation) (A. Schor, N. E. Todreas)

Statement of Task:

Development of a two-velocity, thermal equilibrium, multidimensional two-phase flow model for the analysis of sodium boiling phenomena in LMFR's.

Accomplishments to Date:

- 1) Mathematical and Numerical Formulation
 - a) Refined the pressure field solution and provided solvers optimized for various situations (1-D, 2-D, 3-D), open/isolated channels;
 - b) Consistent 4- and 6-equation formulation within the same computational framework;
 - c) Improved boundary condition treatment;
 - d) Developed and implemented a total inlet flow rate boundary condition;
 - e) Better representation of geometrical parameters in the context of porous body formalism.

2) Code Restructuring

- a) Major input modifications toward facilitating the user's effort;
- b) Restructuring of the code for computing efficiency;
- c) Time step control based on accuracy requirements added as an option; particularly important during phase changes when key variables (pressure, density, energy) undergo large and sudden changes.

Work in Progress:

- 1) Study of the characteristics of the 4-equation model.
- 2) Nonequilibrium (superheat) mixture density; possible extension to condensation.
- 3) Conversion to conservative momentum equations to improve mass conservation.
- 4) GR19 steady-state boiling simulation.

Work to be Completed by September 30, 1981:

- 1) Boiling window experiment: W1, run 2 and 7B (low and high power).
- 2) Skewed power distribution effects: Thors, bundle 6A, run 37.

4.4 Assessment of THERMIT-S by Comparison with Single Channel Test Results (K. Huh, N. Todreas)

Statement of Task:

Performing analysis of single channel steady-state and transient sodium boiling experiments with THERMIT-S and comparing results to data and upgrading the constitutive relations as necessary.

Accomplishments to Date:

- 1) Cases for investigation have been defined:
 1. natural convection (sodium and water)
 2. steady-state boiling
 3. loss-of-flow (slow vs. fast)

4. overpower transients

2) Analysis in detail of SBTF cases

For the present, the analysis will be confined to test section with appropriate boundary conditions at each end. (It will later be extended to the whole loop analysis.) The important characteristics of this experiment is that it is a natural convection and loop analysis. Since it is a natural convection, the sodium velocity is extremely low compared with forced convection. The controlled variable is power input to the test section, and the pressure, flow rate and temperatures are measured at the various points.

It has been found that the power that was actually input to the test section is appreciably lower than the values listed in the report, and the fractional error in power input is greater at high powers.

3) Running THERMIT-S for SBTF cases

The steady-state single phase cases were run with THERMIT-S with only 70% of the power listed in the report. It is expected that 70% power is a reasonable estimate of the power. With the calibrated power and the range of the errors in pressure and flow meter, the analysis will be extended to two-phase cases in steady-state. Here the steady-state means the average behavior of the data over longer time steps.

Work in Progress:

1) Modifying the constitutive relations for single tube geometry is almost done. The subroutines related with wall friction and wall heat transfer were modified for the circular geometry. The interfacial mass transfer term is also modified using the same physical concept but with different geometry in mind. The remaining problem is the interfacial

heat transfer term which becomes important in severe transients.

2) Simulating the whole loop with THERMIT-S which is essentially a component code. The most promising scheme for the present is to put a block of a large number of interior nodes in a rectangular array close the inlet and outlet. This will yield the representation of a loop geometry. The remaining problem is how to simulate the upper and lower plenums which maintain the constant temperatures.

3) Replacing the water properties for the sodium properties which already exist in THERMIT-S in order to extend the range of applicability to water experiments.

Work to be Completed by September 30, 1981:

The simulation of the natural convection SBTF case for a loop geometry will be completed.

Appendix A

Properties

The properties used in THERMIT-S, THERMIT-NA, and NATOF were primarily taken from Golden and Tokan. These properties include the following assumptions:

1. A perfect gas behavior for the vapor phase was assumed.
2. A constant sonic speed was taken for the estimation of liquid compressibility.
3. The subcooled effect of liquid was neglected. These assumptions are examined as follows. The equation of state for a real gas can be expressed in the following virial form:

$$\frac{Pv}{RT} = 1 + \frac{B}{v} + \frac{C}{v^2} + \frac{D}{v^3} \dots \quad (A.1)$$

Stone et al. fitted Eq. (A.1) to their P-V-T data and obtained the following expressions for the coefficients:

$$B = -4.44734 \times 10^{-5} T \exp (15554.65/T) : \text{ft}^3/\text{mole}$$

$$C = 0.243388 \exp (24958/T) : \text{ft}^6/\text{mole}^2$$

$$D = -0.811895 \exp (31195/T) : \text{ft}^9/\text{mole}^3$$

The calculated value of $\frac{B}{v}$ in Eq. (A.1) with the typical conditions used in the accident analysis of LMFBR is less than 10^{-3} . In the same conditions one percentage error introduced by the second assumption is less than $10^{-2}\%$.

In order to analyze the effect of the third assumption on the subcooled liquid enthalpy, we use Maxwell's relation and get the following expression

$$\begin{aligned}
dh(T,p) &= \left(\frac{\partial h}{\partial T} \right)_p dT + \left(\frac{\partial h}{\partial p} \right)_T dp \\
&= C_p(T,p) dT + \left[v - T \left(\frac{\partial v}{\partial T} \right)_p \right] dp
\end{aligned} \tag{A.2}$$

Integrating Eq. (A.2) from the system pressure, p to the saturation pressure, $p_{\text{sat}}(T)$ corresponding to the system temperature, T and assuming that $\left[v - T \left(\frac{\partial v}{\partial T} \right)_p \right]$ is constant, we obtain the expression for the enthalpy of subcooled liquid

$$h_\ell(T,p) - h_\ell(T,p_{\text{sat}}(T)) + \left[v_\ell - T \left(\frac{\partial v_\ell}{\partial T} \right)_p \right] (p - p_{\text{sat}}(T))$$

The percentage error introduced by the third assumption at $T = 661 \text{ K}$ and $p = 1.5 \times 10^5 \text{ N/m}^2$ is less than 0.1%

Therefore, the above three assumptions added less error than the amounts of deviation of the equation from data. We conclude that the properties used in our codes represent well the real properties.

Appendix B

THE EFFECT OF VIRTUAL MASS ON THE CHARACTERISTICS
AND THE NUMERICAL STABILITY IN TWO-PHASE FLOWS

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April 1981

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THE EFFECT OF VIRTUAL MASS ON THE CHARACTERISTICS AND THE NUMERICAL STABILITY IN TWO-PHASE FLOWS

ABSTRACT

It is known that the typical six equation two-fluid model of the two-phase flow possesses complex characteristics, exhibits unbounded instabilities in the short-wavelength limit and constitutes an ill-posed initial value problem. Among the suggestions to overcome these difficulties, one model for the virtual mass force terms was studied here, because the virtual mass represents real physical effects to accomplish the dissipation for numerical stability. It was found that the virtual mass has a profound effect upon the mathematical characteristic and numerical stability. Here a quantitative bound on the coefficient of the virtual mass terms was suggested for mathematical hyperbolicity and numerical stability. It was concluded that the finite difference scheme with the virtual mass model is restricted only by the convective stability conditions with the above suggested value.

INTRODUCTION

Transient two-phase flow analysis is of importance in nuclear reactors under various accident conditions. Among the several models of two-phase flow the two-fluid model offers the most detailed and general description of two-phase flow. But it was reported in early work that the model has inherent instability problems. In 1965 Jarvis [1] tried to solve the two-phase equations using the two-fluid model for modeling the cooldown process in cryogenics and faced instability problems. He attributed the instabilities he encountered to the fact that the system was found to be nonhyperbolic. In 1967 Richtmyer and Morton [2] showed that, if the IVP is ill-posed, then no difference scheme that is consistent with the problem can be stable. In 1971 Siegmann encountered numerical instabilities in his transient sodium boiling code called MOC. In 1973 Boure appears to have encountered severe stability difficulties in the GEVATRAN code. In a round table discussion during the Fifth International Heat Transfer Conference [5], the instability problems in the two-fluid model was shown to be caused by an ill-posed problem. In 1976 Bryce [6] experienced large-scale pressure oscillations with RELAP-UK code. He demonstrated that simple two-fluid model with complex characteristics does not give solutions which converge as the mesh size and time step size are refined. He concluded that, though the solution may be obtained by using numerical technique, the significance of these solutions is not clear. In 1976 Lyczkowski [7] did a sample calculation illustrating error growth caused by complex characteristics. In the 1977 ANS topical meeting of thermal reactor safety, Anderson [8] reported his attempt to achieve stability by using virtual mass terms in his RISQUE code which was proposed

by Hilprecht. It was found that the virtual mass has a profound effect on the dynamics of two-phase flow. His code was used to perform numerical calculations on the behavior of interfacial waves. It was observed that for no virtual mass, the wave amplitude grows rapidly, but the computed result with virtual mass shows a stable oscillatory behavior for the wave amplitude. In 1979 Rivard and Travis [9] successfully dealt with critical flow with the two-fluid code, K-FIX. But for the isolated results presented, the value of the interfacial drag function was chosen sufficiently large that there is no mean relative motion between the phases. In 1979 investigators at R.P.I. [10] showed that without virtual mass not only was it more costly to run the problem, but they could not even run the complete problem using their code, GEAR. Computer running time without virtual mass was forty times longer than that with virtual mass.

It may well be argued that equation sets with complex characteristics may still be adequate for a range of phenomena if the numerical method introduces sufficient dissipation to damp the high frequency instabilities. But obviously there are real physical effects to accomplish the dissipation needed for numerical stability. Among several candidates suggested for numerical stability, Cheng and Lahey's model [10] with the virtual mass terms will be studied here.

A. ILL-POSED PROBLEMS AND MATHEMATICAL STABILITY OF THE INITIAL VALUE PROBLEM (IVP)

Quantitative physical laws are idealizations of reality. As knowledge grows, we observe that a given physical situation can be idealized mathematically in a number of different ways. It is therefore important to characterize those reasonable ideal formulations. Hadamard [11] examined this problem, asserting that a physical formulation is well-posed if its solution exists, is unique, and depends continuously on the external (boundary) conditions. Existence and uniqueness are an affirmation of its principle of determination without which experiments could not be repeated with the expectation of consistent data. The continuous dependence criterion is an expression of the stability of the solution; a small change in any of the problem's data should produce only a correspondingly small change in the solution. The general one-dimensional problem can be represented in matrix form, as follows:

$$A(x) \frac{\partial x}{\partial t} + B(x) \frac{\partial x}{\partial z} + C(x) = 0, \quad (A.1)$$

where

x is a column vector of independent variables,
 $A(x)$ and $B(x)$ are the coefficient matrices, and
 $C(x)$ is a source or sink vector.

The IVP under consideration is to find a solution of Eq. (A.1) in some region

$$a \leq z \leq b,$$

subject to the initial condition

$$x(0,z) = u_0(z), \quad (\text{A.2})$$

and the value of x or its derivatives prescribed on the boundaries,

$$z = a \text{ and } z = b. \quad (\text{A.3})$$

When the definition of a well-posed problem by Hadamard is applied to the above system, the IVP is said to be well-posed if the solution of Eq. (A.1) exists, is unique, and depends continuously on both of the initial condition, Eq. (A.2) and the boundary conditions Eq. (A.3). Also system Eq. (A.1) is defined as hyperbolic if all values of the characteristics of Eq. (A.1) are distinct and real. According to Lax [12] the requirement of a well-posed problem in the linear partial differential equations with the form like Eq. (A.1) is the same as that of hyperbolicity. Along characteristic curves the highest-order derivatives in a PDE are indeterminate; characteristics may separate discontinuities in the solution. Therefore characteristics are trajectories of discontinuities in the solution. Physically it represents the travel of information in a physical system. The typical two-fluid models have four real characteristics (two convective velocities of liquid and vapor, and two relative sound velocities to both convective velocities of liquid and vapor through system) and two imaginary characteristics. The two imaginary characteristics may represent two relative velocities of interfacial wave propagation to both liquid and vapor with complex values which indicate an instability of interfacial waves.

Characteristics and stability are shown to be related only in the limit of large frequency. The characteristics, μ of Eq. (A.1) are defined by the equation

$$\text{Det } (\mu A - B) = 0 \quad (\text{A.4})$$

The local linear stability behavior of Eq. (A.1) is examined. x is replaced by $x_0 + \delta x$, and the result is linearized with respect to δx , resulting in

$$\begin{aligned} A(x_0) \frac{\partial(\delta x)}{\partial t} + B(x_0) \frac{\partial(\delta x)}{\partial z} + (\delta x) \left(\frac{\partial A}{\partial x} \right)_{x_0} \frac{\partial x_0}{\partial t} \\ + (\delta x) \left(\frac{\partial B}{\partial x} \right)_{x_0} \frac{\partial x_0}{\partial z} + \delta x \left(\frac{\partial C}{\partial x} \right)_{x_0} = 0 \end{aligned} \quad (\text{A.5})$$

As Eq. (A.5) describes small perturbations, δx about an unperturbed solution, a uniform steady-state unperturbed solution is assumed, which means that x_0 is independent of z and t . We take a wave form for the perturbed amount, δx

$$\delta x = \delta x_0 \exp[i(kz - \omega t)] \quad (\text{A.6})$$

Then Eq. (A.5) becomes

$$-i\omega A(x_0) \delta x_0 + ik B(x_0) \delta x_0 + \delta x_0 \left(\frac{\partial C}{\partial x} \right)_{x_0} = 0 \quad (\text{A.7})$$

Equation (A.7) is a homogeneous linear system in the components of δx . For a nontrivial solution the determinant of the coefficient matrix must vanish.

$$\text{Det}(-i\omega A + ikB + D) = 0, \quad (\text{A.8})$$

where

$$D = \left(\frac{\partial C}{\partial x} \right)_{x_0}^T, \quad (\text{A.9})$$

the superscript T denoting the matrix transpose. For nonzero k Eq. (A.8) can be rewritten as

$$\text{Det}\left(\frac{\omega}{k} A - B + \frac{i}{k} D\right) = 0 \quad (\text{A.10})$$

Inspection of Eq. (A.6) shows that the condition for stability is that $\text{Im}(\omega) \leq 0$ for all roots ω . Comparison of Eq. (A.4) and Eq. (A.10) shows that, if $D = 0$ or $k \rightarrow \infty$, the dispersion relations can be obtained from the characteristics simply by equating ω/k to μ . Physically the instability at long wavelengths is the well-known Helmholtz instability. But the system which possesses complex characteristics exhibits unphysical and unbounded instabilities in the short-wavelength limit. One of the reasons why the basic system behaves like the above lies in the simplest choice for the interfacial pressure distribution in which it is constant and equal to the bulk pressure. Physically the above choice is consistent with the assumption that the two fluid velocities are equal. Therefore

this leads to an equation set possessing real characteristics only when the two fluid velocities are equal. This is the reason for the present study of the effect of the virtual mass here.

B. CHARACTERISTIC AND STABILITY ANALYSIS

The effect of virtual mass on the characteristics can be illustrated by adding the virtual mass terms to the one-dimensional momentum equations of both liquid and vapor. Here Cheng and Lahey's model for the virtual mass terms is considered. The conservation equations are as follows:

Conservation of vapor mass:

$$\frac{\partial}{\partial t} (\alpha_v \rho_v) + \frac{\partial}{\partial z} (\alpha_v \rho_v V_v) = \Gamma \quad (\text{B.1})$$

Conservation of liquid mass:

$$\frac{\partial}{\partial t} (\alpha_l \rho_l) + \frac{\partial}{\partial z} (\alpha_l \rho_l V_l) = -\Gamma \quad (\text{B.2})$$

Conservation of vapor momentum:

$$\begin{aligned} \alpha_v \rho_v \frac{\partial V_v}{\partial t} + \alpha_v \rho_v V_v \frac{\partial V_v}{\partial z} \\ = \Gamma (V_i - V_v) - \alpha_v \frac{\partial P}{\partial z} - \alpha_v [F_s + F_v] - \alpha_v \rho_v g - F_{wv}, \end{aligned} \quad (\text{B.3})$$

where

V_i , F_s , F_v and F_{wv} are the interfacial velocity, the standard drag force per unit volume, the virtual mass force per unit volume, the vapor wall friction force per unit volume respectively.

$$F_s = k(V_v - V_l) \quad (\text{B.4})$$

$$F_v = \rho_l C_v \left\{ \frac{\partial v_v}{\partial t} - \frac{\partial v_l}{\partial t} + v_v \frac{\partial}{\partial z} (v_v - v_l) + (v_v - v_l) \left[(\lambda - 2) \frac{\partial v_v}{\partial z} + (1 - \lambda) \frac{\partial v_l}{\partial z} \right] \right\} \quad (B.5)$$

Conservation of liquid momentum

$$\alpha_l \rho_l \frac{\partial v_l}{\partial t} + \alpha_l \rho_l v_l \frac{\partial v_l}{\partial z} = \Gamma(v_l - v_i) - \alpha_l \frac{\partial P}{\partial z} + \alpha_v [F_s + F_v] - \alpha_l \rho_l g - F_{wl}, \quad (B.6)$$

where F_{wl} is the liquid with friction force per unit volume. For convenience by summing up Eq. (B.3) and Eq. (B.6), we can get the momentum equation

$$\alpha_v \rho_v \frac{\partial v_v}{\partial t} + \alpha_l \rho_l \frac{\partial v_l}{\partial t} + \alpha_v \rho_v v_v \frac{\partial v_v}{\partial z} + \alpha_l \rho_l v_l \frac{\partial v_l}{\partial z} = -\Gamma(v_l - v_v) - \frac{\partial P}{\partial z} - (\alpha_v \rho_v + \alpha_l \rho_l) g - (F_{wv} + F_{wl}) \quad (B.7)$$

By multiplying Eq. (B.3) by α_l and Eq. (B.6) by α_v and then subtracting each other, we obtain

$$\begin{aligned} & \alpha_v (\alpha_l \rho_v + \rho_l C_v) \frac{\partial v_v}{\partial t} - \alpha_v (\alpha_l \rho_l + \rho_l C_v) \frac{\partial v_l}{\partial t} \\ & + \{ [\alpha_l \alpha_v \rho_v + \alpha_v \rho_l C_v (\lambda - 1)] v_v - \alpha_v \rho_l C_v (\lambda - 2) v_l \} \frac{\partial v_v}{\partial z} \\ & - \{ [\alpha_l \alpha_v \rho_l + \alpha_v \rho_l C_v (1 - \lambda)] v_l + \alpha_v \rho_l C_v \lambda v_v \} \frac{\partial v_l}{\partial z} \\ & = \Gamma(v_i - \alpha_v v_l - \alpha_l v_v) - \alpha_v F_s + \alpha_l \alpha_v (\rho_l - \rho_v) g - \alpha_l F_{wv} + \alpha_v F_{wl} \end{aligned} \quad (B.8)$$

Conservation of vapor energy:

$$\frac{\partial}{\partial t} (\alpha_v \rho_v e_v) + \frac{\partial}{\partial z} (\alpha_v \rho_v e_v V_v) + P \frac{\partial}{\partial z} (\alpha_v V_v) + P \frac{\partial \alpha_v}{\partial t} = Q_{wv} + Q_i \quad (B.9)$$

where Q_{wv} and Q_i are the heat transfer per unit volume from wall to vapor and from interfacial, respectively.

Conservation of liquid energy:

$$\frac{\partial}{\partial t} (\alpha_l \rho_l e_l) + \frac{\partial}{\partial z} (\alpha_l \rho_l e_l V_l) + P \frac{\partial}{\partial z} (\alpha_l V_l) - P \frac{\partial \alpha_v}{\partial t} = Q_{wl} - Q_i, \quad (B.10)$$

where Q_{wl} is the heat transfer per unit volume from wall to liquid.

Let us construct the form of

$$A(x) \frac{\partial x}{\partial t} + B(x) \frac{\partial x}{\partial z} + C(x) = 0, \quad (B.11)$$

where x is a column vector of independent variables,

$$x = (\alpha, P, V_v, V_l, e_v, e_l)^T$$

It is assumed that the density of both liquid and vapor is the function of pressure. Here a_v and a_l denote the sound velocity of vapor and liquid, respectively.

$$A = \begin{bmatrix} \rho_v & \alpha_v a_v^{-2} & & & & \\ -\rho_\ell & \alpha_\ell a_\ell^{-2} & & & & \\ & & \alpha_v \rho_v & \alpha_\ell \rho_\ell & & \\ & & \alpha_\ell \rho_v + \rho_\ell C_v & -\alpha_\ell \rho_\ell - \rho_\ell C_v & & \\ \rho_v e_v + P & \alpha_v e_v a_v^{-2} & & & \alpha_v \rho_v & \\ -\rho_\ell e_\ell - P & \alpha_\ell \rho_\ell a_\ell^{-2} & & & & \alpha_\ell \rho_\ell \end{bmatrix} \quad (B.12)$$

$$B = \begin{bmatrix} \rho_v V_v & \alpha_v V_v a_v^{-2} & \alpha_v \rho_v & & \\ -\rho_\ell V_\ell & \alpha_\ell V_\ell a_\ell^{-2} & & \alpha_\ell \rho_\ell & \\ & 1 & \alpha_v \rho_v V_v & \alpha_\ell \rho_\ell V_\ell & \\ & & [\alpha_\ell \rho_v + \rho_\ell C_v (\lambda-1)] V_v & -[(1-\alpha) \rho_\ell + \rho_\ell C_v (1-\lambda)] V_\ell & \\ & & -\rho_\ell C_v (\lambda-2) V_\ell & -\rho_\ell C_v V_v \lambda & \\ \rho_v e_v V_v + P V_v & \alpha_v e_v V_v a_v^{-2} & \alpha_v \rho_v e_v + P \alpha_v & & \alpha_v \rho_v V_v \\ -\rho_\ell e_\ell V_\ell - P V_\ell & (\alpha_\ell) e_\ell V_\ell a_\ell^{-2} & & \alpha_\ell \rho_\ell e_\ell - \alpha_v P & \alpha_\ell \rho_\ell V_\ell \end{bmatrix} \quad (B.13)$$

In order to get the characteristics from Eq. (A.2)

$$\text{Det} (\mu A - B) = 0 \quad (B.14)$$

Reducing the terms related to the energy equations from Eq. (B.12),
we get

$$\alpha_v \rho_v (\mu - V_v) \times \alpha_l \rho_l (\mu - V_l) \times$$

$$\text{Det} \begin{bmatrix} \rho_v (\mu - V_v) & \alpha_v a_v^{-2} (\mu - V_v) & -\alpha_v \rho_v & \\ -\rho_l (\mu - V_l) & \alpha_l a_l^{-2} (\mu - V_l) & & -\alpha_l \rho_l \\ 0 & -1 & \alpha_v \rho_v (\mu - V_v) & \alpha_l \rho_l (\mu - V_l) \\ 0 & 0 & (\alpha_l \rho_v + \rho_l C_v) (\mu - V_v) - (\alpha_l \rho_l + \rho_l C_v) (\mu - V_l) & \\ & & -\rho_l C_v (\lambda - 2) (V_v - V_l) & +\rho_l C_v \lambda (V_v - V_l) \end{bmatrix} = 0 \quad (\text{B.15})$$

From the terms related to the energy equations we obtained $\mu = V_v, V_l$. Physically this means that energy is transferred by only convection. For simplicity with the assumption that $a_v \gg V_v$ and $a_l \gg V_l$, that is, incompressible flow is used, we neglect

$$\alpha_l a_l^{-2} (\mu - V_l) \text{ and } \alpha_v a_v^{-2} (\mu - V_v)$$

We may predict that these terms are related to the sonic velocities transferred through liquid and vapor.

Now we get the simplified determinant form

$$\text{Det} \begin{bmatrix} \rho_v (\mu - V_v) & -\alpha_v \rho_v & 0 \\ -\rho_l (\mu - V_l) & 0 & -\alpha_l \rho_l \\ 0 & (\alpha_l \rho_v + \rho_l C_v) (\mu - V_v) - (\alpha_l \rho_l + \rho_l C_v) (\mu - V_l) \\ & -\rho_l C_v (\lambda - 2) (V_v - V_l) & +\rho_l C_v \lambda (V_v - V_l) \end{bmatrix} = 0 \quad (\text{B.16})$$

From Eq. (B.16) we obtain an algebraic equations of the form

$$aq^2 + bq + c = 0, \quad (B.17)$$

where $a = \alpha_\ell^2 \rho_v + \alpha_v \alpha_\ell \rho_\ell + \gamma$

$$b = [\gamma(2-\lambda) + 2\alpha_v \alpha_\ell \rho_\ell] V_r$$

$$c = [\alpha_v \alpha_\ell \rho_\ell + \alpha_v \gamma(1-\lambda)] V_r^2$$

$$\gamma = \rho_\ell C_c$$

$$q = \mu - V_v$$

$$V_r = V_v - V_\ell$$

In order for our system to have real and distinct characteristics

$$b^2 - 4ac > 0$$

$$\frac{b^2 - 4ac}{V_r^2} = [\gamma^2(\lambda^2 + 4(1-\lambda)\alpha_\ell) + 4\gamma(1-\lambda)\alpha_v \alpha_\ell^2(\rho_\ell - \rho_v) - 4\alpha_v \alpha_\ell^3 \rho_v \rho_\ell] \quad (B.18)$$

i) There is no virtual mass; $C_v = 0$

$$b^2 - 4ac = -4\alpha_v \alpha_\ell^3 \rho_v \rho_\ell$$

Therefore the system with no virtual mass always has two complex characteristics except the single phase region

$$(\alpha_v = 0 \text{ or } \alpha_\ell = 0)$$

$$\text{ii) If } \alpha_v \rightarrow 0, b^2 - 4ac = \gamma^2(\lambda - 2)^2$$

$$\text{If } C_v \neq 0 \text{ and } \lambda \neq 2, \lim_{\alpha_v \rightarrow 0} (b^2 - 4ac) > 0$$

$$\text{iii) If } \alpha_\ell \rightarrow 0, b^2 - 4ac = \gamma^2 \lambda^2$$

$$\text{If } C_v \neq 0 \text{ and } \lambda \neq 0, \lim_{\alpha_\ell \rightarrow 0} (b^2 - 4ac) > 0$$

$$\text{iv) If } \lambda = 1, b^2 - 4ac = \gamma^2 - 4\alpha_v \alpha_\ell^3 \rho_v \rho_\ell$$

$$\text{If } C_v > \sqrt{\frac{4 \alpha_v \alpha_\ell^3 \rho_v}{\rho_\ell}}, b^2 - 4ac > 0 \quad (\text{B.19})$$

$$\begin{aligned} \text{v) If } 0 \leq \lambda < 1, C_v^2 [\lambda^2 + 4(1-\lambda)\alpha_\ell] + 4 \alpha_v \alpha_\ell^2 \gamma(1-\lambda)(1 - \frac{\rho_v}{\rho_\ell}) \\ > 4 \alpha_v \alpha_\ell^3 \frac{\rho_v}{\rho_\ell} \end{aligned}$$

$$\text{As } 4(1-\lambda)\alpha_\ell > 0 \text{ and } 4\alpha_v \alpha_\ell^2 \gamma(1-\lambda)(1 - \frac{\rho_v}{\rho_\ell}) > 0,$$

$$C_v^2 \lambda^2 > 4\alpha_v \alpha_\ell^3 \frac{\rho_v}{\rho_\ell}$$

$$(C_v \lambda) > \sqrt{4\alpha_v \alpha_\ell^3 \frac{\rho_v}{\rho_\ell}}$$

$$\text{vi) If } 1 < \lambda < 2, C_v^2 (\lambda - 2)^2 - 4\alpha_v \alpha_\ell^2 C_v > 4\alpha_v \alpha_\ell^3 \frac{\rho_v}{\rho_\ell}$$

$$C_v^2 (\lambda - 2)^2 > 4\alpha_v \alpha_\ell^2 [C_v + \alpha_\ell \frac{\rho_v}{\rho_\ell}]$$

Note that as λ approaches 2, the required value of C_v becomes larger and larger.

The effect of virtual mass on stability can be simply illustrated by using the characteristics analysis. If the mass exchange, standard drag force, wall friction, and the energy equations are neglected in order to know clearly the effect of virtual mass on stability, D in Eq. (A.10) reduces to zero.

Now as stated before, the dispersion relationship can be obtained from the characteristics simply by equating $\frac{w}{k}$ to μ .

$$w = k\mu$$

The stability condition, $\text{Im}(w) \leq 0$ for all roots w is the same as the condition for the real and distinct characteristics.

C. NUMERICAL STABILITY

Here we shall examine a difference scheme formed in a similar way to THERMIT [13]. For the same reason as in the stability analysis, mass exchange, standard drag force, and wall friction are neglected. Also energy equations are dropped because they only represent convective properties as we say in the characteristic analysis

vapor mass

$$\frac{(\alpha_v \rho_v)_j^{n+1} - (\alpha_v \rho_v)_j^n}{\Delta t} + \frac{1}{\Delta z} [(\alpha_v \rho_v^n v_v^{n+1})_{j+1/2} - (\alpha_v \rho_v^n v_v^{n+1})_{j-1/2}] = 0 \quad (C.1)$$

liquid mass

$$\frac{(\alpha_\ell \rho_\ell)^{n+1} - (\alpha_\ell \rho_\ell)^n}{\Delta t} + \frac{1}{\Delta z} [(\alpha_\ell \rho_\ell^n v_\ell^{n+1})_{j+1/2} - (\alpha_\ell \rho_\ell^n v_\ell^{n+1})_{j-1/2}] = 0 \quad (C.2)$$

momentum equations

$$\begin{aligned} & (\alpha_v \rho_v)^n \frac{(v_v^{n+1} - v_v^n)_{j+1/2}}{\Delta t} + (\alpha_\ell \rho_\ell)^n \frac{(v_\ell^{n+1} - v_\ell^n)_{j+1/2}}{\Delta t} + (\alpha_v \rho_v v_v \frac{\Delta v_v}{\Delta z})_{j+1/2}^n \\ & + (\alpha_\ell \rho_\ell v_\ell \frac{\Delta v_\ell}{\Delta z})_{j+1/2}^n + \frac{(p_{j+1} - p_j)^{n+1}}{\Delta z} = 0 \end{aligned} \quad (C.3)$$

$$\begin{aligned} & \alpha_v^n [\alpha_\ell \rho_v + \rho_\ell C_v]^n \frac{(v_v^{n+1} - v_v^n)_{j+1/2}}{\Delta t} - \alpha_v^n [\alpha_\ell \rho_\ell + \rho_\ell C_v]^n \frac{(v_\ell^{n+1} - v_\ell^n)_{j+1/2}}{\Delta t} \\ & + \{ \langle \alpha_\ell \alpha_v \rho_v + \alpha_v \rho_\ell C_v (\lambda - 1) \rangle v_v - \alpha_v \rho_\ell C_v (\lambda - 2) v_\ell \rangle \frac{\Delta v_v}{\Delta z} \\ & - \langle [\alpha_\ell \alpha_v \rho_\ell + \alpha_v \rho_\ell C_v (1 - \lambda)] v_\ell + \alpha_v \rho_\ell C_v \lambda v_v \rangle \frac{\Delta v_\ell}{\Delta z} \}^n_{j+1/2} = 0 \end{aligned} \quad (C.4)$$

The above numerical scheme can be modified as follows:

vapor mass

$$\begin{aligned} & \rho_{vj}^{n+1} (\alpha_v^{n+1} - \alpha_v^n)_j + \alpha_{vj}^n (\rho_v^{n+1} - \rho_v^n)_j + \frac{\Delta t}{\Delta z} [\rho_{vj}^n v_{vj+1/2}^{n+1} \\ & (\alpha_{vj}^n - \alpha_{vj-1}^n) + \alpha_{vj-1}^n v_{vj+1/2}^{n+1} (\rho_{vj}^n - \rho_{vj-1}^n) \\ & + \alpha_{vj-1}^n \rho_{vj-1}^n (v_{vj+1/2}^{n+1} - v_{vj-1/2}^{n+1})] = 0 \end{aligned} \quad (C.6)$$

momentum equations

$$\begin{aligned} & (\alpha_v \rho_v)^n (v_v^{n+1} - v_v^n)_j + (\alpha_\ell \rho_\ell)^n (v_\ell^{n+1} - v_\ell^n)_j \\ & + \frac{\Delta t}{\Delta z} [(\alpha_v \rho_v v_v)^n (v_v - v_{vj-1})^n + (\alpha_\ell \rho_\ell v_\ell)^n (v_\ell - v_{\ell j-1})^n \\ & + (p_{j+1/2} - p_{j-1/2})^{n+1}] = 0 \end{aligned} \quad (C.7)$$

$$\begin{aligned} & [\alpha_\ell \rho_v + \rho_\ell C_v]^n (v_v^{n+1} - v_v^n)_j - [\alpha_\ell \rho_\ell + \rho_\ell C_v]^n (v_\ell^{n+1} - v_\ell^n)_j \\ & + \{ < [\alpha_\ell \rho_v + \rho_\ell C_v (\lambda - 1)] v_v - \rho_\ell C_v (\lambda - 2) v_\ell > \frac{\Delta t}{\Delta z} (v_{vj} - v_{vj-1}) \\ & - < [\alpha_\ell \rho_\ell + \rho_\ell C_v (1 - \lambda)] v_\ell + \rho_\ell C_v \lambda v_v > \frac{\Delta t}{\Delta z} (v_{\ell j} - v_{\ell j-1}) > \}^n \end{aligned} \quad (C.8)$$

Treating the coefficients as constants and using Von Neumann local stability analysis, $U_j^n = \zeta^n \exp(ikj\Delta z)$, we obtain the determinant form for a nontrivial solution.

$$\text{Det} \begin{bmatrix} \rho_v(\zeta-1+\tilde{V}_v) & a_v^{-2}\alpha_v(\zeta-1+\tilde{V}_v) & \alpha_v\rho_v\zeta q & 0 \\ -\rho_\ell(\zeta-1+\tilde{V}_\ell) & a_\ell^{-2}\alpha_\ell(\zeta-1+\tilde{V}_\ell) & 0 & \alpha_\ell\rho_\ell\zeta q \\ 0 & \zeta q & \alpha_v\rho_v(\zeta-1+\tilde{V}_v) & \alpha_\ell\rho_\ell(\zeta-1+\tilde{V}_\ell) \\ 0 & 0 & (\alpha_\ell\rho_v+\rho_\ell C_v)(\zeta-1) + \{[\alpha_\ell\rho_v+\rho_\ell C_v(\zeta-1)]\tilde{V}_v - \rho_\ell C_v(\lambda-2)\tilde{V}_\ell\} & -(\alpha_\ell\rho_\ell+\rho_\ell C_v)(\zeta-1) - \{(\alpha_\ell\rho_\ell+\rho_\ell C_v(1-\lambda)]\tilde{V}_\ell + \rho_\ell C_v\lambda\tilde{V}_v\} \end{bmatrix} = 0 \quad (\text{C.9})$$

where $\tilde{V}_v = \frac{\Delta t}{\Delta z} V_v (1 - \exp(-ik\Delta z))$

$\tilde{V}_\ell = \frac{\Delta t}{\Delta z} V_\ell (1 - \exp(-ik\Delta z))$

$q = 2i \frac{\Delta t}{\Delta z} \sin\left(\frac{k\Delta z}{2}\right)$

$k = \frac{\pi}{n\Delta z} : n=1 \sim J : J: \text{ the number of axial mesh}$

For simplicity we assume in the same way as in the previous analysis that $a_v \gg V_v$, $a_\ell \gg V_\ell$

Then the determinant reduces to the 3 x 3 determinant

$$\text{Det} \begin{bmatrix} \rho_v(\zeta-1+\tilde{V}_v) & \alpha_v\rho_v\zeta q & 0 \\ -\rho_\ell(\zeta-1+\tilde{V}_\ell) & 0 & \alpha_\ell\rho_\ell\zeta q \\ (\alpha_\ell\rho_v+\rho_\ell C_v)(\zeta-1+\tilde{V}_v) + \rho_\ell C_v(\lambda-2)(\tilde{V}_v-\tilde{V}_\ell) & -(\alpha_\ell\rho_\ell+\rho_\ell C_v)(\zeta-1+\tilde{V}_\ell) & -\rho_\ell C_v\lambda(\tilde{V}_v-\tilde{V}_\ell) \end{bmatrix} = 0 \quad (\text{C.10})$$

As we notice that ζq appears as a common coefficient, resolving the determinant we obtain the modified determinant

$$\text{Det} \begin{bmatrix} \rho_V(\mu - \tilde{V}_V) & -\alpha_V \rho_V & 0 \\ -\rho_\ell(\mu - \tilde{V}_\ell) & 0 & -\alpha_\ell \rho_\ell \\ 0 & (\alpha_\ell \rho_V + \rho_\ell C_V)(\mu - \tilde{V}_V) - (\alpha_\ell \rho_\ell + \rho_\ell C_V)(\mu - \tilde{V}_\ell) \\ & -\rho_\ell C_V(\lambda - 2)(\tilde{V}_V - \tilde{V}_\ell) & +\rho_\ell C_V \lambda(\tilde{V}_V - \tilde{V}_\ell) \end{bmatrix} = 0 \quad (\text{C.11})$$

where $\mu = -(\zeta - 1)$

The above determinant, Eq. (C.11) is in the exact same form as Eq. (B.16).

Therefore we can use Eq. (B.17) as follows:

$$aq^2 + b\tilde{V}_r q + c\tilde{V}_r^2 = 0 \quad (\text{C.12})$$

where

$$\begin{aligned} a &= \alpha_\ell^2 \rho_V + \alpha_V \alpha_\ell \rho_\ell + \rho_\ell C_V \\ b &= \rho_\ell C_V(2 - \lambda) + 2\alpha_V \alpha_\ell \rho_\ell \\ c &= \alpha_V \alpha_\ell \rho_\ell + \alpha_V \rho_\ell C_V(1 - \lambda) \\ q &= \mu - \tilde{V}_V \\ \tilde{V}_r &= \tilde{V}_V - \tilde{V}_\ell \end{aligned}$$

For numerical stability the absolute value of the growth factor, $|\zeta|$, should be less than 1

From Eq. (C.12)

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tilde{V}_r \quad (\text{C.13})$$

$$\zeta - 1 = -\mu = -(\tilde{V}_V + q) \quad (C.14)$$

$$\zeta - 1 = - \left[\frac{(2a - b \pm \sqrt{b^2 - 4ac})}{2a} \tilde{V}_V + \frac{b \mp \sqrt{b^2 - 4ac}}{2a} \tilde{V}_\ell \right] \quad (C.15)$$

If $\lambda = 1$ and $\rho_\ell^2 C_V^2 >> 4\alpha_V \alpha_\ell^3 \rho_\ell \rho_V$, for $+\sqrt{b^2 - 4ac}$

$$\zeta - 1 = - \frac{(\alpha_\ell^2 \rho_V + \rho_\ell C_V) \tilde{V}_V + (\alpha_V \alpha_\ell \rho_\ell) \tilde{V}_\ell}{\alpha_\ell^2 \rho_V + \alpha_V \alpha_\ell \rho_\ell + \rho_\ell C_V}$$

$$= -p + p \exp(-ik\Delta z) \quad (C.16)$$

where $p = \frac{(\alpha_\ell^2 \rho_V + \rho_\ell C_V) \frac{\Delta t}{\Delta z} V_V + (\alpha_V \alpha_\ell \rho_\ell) \frac{\Delta t}{\Delta z} V_\ell}{\alpha_\ell^2 \rho_V + \alpha_V \alpha_\ell \rho_\ell + \rho_\ell C_V}$

For numerical stability $0 < P < 1$

$$\begin{aligned}
 0 &< (\alpha_{\ell}^2 \rho_v + \rho_{\ell} C_v) \frac{\Delta t}{\Delta z} V_v + (\alpha_v \alpha_{\ell} \rho_{\ell}) \frac{\Delta t}{\Delta z} V_{\ell} \\
 &< (\alpha_{\ell}^2 \rho_v + \rho_{\ell} C_v + \alpha_v \alpha_{\ell} \rho_{\ell})
 \end{aligned} \tag{C.17}$$

If $\frac{\Delta t}{\Delta z} V_v < 1$ and $\frac{\Delta t}{\Delta z} V_{\ell} < 1$, the above inequality is always met.

For $-\sqrt{b^2 - 4ac}$

$$\begin{aligned}
 \zeta - 1 &= - \left[\frac{(\alpha_{\ell}^2 \rho_v) \frac{\Delta t}{\Delta z} \tilde{V}_v + (\rho_{\ell} C_v + \alpha_v \alpha_{\ell} \rho_{\ell}) \frac{\Delta t}{\Delta z} \tilde{V}_{\ell}}{\alpha_{\ell}^2 \rho_v + \rho_{\ell} C_v + \alpha_v \alpha_{\ell} \rho_{\ell}} \right] \\
 &= -r + r \exp(-ik\Delta z)
 \end{aligned} \tag{C.18}$$

$$\text{where } r = \frac{(\alpha_{\ell}^2 \rho_v) \frac{\Delta t}{\Delta z} V_v + (\rho_{\ell} C_v + \alpha_v \alpha_{\ell} \rho_{\ell}) \frac{\Delta t}{\Delta z} V_{\ell}}{\alpha_{\ell}^2 \rho_v + \rho_{\ell} C_v + \alpha_v \alpha_{\ell} \rho_{\ell}}$$

For numerical stability $0 < r < 1$

$$\begin{aligned}
 0 &< (\alpha_{\ell}^2 \rho_v) \frac{\Delta t}{\Delta z} V_v + (\rho_{\ell} C_v + \alpha_v \alpha_{\ell} \rho_{\ell}) \frac{\Delta t}{\Delta z} V_{\ell} \\
 &< (\alpha_{\ell}^2 \rho_v + \rho_{\ell} C_v + \alpha_v \alpha_{\ell} \rho_{\ell})
 \end{aligned} \tag{C.19}$$

Also if $\frac{\Delta t}{\Delta z} V_v < 1$ and $\frac{\Delta t}{\Delta z} V_{\ell} < 1$, the above inequality is always met.

Therefore, it can be concluded that, if $\lambda = 1$ and

$C_v > \sqrt{4\alpha_v \alpha_{\ell}^3 \rho_v / \rho_{\ell}}$, the finite difference scheme is restricted only by the convective stability conditions.

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