NONLOCAL BOLTZMANN THEORY OF PLASMA CHANNELS

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ABSTRACT

The mathematical framework for the LLNL code NUTS is developed. This code is designed to study the evolution of an electron-beam-generated plasma channel at all pressures. The Boltzmann treatment of the secondary electrons presented include all inertial, nonlocal, electric and magnetic effects, as well as effects of atomic collisions. Field equations are advanced simultaneously and self-consistently with the evolving plasma currents.

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1. Introduction

An intense electron-beam propagating through a gaseous medium generates secondary electrons by direct ionization and avalanching. The resultant plasma channel evolves rapidly under the combined effects of the electromagnetic fields and atomic collisions. The characteristics of the plasma channel are strongly dependent on gas pressure. Conventionally, plasma channel evolution has been studied only in two extreme limits: at the high pressure end, where Ohm's Law is assumed to hold, and at the very low pressures (the ion-focused regime), where electrons are assumed to be instantaneously expelled from the beam region. It has been argued¹ that there exists a large intermediate pressure regime which belongs to neither of the two extreme cases. In fact, Ohm's Law is expected to break down at rather high pressures (≤ 100 torr). At the highest pressures, where Ohm's Law holds for the bulk electrons, there remains a small population of high energy runaways (delta rays) that are not mobility-limited.

Our goal is to understand plasma channel evolution at all pressures. The proposed description should reduce to the special limits of Ohm's Law and ion-focused regime at the appropriate pressures, and should give a reasonable description in the intermediate pressure regimes.

The scalar conductivity model is based on the assumption of approximate balance of the electric force against collisional drag

$$eE \sim v_m v_d$$

The plasma current at any point in time and space is proportional to the instantaneous and local electric field. A generalized description of the channel must include the following additional effects:

(1) Inertial effects. Away from the mobility-limited regime, the plasma current will not be proportional to the instantaneous local electric field, but will depend on the past history of forces. This effect can be incorporated by relaxing the approximate force balance assumption.

$$\frac{dv_d}{dt} \approx eE - v_m v_d \neq 0$$

(2) Nonlocal effects. The model must allow the secondary electrons to move from one point in space to another.

(3) Magnetic effects. Magnetic fields can also have an effect on plasma currents. In special limits, these effects have been labeled Hall currents, or tensor conductivity effects. In some regions, the electric field can be very small, while the magnetic fields are large. Thus, it is highly unjustified to drop the magnetic effects from the force law.

The importance of the non-ohmic effects is determined to a large extent by the magnitude of the momentum transfer rate v_m . The scalar conductivity model fails when v_m becomes sufficiently small. A key observation regarding v_m is that it is strongly electron-energy dependent. The collision frequency drops rapidly with energy, with an asymptotic drop-off rate of $\epsilon^{-3/2}$. Thus, at a given pressure, the motion of high energy electrons could be radically different from the low energy electrons. A reasonable model of plasma channel evolution must therefore be able to differentiate between electrons of different energies.

The inclusion of all non-ohmic effects and the necessity for electron energy differentiation are the key ideas which motivated the Boltzmann treatment that we will present in the remainder of this paper.

Our Boltzmann theory is based on the following approximations:

(1) Nonrelativistic electron motion. For ETA-type beam parameters, secondary electrons can be accelerated by the field. However, potential energy gain associated with the fields is not sufficient to make the bulk of electrons relativistic. While some electrons may be born with relativistic energies (very high energy delta rays), they are few in number, and generally escape from the beam region unless the magnetic field becomes high. Under these circumstances the nonrelativistic version is adequate for describing the bulk of low energy electrons, as well as the delta rays.

(2) Axisymmetry.

(3) Frozen field approximation. This is the approximation used by E. P. Lee in the "New Field Equations,"² and assumes that the pattern of fields associated with the propagating beam is slowly varying in z . Hence, the fields as well as the currents and distribution functions are, to leading order, functions only of $x = z - ct$.

(4) Dipole expansion in velocity space. The so-called two-term approximation is a key assumption in our theory, and will be further discussed in the next section.

The plan for the paper is as follows. In Section II, we derive the Boltzmann Equations for the secondary electrons in the two-term approximation. The relations of the distribution functions to macroscopic variables (Plasma currents $\vec{J}_p$, plasma density n_e , and plasma temperature T_e) are also given. In Section III we make several changes of variables to recast the Boltzmann equations into forms which are convenient for numerical integration. Appropriate moments of the reduced Boltzmann Equations give rise to fluid equations which

are given in Section IV. To ensure that bulk conservation laws are obeyed, numerical schemes for advancing the Boltzmann equation are constructed to obey the fluid equations explicitly. Plasma currents derived from the Boltzmann Equations are used as sources to determine the fields. The relevant field equations are derived in Section V. The equations would reduce to the New Field Equations if the assumption of $J = \sigma E$ were made. A numerical code NUTS (Nonlocal Undulating Transport Solver) has been constructed, in which the field equations were advanced simultaneously and self-consistently with the Boltzmann Equations to obtain the history (in x) of the fields and plasma currents.

II. Boltzmann Equations in the Two-Term Approximation

In a Boltzmann description of the plasma channels, $f(\vec{v}, \vec{r}, t) d^3v d^3r$ represents the number of secondary electrons (all electrons other than beam) at velocity $\vec{v}$, and spatial position $\vec{r}$ at time t . The evolution of the distribution function is described in general by the Boltzmann Equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \vec{a} \cdot \vec{\nabla}_v f = L \quad (1)$$

where the acceleration vector is

$$\vec{a} = \frac{e}{m} (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \quad (2)$$

and L represents all atomic collisions. We consider the velocity space in spherical polar coordinates (v, θ, ϕ) . With no loss of generality, the distribution functions can be expressed as an infinite sum of spherical harmonics $Y_{lm}(\theta, \phi)$, with coefficients which depend only on the magnitude of the velocity v . A full treatment of the angular decomposition of the Boltzmann

Equations for the ϕ -independent case (involving Legendre functions) was presented in Ref. (3). A full treatment of the general ϕ -dependent case in terms of spherical harmonics could be made similarly. Here, we will simply adopt as our ansatz the following approximation:

$$f(\vec{v}, \vec{r}, t) = f_0 + f_r \sin \theta \cos \phi + f_z \cos \theta \quad (3)$$

The three functions f_0 , f_r , and f_z are assumed to be functions of v , the magnitude of the velocity, and of $\vec{r}$ and t . The approximation is equivalent to setting all coefficients with $\ell \sim 1$ to zero in a spherical harmonics expansion. The so-called two-term approximation ($\ell = 0$ and 1) is rigorously justified when the system is nearly isotropic in velocity space. This is a reasonable assumption at the high pressures where electron motion is collision-dominated. However, in several test examples, the two-term expansion has been found to be surprisingly good even when the system is highly anisotropic.⁴ These examples are by no means exhaustive, and the validity of the two-term approximation remains an open question.

There are in principle three independent coefficients for $\ell = 1$, corresponding to $m = -1, 0, +1$. However, we have introduced only two independent functions f_r and f_z in our ansatz. This is the assumption of axisymmetry. A nonaxisymmetric description would include an additional function f_θ . The particular angular combinations are constructed so that the axial current J_z and radial current J_r (the two macroscopic variables of interest) are related simply to weighted moments of f_z and f_r respectively.

$$J_z \equiv e \int d^3v \, v \cos \theta \, f(\vec{v}, \vec{r}, t) = \frac{4\pi}{3} e \int_0^\infty dv \, v^3 f_z \quad (4)$$

$$J_r \equiv e \int d^3v \, v \sin \theta \cos \phi \, f(\vec{v}, \vec{r}, t) = \frac{4\pi}{3} e \int_0^\infty dv \, v^3 f_r \quad (5)$$

The plasma number density n_e and energy density $\bar{\epsilon}$ are moments of f_0 .

$$n_e \equiv \int d^3v \, f(\vec{v}, \vec{r}, t) = 4\pi \int_0^\infty dv \, v^2 f_0 \quad (6)$$

$$\bar{\epsilon} \equiv \int d^3v \left(\frac{1}{2} m v^2 \right) f(\vec{v}, \vec{r}, t) = 4\pi \int_0^\infty dv \left(\frac{1}{2} m v^2 \right) v^2 f_0 \quad (7)$$

In contrast to Eq. (3), the moment relations of Eqs. (4) to (7) are exact, and are valid in the full spherical harmonics expansion with no approximation.

One point of possible confusion should be clarified. While the spatial variable is most conveniently expressed in cylindrical coordinates (r, θ_r, z) because of the beam geometry, the velocity variable is expressed in spherical polar coordinates (v, θ, ϕ) . There is no contradiction because velocity space and coordinate space are mutually orthogonal. However, care has to be exercised in matching the two coordinate systems.

Having prescribed an explicit angular form for the distribution function, the next step is to express the operators $\vec{v} \cdot \vec{\nabla}$, $\vec{a} \cdot \vec{\nabla}_v$ and L in polar coordinates. A detailed treatment of the angular decomposition of L is given in Ref. (3), and its application to the present problem is straightforward. We will therefore not repeat the derivation of the atomic collisions operator, but simply quote the relevant results as we go along. In polar coordinates, we have

$$\begin{aligned}\vec{v} \cdot \vec{v} &= v_r \frac{\partial}{\partial r} + v_z \frac{\partial}{\partial z} \\ &= v \sin \theta \cos \phi \frac{\partial}{\partial r} + v \cos \theta \frac{\partial}{\partial z}\end{aligned}\quad (8)$$

The acceleration operator $\vec{a} \cdot \vec{v}$ requires a little more work. First, for the electric component, we have

$$\vec{E} \cdot \vec{v} = E_r \frac{\partial}{\partial v_r} + E_z \frac{\partial}{\partial v_z} \quad (9)$$

By straight-forward algebra, we obtain a representation of the two differential operators in polar coordinates:

$$\frac{\partial}{\partial v_r} = \sin \theta \cos \phi \frac{\partial}{\partial v} - \cos \theta \sin \theta \cos \phi \frac{1}{v} \frac{\partial}{\partial \cos \theta} + \frac{\sin \phi}{v \sin \theta} \frac{\partial}{\partial \cos \phi} \quad (10)$$

and

$$\frac{\partial}{\partial v_z} = \cos \theta \frac{\partial}{\partial v} + \frac{\sin^2 \theta}{v} \frac{\partial}{\partial \cos \theta} \quad (11)$$

The magnetic component of the acceleration operator is proportional to

$$(\vec{v} \times \vec{B}) \cdot \vec{v} = B_\theta \left(v_r \frac{\partial}{\partial v_z} - v_z \frac{\partial}{\partial v_r} \right) \quad (12)$$

Equations (8) to (12) can now be used to rewrite the Boltzmann Equation (1) in a form where all angular dependence is explicit (though messy). Our next step is to eliminate all angular dependence by taking appropriate angular moments of the Boltzmann Equation. Three different moments are taken:

$$(1) \quad \frac{1}{4\pi} \int d\Omega \quad \times \quad \text{Eq. (1)}$$

$$(2) \quad -\frac{1}{4\pi} \int \sin \theta \cos \phi \, d\Omega \quad \times \quad \text{Eq. (1)}$$

$$(3) \quad -\frac{1}{4\pi} \int \cos \theta \, d\Omega \quad \times \quad \text{Eq. (1)} \quad .$$

The resulting angular integrals are many, but they are all easily evaluated.

The net result is a set of three equations which depend only on v , and are independent of the angular variables. They are as follows:

(1) Scalar equation

$$\frac{\partial f_0}{\partial t} + v \left(\frac{1}{r} \frac{\partial f_r}{\partial r} + \frac{\partial f_z}{\partial z} \right) + \frac{1}{3} \left[\frac{eE_r}{mv^2} \frac{\partial}{\partial v} (v^2 f_r) + \frac{eE_z}{mv^2} \frac{\partial}{\partial v} (v^2 f_z) \right] = L_0 \quad (13)$$

(2) Radial equation

$$\frac{\partial f_r}{\partial t} + v \frac{\partial f_0}{\partial r} + \frac{eE_r}{m} \frac{\partial f_0}{\partial v} + \frac{eB_\theta}{mc} f_z = 3 L_r \quad (14)$$

(3) Axial equation

$$\frac{\partial f_z}{\partial t} + v \frac{\partial f_0}{\partial z} + \frac{eE_z}{m} \frac{\partial f_0}{\partial v} - \frac{eB_\theta}{mc} f_r = 3 L_z \quad (15)$$

where the atomic collision operators are defined by

$$L_0 = \frac{1}{4\pi} \int d\Omega \, L \quad (16)$$

$$L_r = \frac{1}{4\pi} \int d\Omega \sin \theta \cos \phi \, L \quad (17)$$

$$L_z = \frac{1}{4\pi} \int d\Omega \cos \theta \, L \quad (18)$$

The three coupled equations (13) to (15) are the Boltzmann Equations in the two-term approximation. The angular decomposition in this section has led to the replacement of the full Boltzmann Equation (1) for the distribution

function $f(\vec{v})$, (which is dependent on three velocity variables) by a set of coupled equations for three functions f_0, f_r, f_z , which depend only on the magnitude of the velocity vector v . The dimensions of the problem have therefore been reduced by two.

Information about number and energy density changes is carried by the scalar equation while the current flow is described by the vector equations (the radial and axial equations). Note that spatial advection has been retained in all three equations. Both electric effects as well as magnetic effects enter into the vector equations. The magnetic term couples the two vector equations, reflecting the tensorial characteristics of the current flow. Since magnetic fields do not change the number nor the energy of secondary electrons, they do not enter into the scalar equation.

III. Transformation of Variables

In this section, we will perform several transformations on both the dependent variables as well as the independent variables. These manipulations are motivated in part by physics, and in part by numerics. The objective is convenience. The physical contents of the Boltzmann Equations remain unchanged under the mathematical manipulations. The transformations in this section will be carried out in two steps. In the first step, we replace the magnitude of velocity v with the energy variable ϵ . The explicit form of atomic collisions operators in the transformed ϵ -dependent Boltzmann Equations will then be given. In the second step, we transform to a variable y which is logarithmic in energy. The resultant Boltzmann Equations in y are much easier to integrate numerically.

Step 1: The ϵ variable and the atomic collision operators

First, the variable v will be replaced by the energy variable ϵ via the nonrelativistic relation

$$\epsilon = \frac{1}{2} m v^2 \quad (19)$$

Since most of the atomic data, (and associated intuition), are functions of energy, the transformation allows us to write the atomic collision operators in explicit form.

Associated with the transformation to energy variables, we will also introduce transformed dependent variables $\bar{f}_0$, g_r , and g_z :

$$\bar{f}_0 d\epsilon \equiv 4\pi v^2 f_0 dv \quad (20)$$

$$g_z d\epsilon = \frac{4\pi}{3} \frac{v^3}{c} f_z dv \quad (21)$$

$$g_r d\epsilon = \frac{4\pi}{3} \frac{v^3}{c} f_r dv \quad (22)$$

The three new variables represent the differential number density and the axial and radial current densities in energy space. They are related to the macroscopic variables of Eqs. (4) to (7) by

$$J_z = ec \int d\epsilon g_z \quad (23)$$

$$J_r = ec \int d\epsilon g_r \quad (24)$$

$$n_e = \int d\epsilon \bar{f}_0 \quad (25)$$

$$\bar{\epsilon} = \int d\epsilon \epsilon \bar{f}_0 \quad (26)$$

Note that $\bar{f}_0$, g_r and g_z have the same dimensions.

In the transformed variables, the Boltzmann Equations (13) to (15) take on a slightly different form

$$\frac{\partial \bar{f}_0}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} r c g_r + \frac{\partial c g_z}{\partial z} + e c \left(E_r \frac{\partial g_r}{\partial \epsilon} + E_z \frac{\partial g_z}{\partial \epsilon} \right) = 4\pi \left(\frac{v}{m} \right) L_0 \quad (27)$$

$$\frac{\partial c g_r}{\partial t} + \frac{2e}{3m} \frac{\partial \bar{f}_0}{\partial r} + \frac{2e}{3m} E_r \epsilon^{3/2} \frac{\partial}{\partial \epsilon} \left(\frac{\bar{f}_0}{\epsilon^{1/2}} \right) + \frac{e B_\theta}{m c} c g_z = 4\pi \left(\frac{v}{m} \right) L_r \quad (28)$$

$$\frac{\partial c g_z}{\partial t} + \frac{2e}{3m} \frac{\partial \bar{f}_0}{\partial z} + \frac{2e}{3m} E_z \epsilon^{3/2} \frac{\partial}{\partial \epsilon} \left(\frac{\bar{f}_0}{\epsilon^{1/2}} \right) - \frac{e B_\theta}{m c} c g_r = 4\pi \left(\frac{v}{m} \right) L_z \quad (29)$$

The atomic collision operators L_0 and L_z have been treated in detail in Ref. (3). The explicit form of these operators in terms of cross sections are given in Eqs. (49) to (51) of that paper. The operator L_z is proportional to the momentum transfer rate coefficient $\tilde{v}_m$ and, in the notations of the present paper, is given by

$$4\pi \left(\frac{v}{m} \right) L_z = - n_g \tilde{v}_m c g_z \quad (30)$$

The operator L_r has a similar form, but the momentum transfer rate involves a slightly different weighting of the differential cross sections (with $\cos \theta$ replaced by $\sin \theta \cos \phi$). Consistent with the two-term approximation, we will set these two momentum transfer rates equal, and write

$$4\pi \left(\frac{v}{m} \right) L_r = - n_g \tilde{v}_m c g_r \quad (31)$$

The operator L_0 consists of contributions from

- (1) Source term from direct ionization by beam electrons.
- (2) Source term from ionization by secondary electrons.

(3) Sink term from recombination.

(4) Energy loss terms from excitation processes.

The energy loss term generally consists of a sum over all discrete excitations. The exact treatment in terms of discrete sums is given in Ref. (3). In the present version of NUTS, we have chosen to represent the energy loss by a single loss function in a "continuous slowing down" approximation. The validity of this simplified description has yet to be checked. The atomic collisions operator then takes on the form

$$4\pi \left(\frac{v}{m} \right) L_0(\epsilon) = \left(\frac{J_b}{e} \right) n_g \frac{d\sigma}{d\epsilon}(\epsilon_b, \epsilon) + \int_{\epsilon}^{\epsilon_{\max}} d\epsilon' \bar{f}_0(\epsilon') n_g v' \frac{d\sigma}{d\epsilon}(\epsilon', \epsilon) \\ - \bar{f}_0(\epsilon) n^+ v \sigma^{\text{rec}}(\epsilon) + \frac{\partial}{\partial \epsilon} \left[\bar{f}_0(\epsilon) n_g v \left(\sum_i \epsilon_i \sigma_i(\epsilon) \right) \right] \quad (32)$$

where $\frac{d\sigma}{d\epsilon}(\epsilon', \epsilon)$ is the differential cross section for the production of a secondary electron with energy ϵ by an incident electron of energy ϵ' .

In the first term on the RHS, ϵ_b is the beam energy, $\sigma^{\text{rec}}(\epsilon)$ is the cross section for the recombination of electrons with ions (ion density n^+), and σ_i and ϵ_i represent the cross section and excitation energy for electron excitation via the i^{th} inelastic channel. In the present formulation, we include in the sum all inelastic channels, including ionization processes. Therefore the effects of the secondary ionization processes on energy loss by the incident electron (the 4th term) and on the production of additional electrons (the 2nd term) are separately accounted for.

It is also possible to add a source term to the axial equation to represent the forward component of delta ray current at birth. This source term, added to the RHS of Eq. (30), takes on the form

$$\left(\frac{j_b}{e}\right) n_g \frac{d\sigma}{d\epsilon} (\epsilon_b, \epsilon) v \cos \theta_c ,$$

where θ_c is the angle of ejection of the secondary electron (energy ϵ) with respect to the incident electron (energy ϵ_b)

Ions are assumed to be immobile. They are followed by rate equations

$$\begin{aligned} \frac{dn^+}{dt} = & \left(\frac{j_b}{e}\right) n_g \sigma(\epsilon_b) + \int_0^{\epsilon_{\max}} d\epsilon' f_0(\epsilon') n_g v^l \sigma(\epsilon') \\ & - \int_0^{\epsilon_{\max}} d\epsilon' f_0(\epsilon') n^+ v^l \sigma^{\text{rec}}(\epsilon') \end{aligned} \quad (33)$$

where $\sigma(\epsilon)$ is the integrated ionization cross section. Generalizations to multiple ion species is straightforward and was performed in Ref. (3).

Step 2: The y variable

Although Eqs. (27) to (29) give the Boltzmann Equations in a closed form, direct integration of this system is a rather cumbersome task. The ϵ -space of interest spans six orders of magnitude (0.1 eV to 100 keV). The atomic cross sections are in general large and rich in structure at the low energy end, and decreases over several orders of magnitude at the high energy end. To make the problem numerically tractable, we introduce the new variable

$$y = \ln (\epsilon + \epsilon_s)$$

where ϵ_s is an externally specified parameter. The domain of integration in y is of course much smaller than in ϵ , and the atomic data, when suitably defined in y -space, are relatively more uniform, and therefore easier to treat numerically.

In analogy to Step 1, we introduce transformed dependent variables F_0 , G_r , and G_z :

$$F_0(y)dy = \bar{F}_0(\epsilon) d\epsilon \quad (34)$$

$$G_z(y)dy = g_z(\epsilon) d\epsilon \quad (35)$$

$$G_r(y)dy = g_r(\epsilon) d\epsilon \quad (36)$$

We will also at this point make the frozen field approximation² with

$$x = z - ct \quad (37)$$

and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial ct} = - \frac{\partial}{\partial z} \quad (38)$$

The complete set of Boltzmann Equations in NUTS solves for three quantities F_0 , G_z and G_r as functions of three variables x , r and $y = \ln(\epsilon + \epsilon_s)$. They are as follows:

$$\begin{aligned} \frac{\partial F_0}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} r G_r - \frac{\partial G_z}{\partial x} + \frac{\partial}{\partial y} \left\{ \frac{1}{(\epsilon + \epsilon_s)} \left[e E_r G_r + e E_z G_z \right] \right\} \\ = \left(\frac{j_b}{ec} \right) n_g S^{dir}(\epsilon) + \int_y^{y_{max}} dy' F_0(y') n_g S^{ion}(\epsilon', \epsilon) \\ - F_0 n^+ S^{rec}(\epsilon) + \frac{\partial}{\partial y} \left(F_0 n_g S^{loss}(\epsilon) \right) \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{\partial G_r}{\partial x} + \left(\frac{2\epsilon}{3mc^2} \right) \frac{\partial F_0}{\partial r} + \frac{2eE_r}{3mc^2} \left[\epsilon \frac{\partial}{\partial y} \left(\frac{F_0}{\epsilon + \epsilon_s} \right) - \frac{F_0}{2} \right] + \frac{eB_\theta}{mc^2} G_z \\ = - G_r n_g S^m(\epsilon) \end{aligned} \quad (40)$$

$$\begin{aligned} \frac{\partial G_z}{\partial x} - \left(\frac{2\epsilon}{3mc^2} \right) \frac{\partial F_0}{\partial x} + \frac{2eE_z}{3mc^2} \left[\epsilon \frac{\partial}{\partial y} \left(\frac{F_0}{\epsilon + \epsilon_s} \right) - \frac{F_0}{2} \right] - \frac{eB_\theta}{mc^2} G_r \\ = - G_z n_g S^m(\epsilon) + \left(\frac{j_b}{ec} \right) n_g S_z^{dir}(\epsilon). \quad (41) \end{aligned}$$

The input atomic data consists of six sets of atomic data, five of which are 1-D arrays, and one set is 2-D. They are related to the cross sections by

$$S^{dir}(\epsilon) = (\epsilon + \epsilon_s) \frac{d\sigma}{d\epsilon}(\epsilon_b, \epsilon) \quad (42)$$

where $\frac{d\sigma}{d\epsilon}$ is the ionization cross section with ϵ_b the beam energy;

$$S^{ion}(\epsilon^I, \epsilon) = (\epsilon + \epsilon_s) \left(\frac{v}{c} \right) \frac{d\sigma}{d\epsilon}(\epsilon^I, \epsilon) \quad (43)$$

$$S^{rec}(\epsilon) = \left(\frac{v}{c} \right) \sigma^{rec}(\epsilon) \quad (44)$$

where σ^{rec} is the cross section for recombination;

$$S^{loss}(\epsilon) = \frac{1}{(\epsilon + \epsilon_s)} \left(\frac{v}{c} \right) \sum_i \epsilon_i \sigma_i \quad (45)$$

where σ_i and ϵ_i are the excitation cross section and excitation energy respectively, and the sum is performed over all inelastic channels;

$$S^m(\epsilon) = \left(\frac{v}{c} \right) \sigma^m(\epsilon) \quad (46)$$

where σ^m is the cross section for momentum transfer;

and

$$S_z^{dir}(\epsilon) = S^{dir}(\epsilon) \left(\frac{v}{c} \right) \cos \theta_c \quad (47)$$

where θ_e is the angle of the secondary electron with respect to the incident beam electron at birth. Note that all six sets of atomic data have the dimensions of area.

To close the system, we need an ion rate equation given by

$$\begin{aligned} \frac{\partial n^+}{\partial x} = & \left(\frac{J_b}{ec} \right) n_g \sigma^{\text{ion}}(\epsilon_b) + \int_{y_{\min}}^{y_{\max}} dy' F_0(y') n_g \bar{S}^{\text{ion}}(\epsilon') \\ & - \int_{y_{\min}}^{y_{\max}} dy' F_0(y') n^+ S^{\text{rec}}(\epsilon') \end{aligned} \quad (48)$$

where

$$\sigma^{\text{ion}}(\epsilon_b) \equiv \int dy S^{\text{dir}}(\epsilon) \quad (49)$$

$$\text{and } \bar{S}^{\text{ion}}(\epsilon') = \int dy S^{\text{ion}}(\epsilon' \epsilon) = \frac{v'}{c} \sigma^{\text{ion}}(\epsilon') \quad (50)$$

where σ^{ion} is the total ionization cross section.

The scaled distribution functions F_0 , G_r , G_z are related to macroscopic variables of interest by

$$n_e = \int dy F_0(y) \quad (51)$$

$$J_r = ec \int dy G_r \quad (52)$$

$$J_z = ec \int dy G_z \quad (53)$$

IV. Fluid Equations

In this section, we will derive fluid equations by taking various velocity moments of the reduced Boltzmann Equations. These macroscopic equations do not form a closed system; nor do we expect them to be closed, since, as was discussed in the Introduction, the plasma channels possess some fundamental non-fluid characteristics. However, these fluid equations are useful in two ways. First, they can provide us with additional insight into the nature of the two-term expansion; in particular the extent to which the approximation preserves or modifies the macroscopic structure of the system. Secondly, the fluid equations are in essence the conservation laws obeyed by the system. Numerical algorithms for advancing the Boltzmann Equations are constructed to explicitly obey these conservation laws whenever possible.

To derive the fluid equations, we return to the reduced Boltzmann Equations before transformation [Eq. (13) to (15)]. Mass conservation is obtained simply by integrating the scalar equation (13) over all velocity space.

$$4\pi \int_0^{\infty} v^2 dv \times \text{Eq. (13)} \quad (54)$$

gives rise to the continuity equation

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot \left(\frac{\vec{j}}{e} \right) = B_0 \quad (55)$$

where we have used the relations between the macroscopic variables and the distributions in Eqs. (4) to (7). B_0 is the effective source term given by

$$B_0 = 4\pi \int_0^{\infty} v^2 dv L_0 \quad (56)$$

By taking the energy moment of the scalar Boltzmann equation, the fluid equation for energy conservation is obtained

$$\frac{\partial \bar{\epsilon}}{\partial t} + \nabla \cdot \vec{Q} - \vec{E} \cdot \vec{J} = U_0 \quad (57)$$

where the heat flux term is given by

$$\vec{Q} = \frac{4\pi}{3} \int_0^\infty dv \left(\frac{1}{2} m v^2 \right) v^3 \vec{f}_1 \quad (58)$$

We have introduced the vector rotation $\vec{f}_1 = (f_1, f_2)$. The energy source term is the corresponding energy moment of the collision operator

$$U_0 = 4\pi \int_0^\infty dv \left(\frac{1}{2} m v^2 \right) v^2 L_0 \quad (59)$$

Momentum conservation laws are obtained by integrating over the vector equations. The resulting fluid equations for current flow are given by:

$$\frac{\partial}{\partial t} \left(\frac{\vec{J}}{e} \right) + \nabla \pi - \frac{e \vec{E} n}{m} - \frac{e}{mc} (\vec{J} \times \vec{B}) = \vec{B}_1 \quad (60)$$

where the "pressure" is proportional to the electron energy density

$$\pi = \frac{2}{3m} \bar{\epsilon} \quad (61)$$

and the drag term is an integral over the vector components of the collision operators

$$\vec{B}_1 = 4\pi \int_0^\infty v^3 dv \vec{L}_1 \quad (62)$$

We will first consider the left hand side of the fluid equations (55) to (62). The continuity equation (55) is exact. The two-term expansion has led to no loss of generality at this level. In the energy equation and the momentum equations, terms involving the electromagnetic fields are exact, but the terms involving spatial derivatives are only approximate. The heat flux "Q" and the "pressure" π are different from their usual definitions in standard hydrodynamic theories since they do not involve averages about the mean. In the two-term expansion, it is not possible to make the usual rearrangement of terms to cast the fluid equations in forms involving convective derivatives. The dipole approximation led naturally to a closure relation for the pressure π , but the heat flux Q is expressible only as an integral over distribution functions.

To go a step further with the atomic collision terms B_0 , U_0 , and B_1 , we resort to the transformed Boltzmann Equations in the y variable Eqs. (39) to (41). The macroscopic terms are readily obtainable by integration over y . We obtain the source term as

$$\begin{aligned} \frac{B_0}{c} = & \left(\frac{j_b}{ec} \right) n_g \sigma^{\text{ion}}(\epsilon_b) + \int_{y_{\min}}^{y_{\max}} dy' F_0(y') n_g S^{\text{ion}}(\epsilon') \\ & - \int_{y_{\min}}^{y_{\max}} dy' F_0(y') n^+ S^{\text{rec}}(\epsilon') \end{aligned} \quad (63)$$

which is identical to the source term in the ion equation (48). This identity is a simple consequence of charge conservation, but to enforce this identity numerically is an important and nontrivial task.

The energy collision term U_0 is similarly obtained as

$$\begin{aligned} \frac{U_0}{c} = & \left(\frac{j_b}{ec} \right) n_g \int dy \epsilon S^{dir}(\epsilon) + \int dy' F_0(y') n_g \int dy S^{ion}(\epsilon' \epsilon) \epsilon \\ & - \int dy' F_0(y') \epsilon' S^{rec}(\epsilon') n^+ - \int dy (\epsilon + \epsilon_s) S^{loss}(\epsilon) F_0(\epsilon) n_g \end{aligned} \quad (64)$$

where the last term involving S^{loss} is derived by an integration by part.

Finally, the drag terms are simply

$$\frac{B_r}{c^2} = - \int dy G_r S^m(\epsilon) n_g \quad (65)$$

$$\frac{B_z}{c^2} = - \int dy G_z S^m(\epsilon) n_g + \left(\frac{j_b}{ec} \right) n_g \int dy S_z^{dir}(\epsilon) \quad (66)$$

V. Field Equations

The Boltzmann Equations allow us to calculate the plasma currents, given the electromagnetic fields. To obtain a closed system of equations, we will derive in this section the relevant field equations to calculate the electromagnetic fields, given the currents. In the context of this paper, the beam current is considered as an external source.

For the axisymmetric problem, the relevant Maxwell's Equations are the θ -component of Faraday's Law

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = - \frac{1}{c} \frac{\partial B_\theta}{\partial t} \quad (67)$$

and the axial and radial components of Ampere's Law

$$\frac{1}{r} \frac{\partial}{\partial r} \{ r B_{\theta} \} = \frac{1}{c} \frac{\partial E_z}{\partial t} + \frac{4\pi}{c} \{ J_{bz} + J_{pz} \} \quad (68)$$

$$- \frac{\partial B_{\theta}}{\partial z} = \frac{1}{c} \frac{\partial E_r}{\partial t} + \frac{4\pi}{c} J_{pr} \quad (69)$$

We have dropped J_{br} , following the ordering scheme of E. P. Lee.²

Again, employing the frozen field approximation

$$\frac{\partial}{\partial x} = \frac{1}{c} \frac{\partial}{\partial t} = - \frac{\partial}{\partial z} , \quad (70)$$

we can rewrite the axial component of Ampere's Law as

$$\frac{1}{r} \frac{\partial}{\partial r} r B_{\theta} = \frac{4\pi}{c} (J_{bz} + J_{pz}) + \frac{\partial E_z}{\partial x} \quad (71)$$

and the radial component of Ampere's Law as

$$\frac{\partial F}{\partial x} = - \frac{4\pi}{c} J_{pr} , \quad (72)$$

where F is the pinch field

$$F = E_r - B_{\theta} . \quad (73)$$

Finally, we have Faraday's Law as

$$\frac{\partial E_z}{\partial r} = - \frac{\partial F}{\partial x} = \frac{4\pi}{c} J_{pr} \quad (74)$$

The boundary conditions at the origin are:

$$B_{\theta}(r=0) = F(r=0) = E_r(r=0) = 0 \quad (75)$$

and at the outer boundary $r = b$:

$$E_z(r=b) = 0 \quad (76)$$

Equations (71) to (74) are advanced in x simultaneously and self-consistently with the Boltzmann Equations to obtain B_0 , E_z , and E_r . This set of field equations is a natural generalization to non-ohmic currents, and reduces to the monopole component of Ed Lee's "New Field Equations" if we make two additional approximations:

$$(1) \quad \vec{j}_p = \sigma \vec{E} \quad (77)$$

$$(2) \quad \frac{\partial E_z}{\partial x} = 0 \quad \text{in Eq. (71)} \quad (78)$$

VI. Conclusions

We have presented the mathematical framework for the LLNL code NUTS. The six sets of atomic data (Eqs. (42) to (47)) are inputs into the code. Equations (39) to (41) are then advanced simultaneously to obtain the three distribution functions F_0 , G_r , and G_z . The currents obtained by integrating G_r and G_z (Eqs. (52) and (53)) are then used as source terms to determine the fields B_0 , E_r , and E_z via Equations (71) to (74). The resulting fields are fed back into the Boltzmann Equations (39) to (41).

Construction of the NUTS code is now complete and operational. Many numerical instabilities were encountered along the way. Numerical methods that have been useful in controlling the numerical instabilities will be left for future discussions.

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