THEORY
OF
ASYMMETRIC ARRAYS OF CONTROL RODS
IN
NUCLEAR REACTORS

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Theory of Asymmetric Arrays of Control Rods in Nuclear Reactors

Raymond L. Murray

1. INTRODUCTION.

Seldom does the actual arrangement of control elements in a nuclear reactor conform to the ideal and convenient mathematical array. In order to achieve shim control, safety and regulation, it is desirable to design with rods of different sizes and materials. With a given fuel element arrangement, typically in square or hexagonal lattice spacing, there will be rods located at different distances from the center of the core and from each other. As the reactor operates, absorbers will be withdrawn, leaving further asymmetries in the location of those remaining. A few typical situations that may be encountered are sketched in Figure 1. The scope of reactor theory for prediction of reactivity values and critical conditions would be considerably broadened by the availability of a method to describe any situation of the types discussed above. It is the purpose of this report to develop in detail a two-group diffusion theory with as complete generality as possible. The method is as yet restricted to the unreflected core, or to the reflected core by use of reflector savings and bare equivalent geometries. The physical state of the system for which criticality conditions are sought is:

TWO "RINGS" WITH
CENTRAL ROD

RODS OF SEVERAL SIZES
AND TYPES

ARRAY OF FIVE RODS
WITH TWO REMOVED

ARRAY CONFORMING TO
FUEL ELEMENT PATTERN

FIGURE 1. ASYMMETRIC ROD ARRAYS

(a) Cylindrical homogenized core of height H and radius R , where dimensions may be considered as extrapolated or bare equivalent.

(b) Two-group diffusion theory (or modified two-group) represents the behavior of fast and thermal neutrons. Physical boundary conditions on both groups are to be applied at rod surfaces.

(c) The core is traversed through its full height by a total of $M + 1$ control "rods", the shapes of which are immaterial, so long as one may define extrapolation distances at surfaces of equivalent circular rods. One of these rods may be along the central axis, the remaining M may be located anywhere.

(d) Every rod in the system may be different in its size, radius a_m , fast and thermal extrapolation distances d_{1m} and d_{2m} , where the subscript m refers to the m th rod.

(e) The radial and angular location of the center of each of the m rods may be designated as d_m , the distance from the central core axis, and β_m , an angle between any pre-determined reference core radial plane and the plane through the rod center.

2. REVIEW OF LITERATURE.

The earliest work on the effect of control rods appears to be that of Wigner, Weinberg and Williamson⁽¹⁾, who treated the case of a central rod in a bare cylinder, according to the one-group model. Adler⁽²⁾ extended the analysis to an eccentric rod and a ring of rods in a reflected core with extrapolation distance at the core boundary. Scalettar and Nordheim⁽³⁾ solved the problem of a ring of identical symmetric rods, using conditions of finite fast flux at the center of each. More than one ring of rods was treated by Garabedian⁽⁴⁾ with the effect of reflector included. Codd and Rennie⁽⁵⁾ introduced the fast flux boundary condition for transparent rods: zero slope at the rod surface. Murray and Niestlie⁽⁶⁾ generalized the treatment of a central rod in bare and reflected cores by use of fast flux boundary conditions deduced from diffusion theory. This approach was extended by Murray⁽⁷⁾ to describe a ring of rods in a bare equivalent core. The treatment of regions that may be multiplying, such as fuel elements in an Argonaut-type reactor, has been given by Avery⁽⁸⁾. Further extensions have been made by Murray, Price and Birken⁽⁹⁾ to include the eccentric rod in a reflected core, ring, and the ring plus a central rod. All of the configurations treated to date have been symmetric arrays of identical rods at equal distances from the axis of the core.

3. GEOMETRY OF SYSTEM AND BASIC THEORY.

Figure 2 shows the necessary coordinates and dimensions for the complete problem. In addition to those listed, it is necessary to specify the location of any point P in the reactor core. In a system without any localized absorbers a single radial measure r is sufficient; with rods the fluxes depend on both r and an angular coordinate θ . Rod designations m and n are indicated, because of the anticipated mutual interaction effects. The index m should be regarded as generally running from 0 to $M + 1$.

It is natural to measure θ from the same reference plane as that defining ϕ_1, ϕ_2 , etc. There is no especial advantage in choosing the reference plane through any rod, in view of the inherent lack of symmetry. It is assumed that the conventional⁽¹⁰⁾ two-group diffusion equations are applicable to the fuel bearing region of the reactor.

$$D_1 \nabla^2 \phi_1 - \phi_1 \Sigma_1 + \phi_2 \Sigma_{f2} = 0 \quad \dots\dots(1)$$

$$D_2 \nabla^2 \phi_2 - \phi_2 \Sigma_2 + p \phi_1 \Sigma_1 = 0 \quad \dots\dots(2)$$

where in cylindrical polar coordinates, the Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{r^2 \partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad \dots\dots(3)$$

It will be convenient to abbreviate this as

$$\nabla_{r, \theta}^2 + \nabla_z^2 \quad \text{or} \quad \nabla_r^2 + \nabla_\theta^2 + \nabla_z^2 \quad \dots\dots(4)$$

Should epithermal or fast fission be significant, constants in

FIGURE 2. NOTATION FOR ANALYSIS

the diffusion equations can be modified without any need for new mathematical form. As usual, it is proposed that there exist solutions of (1) and (2) such that

$$\nabla^2 \phi_i + B^2 \phi_i = 0 \quad \text{.....(5)}$$

with $i = 1$ or 2 for the fast and thermal fluxes respectively. Insertion of these equations in (1) and (2) and application of Cramer's rule yields the buckling roots

$$\mu^2 = \mu_0^2 (1 - \epsilon + 2\epsilon^2 + 5\epsilon^3 + \dots) \quad \text{.....(6)}$$

$$\gamma^2 = \mu^2 + a \quad \text{.....(7)}$$

with

$$\mu_0^2 = \frac{\eta f p - 1}{L^2 + \tau} \quad \text{.....(8)}$$

and

$$a = \frac{1}{L^2} + \frac{1}{\tau}$$

$$\epsilon = \frac{\mu_0^2}{a}$$

The resultant decoupled equations are

$$\nabla^2 \phi_1 + \mu^2 \phi_1 = 0 \quad \text{.....(9)}$$

$$\nabla^2 \phi_1 - \gamma^2 \phi_1 = 0 \quad \text{.....(10)}$$

The method of separation of variables may be applied to effectively eliminate further necessity of attention to z dependence.

Insert

$$\phi_i(r, \theta, z) = \phi_i(r, \theta) Z(z) \quad \text{.....(11)}$$

in equation (9) and rearrange to obtain

$$\frac{\nabla_{r,\theta}^2 \phi_i}{\phi_i} + \frac{\nabla_z^2 Z}{Z} + \mu^2 = 0 \quad \text{.....(12)}$$

Letting the second term on the left be equal to the constant $-B_z^2$ and solving the equation

$$\frac{d^2 Z}{dz^2} + B_z^2 Z = 0 \quad \text{.....(13)}$$

subject to axial symmetry and zero flux at $z = \pm \frac{H}{2}$ gives

$$Z \sim \cos \frac{\pi z}{H} \quad \text{.....(14)}$$

and

$$B_z^2 = \left(\frac{\pi}{H} \right)^2 \quad \text{.....(15)}$$

For the critical reactor only the fundamental cosine is present.

Then, letting $\bar{\mu}^2 = \mu^2 - \left(\frac{\pi}{H} \right)^2$, equation (12) becomes

$$\nabla_{r,\theta}^2 \phi_1 + \bar{\mu}^2 \phi_1 = 0 \quad \text{.....(16)}$$

A similar procedure when applied to equation (10) yields

$$\nabla_{r,\theta}^2 \phi_1 - \bar{\nu}^2 \phi_1 = 0 \quad \text{.....(17)}$$

where

$$\bar{\nu}^2 = \nu^2 + \left(\frac{\pi}{H} \right)^2$$

The variables r, θ are now separated by proposing a solution of the form

$$\phi_1(r, \theta) = R_1(r) \Theta_1(\theta) \quad \text{.....(18)}$$

On substitution in (16) or (17), we obtain

$$\frac{\nabla_r^2 R_1}{R_1} + \frac{\nabla_\theta^2 \Theta_1}{\Theta_1} + \frac{\bar{\mu}^2}{\bar{\nu}^2} = 0 \quad \text{.....(19)}$$

We multiply through by r^2 and let the second term on the left be a negative constant $-s^2$ such that

$$\frac{d^2 \mathbb{H}_1}{d\theta^2} + s^2 \mathbb{H}_1 = 0 \quad \dots\dots(20)$$

The solutions of this equation are either $\cos s\theta$ or $\sin s\theta$, where s is $0, \pm 1, \pm 2, \dots$

The purely radial equations are now

$$\frac{r^2 d^2 \mathcal{R}_1}{dr^2} + \frac{rd \mathcal{R}_1}{dr} + \frac{(\bar{\mu}^2 r^2 - s^2)}{-(\bar{\nu}^2 r^2 + s^2)} \mathcal{R}_1 = 0 \quad \dots\dots(21)$$

which have solutions respectively in terms of ordinary Bessel functions of orders $J_s(\bar{\mu}r)$, $Y_s(\bar{\mu}r)$ and modified Bessel functions of orders $I_s(\bar{\nu}r)$, $K_s(\bar{\nu}r)$.

The most general solution for ϕ_1 and ϕ_2 can be written as linear combinations of all possible solutions.

$$\begin{aligned} \phi_1(r, \theta) = \sum_{s=-\infty}^{\infty} & \left[a_{s1} J_s(\bar{\mu}r) \cos s\theta + b_{s1} J_s(\bar{\mu}r) \sin s\theta \right. \\ & + c_{s1} Y_s(\bar{\mu}r) \cos s\theta + d_{s1} Y_s(\bar{\mu}r) \sin s\theta + e_{s1} I_s(\bar{\nu}r) \cos s\theta \\ & \left. + f_{s1} I_s(\bar{\nu}r) \sin s\theta + g_{s1} K_s(\bar{\nu}r) \cos s\theta + h_{s1} K_s(\bar{\nu}r) \sin s\theta \right] \end{aligned}$$

\dots\dots(22)

At this stage, cognizance must be taken of the presence of the various absorbers. Consider a simple case in which there is a rod at the center of the core, and at no other location. It is evident that there is no angular dependence, i.e. all arbitrary coefficients for $s \neq 0$ must vanish. As well, since $\sin s\theta$ is zero for $s = 0$, there will be no terms at all involving b_{01} , d_{01} , etc.

The fast and thermal fluxes reduce to the simple expression

$$\phi_1(r) = a_{01}J_0(\bar{\mu}r) + c_{01}Y_0(\bar{\mu}r) + e_{01}I_0(\bar{\nu}r) + g_{01}K(\bar{\nu}r).$$

In comparing this with the two-group fluxes without a rod present, it is evident that the flux depression is supplied by the Y_0 and K_0 functions. Next suppose there to be one or more eccentric rods located distances d_m from the origin. The higher order Bessel functions $J_s(\bar{\mu}r)$ and $I_s(\bar{\nu}r)$ with the angular factors will be required to describe the basic flux distribution, along with Y_s and K_s terms that more specifically represent the sink due to the local strong absorption. It will be found convenient because of essential symmetry of the flux in the vicinity of a control rod to transform, using Bessel function addition theorems, the sink functions, (with origin at the core axis) into functions with origin at the axis of the eccentric rod. As shown in the Appendix, no loss of rigor is experienced by initially writing these flux sink terms as $\sum_s Y_s(\bar{\mu}r_m) \cos s\psi_m$, $\sum_s Y_s(\bar{\mu}r_m) \sin s\psi_m$, $\sum_s K_s(\bar{\nu}r_m) \cos s\psi_m$ and $\sum_s K_s(\bar{\nu}r_m) \sin s\psi_m$, where ψ_m is measured around the m^{th} eccentric rod. This procedure may be extended using the principle of superposition. The most general solution is a sum of solutions referred to the $M + 1$ coordinate origins, with coordinates $r, r_1, \dots, r_m, \dots, r_M$.

The general representation of the fluxes depending on the radial and angular coordinates will be written as

$$\phi_1 = X + Y \quad \text{.....(23)}$$

$$\phi_2 = S_1 X + S_2 Y \quad \text{.....(24)}$$

where the axial dependence according to $\cos \frac{\pi z}{H}$ has been suppressed and the various arbitrary coefficients are absorbed in the functions X and Y rather than appearing explicitly, as would be the case without rods. Here

$$X = \sum_s A_s J_s(\mu r) \cos s\theta + \sum_s B_s J_s(\mu r) \sin s\theta \\ + \sum_{m=0}^M \sum_s B_{sm} Y_s(\mu r_m) \cos s\psi_m + \sum_{m=0}^M \sum_s P_{sm} Y_s(\mu r_m) \sin s\psi_m \quad \text{.....(25)}$$

$$Y = \sum_s C_s I_s(\nu r) \cos s\theta + \sum_s D_s I_s(\nu r) \sin s\theta \\ + \sum_{m=0}^M \sum_s E_s K_s(\nu r_m) \cos s\psi_m + \sum_{m=0}^M \sum_s T_s K_s(\nu r_m) \sin s\psi_m \quad \text{.....(26)}$$

The abbreviation $\sum_s$ refers to the $\sum_{s=-\infty}^{s=\infty}$, i.e. over all orders of Bessel functions, $\sum_{m=0}^M$ implies that there is a sum of sink terms, one for each of the $M+1$ rods, including the one at the center, if present. The arbitrary coefficients B_{sm} , P_{sm} , etc. are appropriate to the m^{th} rod. The angle ψ_m is measured counterclockwise from an extension of a radial plane through the m^{th} rod. For convenience, the bars in $\bar{\mu}$ and $\bar{\nu}$ are omitted, it being understood that the usual axial buckling modifications are needed for numerical computations.

4. BOUNDARY CONDITIONS.

The evaluation of the arbitrary coefficients or their elimination to obtain a critical condition is achieved by the assignment of boundary conditions. The physical nature of these is now reviewed.

(a) Flux Zero at Core Boundary.

The assumption is made that both ϕ_1 and ϕ_2 vanish at the same extrapolated boundary of the core. For most thermal reactors the extrapolation distances, of the order of $0.71 \lambda_t$, are small compared with core radii. No serious error is incurred by using the same average or effective extrapolation distance for both groups, and the analysis is greatly simplified.

Formally then,

$$\phi_1(R) = 0 \quad \text{.....(27)}$$

$$\phi_2(R) = 0 \quad \text{.....(28)}$$

which immediately implies from equations (23) and (24), since $S_1 \neq S_2$, that

$$X(R) = 0 \quad \text{.....(29)}$$

$$Y(R) = 0 \quad \text{.....(30)}$$

These conditions must be true at every angle θ on the core surface.

(b) Fluxes Extrapolated into Control Element.

Two methods may be contemplated for treating the thermal flux in the vicinity of the control rod. One as used by Garabedian is

to state that the flux vanishes at an effective rod radius equal to $a - d_2$ where a is the actual radius and d_{2m} is an extrapolation distance appropriate to the m^{th} rod. This is believed to be less meaningful than a second approach, in which one states

$$\frac{\phi_2'(a_m)}{\phi_2(a_m)} = d_{2m} \quad \dots\dots(31)$$

where the prime refers to differentiation with respect to r_m . The numerical results differ for these two approaches. The former in effect prescribes a Bessel function flux variation within the rod, between points $a - d_2$ and a . The latter refers only to conditions at radius a . For thermal absorbers that are black, the thermal extrapolation distances as deduced by Davison and Kushneriuk⁽⁶⁾ using transport theory are recommended; for partially transparent rods, an extension by Kushneriuk and McKay⁽⁷⁾ is available. For some absorbers it may be convenient to solve a separate diffusion equation to deduce fluxes and d_{2m} . The fast group by analogy is assigned an extrapolation distance d_1 such that

$$\frac{\phi_1(a_m)}{\phi_1'(a_m)} = d_{1m} \quad \dots\dots(32)$$

where the value of d_{1m} may be obtained by a separate diffusion analysis of the rod within the core. Presuming that the rod does not contain fissionable material, but consists of elements

with a finite moderating ability, the fast d_1 may be obtained from the relation⁽⁸⁾

$$d_{1m} = \frac{D_{1c} I_0(\kappa_{1m} a_m)}{D_{1m} \kappa_{1m} I_1(\kappa_{1m} a_m)} \quad \dots\dots(33)$$

where subscripts c and m refer to core and m^{th} rod respectively and the inverse fast diffusion lengths includes correction for axial buckling. Regardless of the method of computation used, it will be presumed that values of d_{1m} and d_{2m} peculiar to the rod m are available, and that they are properties of the whole rod, not dependent on angle Ψ_m . It is important to note that boundary conditions (31) and (32) will be applied to every rod in the system, which implies a total of $2(M + 1)$ equations.

A convenient form for (31) and (32) is

$$\left(\frac{X}{d_{1m}} - X'\right) + \left(-\frac{Y}{d_{1m}} - Y'\right) = 0 \quad \dots\dots(34)$$

$$S_1\left(\frac{X}{d_{2m}} - X'\right) + S_2\left(\frac{Y}{d_{2m}} - Y'\right) = 0 \quad \dots\dots(35)$$

where functions and their derivatives are evaluated at $r_m = a_m$, and a total of $M + 1$ sets of equations are implied.

5. APPLICATION OF CORE SURFACE CONDITIONS.

The function X that must be zero at $r = R$ is noted to contain the explicit variable r only in part of the sums, and to involve $r_1, r_2, r_3, \dots, r_M$ in the remainder. To be able to insert R , the latter functions must be transformed, using addition theorems of Bessel functions⁽¹¹⁾. Watson quotes a set of relationships between functions as they pertain to the sides of a triangle as shown in Figure 3.

FIGURE 3. REFERENCE TRIANGLE FOR TRANSFORMATIONS

With the requirement that $Z > z > \omega$, then

$$J_s(\omega) \begin{cases} \cos s\psi \\ \sin s\psi \end{cases} = \sum_k J_s + k(Z)J_k(z) \begin{cases} \cos k\alpha & \dots\dots(36) \\ \sin k\alpha & \dots\dots(37) \end{cases}$$

$$Y_s(\omega) \begin{cases} \cos s\psi \\ \sin s\psi \end{cases} = \sum_k Y_s + k(Z)J_k(z) \begin{cases} \cos k\alpha & \dots\dots(38) \\ \sin k\alpha & \dots\dots(39) \end{cases}$$

$$I_s(\omega) \begin{cases} \cos s\psi \\ \sin s\psi \end{cases} = \sum_k (-1)^k I_s + k(Z)I_k(z) \begin{cases} \cos k\alpha & \dots\dots(40) \\ \sin k\alpha & \dots\dots(41) \end{cases}$$

$$K_s(\omega) \begin{cases} \cos s\psi \\ \sin s\psi \end{cases} = \sum_k K_s + k(Z)I_k(z) \begin{cases} \cos k\alpha & \dots\dots(42) \\ \sin k\alpha & \dots\dots(43) \end{cases}$$

In the above equations $\sum_k$ means $\sum_{k=-\infty}^{k=\infty}$.

FIGURE 4. GEOMETRY FOR TRANSFORMATION r_m to r for $r \approx R$.

The use of the transformations will be illustrated by expressing $Y_s(\mu r_m) \cos s \Psi_m$, applicable to any rod m , in terms of r . We first identify angles and distances according to Figures 3 and 4 and using the table below:

<u>Watson</u>	<u>Here</u>
ω	μr_m
Ψ	Ψ
Z	μr
z	μd_m
α	$\alpha_m = \beta_m = \theta$

From Figure 4,

$$\Psi_m = 2\pi - (\alpha_m + \Psi)$$

Thus,

$$\begin{aligned} \cos s \Psi_m &= \cos s [2\pi - (\alpha_m + \Psi)] = \cos s (\alpha_m + \Psi) \\ \sin s \Psi_m &= \sin s [2\pi - (\alpha_m + \Psi)] = -\sin s (\alpha_m + \Psi) \end{aligned}$$

Multiply both sides of equation (38) by $\cos s \alpha_m$, and both sides of equation (39) by $\sin s \alpha_m$, and subtract the results, using standard trigonometric identities. This yields

$$Y_s(\mu r_m) \cos s \Psi_m = \sum_k Y_s + k(\mu r) J_k(\mu d_m) \cos(k + s) \alpha_m \dots\dots(44)$$

The summation over k may be rewritten by replacing $k + s$ by k' , k by $k' - s$ and summing over k' . Dropping the prime,

$$Y_s(\mu r_m) \cos s \Psi_m = \sum_k Y_k(\mu r) J_{k-s}(\mu d_m) \cos k(\beta_m - \theta) \dots\dots(45)$$

In order to transform $Y_s(\mu r_m) \sin s\psi_m$, we instead multiply (38) by $\sin s\alpha_m$ and (39) by $\cos s\alpha_m$ and add. This gives

$$Y_s(\mu r_m) \sin s\psi_m = -\sum_k Y_k(\mu r) J_{k-s}(\mu d_m) \sin k(\beta_m - \theta) \dots (46)$$

By direct analogy,

$$K_s(\nu r_m) \cos s\psi_m = \sum_k K_k(\nu r) I_{k-s}(\nu d_m) \cos k(\beta_m - \theta) \dots (47)$$

$$K_s(\nu r_m) \sin s\psi_m = -\sum_k K_k(\nu r) I_{k-s}(\nu d_m) \sin k(\beta_m - \theta) \dots (48)$$

Insert equations (45) and (46) into equation (29) using (25)

$$\begin{aligned} X(R) - 0 &= \sum_s A_s J_s(\mu R) \cos s\theta + \sum_s N_s J_s(\mu R) \sin s\theta \\ &+ \sum_{m=0}^M \sum_s B_{sm} \sum_k Y_k(\mu R) J_{k-s}(\mu d_m) \cos k(\beta_m - \theta) \\ &- \sum_{m=0}^M \sum_s P_{sm} \sum_k Y_k(\mu R) J_{k-s}(\mu d_m) \sin k(\beta_m - \theta) \dots (49) \end{aligned}$$

A convenient theorem of summations is invoked:

$$\sum_s a_s \sum_k b_k c_{k-s} = \sum_k b_k \sum_s a_s c_{k-s} \dots (50)$$

The third term, for example, becomes

$$\sum_{m=0}^M \sum_k Y_k(\mu R) \cos k(\beta_m - \theta) \sum_s B_{sm} J_{k-s}(\mu d_m)$$

The cosines and sines of $k(\beta_m - \theta)$ are expanded, and coefficients of all $\cos k\theta$ and $\sin k\theta$ in equation (49) are isolated. (In the first two sums of equation (49), s may be replaced by k .)

$$\begin{aligned}
X(R) = 0 = & \sum_k \left[A_k J_k(\mu R) + Y_k(\mu R) \sum_{s=0}^M \sum_{m=0}^{\infty} B_{sm} J_k - s(\mu d_m) \cos k\beta_m \right. \\
& - Y_k(\mu R) \sum_{s=0}^M \sum_{m=0}^{\infty} P_{sm} J_k - s(\mu d_m) \sin k\beta_m \left. \right] \cos k\theta \\
& + \sum_k \left[N_k J_k(\mu R) + Y_k(\mu R) \sum_{s=0}^M \sum_{m=0}^{\infty} B_{sm} J_k - s(\mu d_m) \sin k\beta_m \right. \\
& + Y_k(\mu R) \sum_{s=0}^M \sum_{m=0}^{\infty} P_{sm} J_k - s(\mu d_m) \cos k\beta_m \left. \right] \sin k\theta \quad \dots\dots(51)
\end{aligned}$$

An analogous expression for $Y(R)$ is readily obtained by replacing $Y_k(\mu R)$ by $K_k(\nu R)$ and $J_k - s(\mu d_m)$ by $I_k - s(\nu d_m)$.

If the flux is to vanish at $r = R$ for all angles θ , i.e. all points on the core perimeter, then the coefficients of $\cos k\theta$ must be zero for every k . This provides relations among the arbitrary coefficients A_k , B_{sm} and P_{sm} , N_k , B_{sm} and P_{sm} , and the corresponding sets of coefficients from $Y(R) = 0$. These are

$$A_k = - \frac{Y_k(\mu R)}{J_k(\mu R)} \left\{ \sum_{m=0}^M \sum_{s=0}^{\infty} J_k - s(\mu d_m) \left[B_{sm} \cos k\beta_m - P_{sm} \sin k\beta_m \right] \right\} \quad \dots\dots(52)$$

$$N_k = - \frac{Y_k(\mu R)}{J_k(\mu R)} \left\{ \sum_{m=0}^M \sum_{s=0}^{\infty} J_k - s(\mu d_m) \left[B_{sm} \sin k\beta_m + P_{sm} \cos k\beta_m \right] \right\} \quad \dots\dots(53)$$

$$C_k = - \frac{K_k(\nu R)}{I_k(\nu R)} \left\{ \sum_{m=0}^M \sum_{s=0}^{\infty} I_k - s(\nu d_m) \left[E_{sm} \cos k\beta_m - I_{sm} \sin k\beta_m \right] \right\} \quad \dots\dots(54)$$

$$Q_k = - \frac{K_k(\nu R)}{I_k(\nu R)} \left\{ \sum_{m=0}^M \sum_{s=0}^{\infty} I_k - s(\nu d_m) \left[E_{sm} \sin k\beta_m + I_{sm} \cos k\beta_m \right] \right\} \quad \dots\dots(55)$$

6. APPLICATION OF ROD SURFACE CONDITIONS.

Consider the application of equations (31) and (32) at the surface of a given rod m . Direct substitution of $r_m = a$ in the functions X and Y is possible only in functions $Y_s(\mu r_m)$ and $K_s(\nu r_m)$. Expressions involving r or r_n where n is not equal to m must be subjected to Bessel function transformations.

(a) Functions of r .

In order to transform $J_s(\mu r)$, the triangle of Watson, Figure 3 is compared with Figure 5.

FIGURE 5. GEOMETRY FOR TRANSFORMATION r to r_m for $r_m \approx a$.

In tabular form:

<u>Watson</u>	<u>Here</u>
ω	μr
Ψ	$\alpha_m - \beta_m - \theta$
Z	μd
z	μr_m
α	$\Psi_m - \pi$

From equations (36) and (37) the addition theorems become

$$J_s(\mu r) \cos s\alpha_m = \sum_k J_s + k(\mu d_m) J_k(\mu r_m) \cos k(\Psi_m - \pi) \dots (56)$$

$$J_s(\mu r) \sin s\alpha_m = \sum_k J_s + k(\mu d_m) J_k(\mu r_m) \sin k(\Psi_m - \pi) \dots (57)$$

We note the fact that $\cos k\pi = (-1)^k$ and $\sin k\pi = 0$. Multiply the first equation by $\cos (-s\beta_m)$ and the second by $\sin (-s\beta_m)$ and subtract the second to yield,

$$\underline{J_s(\mu r) \cos s\theta = \sum_k (-1)^k J_s + k(\mu d_m) J_k(\mu r_m) \cos (s\beta_m - k\Psi_m)} \dots (58)$$

Alternatively, multiply the first by $\sin s\beta_m$ and the second by $\cos s\beta_m$ and subtract the second to obtain

$$\underline{J_s(\mu r) \sin s\theta = \sum_k (-1)^k J_s + k(\mu d_m) J_k(\mu r_m) \sin (s\beta_m - k\Psi_m)} \dots (59)$$

A similar procedure may be applied to obtain the transformations of $I_s(\nu r)$, with attention to signs. The results are

$$\underline{I_s(\nu r) \cos s\theta = \sum_k I_s + k(\nu d_m) I_k(\nu r_m) \cos (s\beta_m - k\Psi_m)} \dots (60)$$

$$\frac{I_s(vr) \sin s\theta - \sum_k \frac{I_k}{k} s + k (vd_m) I_k(vr_m) \sin (s\theta_m - k\psi_m)}{\dots\dots(61)}$$

(a) Functions of r_n , $n \neq m$.

We now construct Figure 6 which includes distances and angles for any pair of rods m and n .

FIGURE 6. GEOMETRY FOR TRANSFORMATION r_n to r_m , for r_n a

Watson's triangle is compared with the one formed by rod centers and the point P. Use the relations

<u>Watson</u>	<u>Here</u>
ω	μr_n
Ψ	a
Z	μr_{mn}
z	μr_m
α	c

It will be useful to label also angles γ and δ . From the figure,

$$a = \Psi_n - \pi - \gamma$$

$$c = \pi - \Psi_m - \delta$$

Apply equations (38) and (39) to the triangle.

$$Y_s(\mu r_n) \cos s[\Psi_n - \pi - \gamma] = \sum_k Y_s + k(\mu r_{mn}) J_k(\mu r_m) \cos k[\pi - \delta - \Psi_m] \quad \text{.....(62)}$$

$$Y_s(\mu r_n) \sin s[\Psi_n - \pi - \gamma] = \sum_k Y_s + k(\mu r_{mn}) J_k(\mu r_m) \sin k[\pi - \delta - \Psi_m] \quad \text{.....(63)}$$

Multiply equation (62) by $\cos s(\pi + \gamma)$ and equation (63) by $\sin s(\pi + \gamma)$ and subtract the latter, giving

$$Y_s(\mu r_n) \cos s\Psi_n = \sum_k Y_s + k(\mu r_{mn}) J_k(\mu r_m) \cos[k(\pi - \delta - \Psi_m) + s(\pi + \gamma)] \quad \text{.....(64)}$$

Following Garabedian's notation, let

$$p_{ks} = k(\pi - \delta) + s(\pi + \gamma) \quad \text{.....(65)}$$

The cosine function becomes $\cos(p_{ks} - k\Psi_m)$. Thus

$$\underline{Y_s(\mu r_n) \cos s\Psi_m = \sum_k Y_s + k(\mu r_{mn}) J_k(\mu r_m) \cos(p_{ks} - k\Psi_m)} \quad \text{.....(66)} \quad 23.$$

Similarly, $Y_s(\mu r_n) \sin s\psi_n$ may be obtained by multiplying equation (62) by $\sin[-s(\pi + \gamma)]$ and (63) by $\cos[-s(\gamma + \pi)]$ and subtracting the first from the second. The result is

$$Y_s(\mu r_n) \sin s\psi_n = \frac{\sum_k Y_{s+k}(\mu r_{mn}) J_k(\mu r_m) \sin(p_{ks} - k\psi_m)}{k s + k} \dots (67)$$

The corresponding transformations of $K_s(vr_n)$ are achieved by analogy, as

$$K_s(vr_n) \cos s\psi_n = \frac{\sum_k K_{s+k}(vr_{mn}) I_k(vr_m) \cos(p_{ks} - k\psi_m)}{k s + k} \dots (68)$$

$$K_s(vr_n) \sin s\psi_n = \frac{\sum_k K_{s+k}(vr_{mn}) I_k(vr_m) \sin(p_{ks} - k\psi_m)}{k s + k} \dots (69)$$

We are now ready to form the X and Y functions near rod m.

Inserting (58) and (59) along with (66) and (67) into (25),

$$\begin{aligned} X(r_m) = & \sum_s A_s \sum_k (-1)^k J_{s+k}(\mu d_m) J_k(\mu r_m) \cos(s\beta_m - k\psi_m) \\ & + \sum_s B_s \sum_k (-1)^k J_{s+k}(\mu d_m) J_k(\mu r_m) \sin(s\beta_m - k\psi_m) \\ & + \sum_s B_{sm} Y_s(\mu r_m) \cos s\psi_m + \sum_s P_{sm} Y_s(\mu r_m) \sin s\psi_m \\ & + \sum_{n \neq m}^M \sum_k \frac{L B_{sn} E Y_{s+k}(\mu r_{mn}) J_k(\mu r_m) \cos(p_{ks} - k\psi_m)}{s s_n k s + k} \\ & + \sum_{n \neq m}^M \sum_k \frac{L P_{sn} E Y_{s+k}(\mu r_{mn}) J_k(\mu r_m) \sin(p_{ks} - k\psi_m)}{s s_n k s + k} \dots (70) \end{aligned}$$

Also, inserting (60) and (61) along with (68) and (69) into (26),

$$\begin{aligned}
Y(r_m) = & \sum_s C_s \sum_k I_k s + k (v d_m) I_k (v r_m) \cos(s \beta_m - k \psi_m) \\
& + \sum_s Q_s \sum_k I_k s + k (v d_m) I_k (v r_m) \sin(s \beta_m - k \psi_m) \\
& + \sum_s E_{sm} K_s (v r_m) \cos s \psi_m + \sum_s T_{sm} K_s (v r_m) \sin s \psi_m \\
& + \sum_{n \neq m}^M \sum_s E_{sn} \sum_k K_k s + k (v r_{mn}) I_k (v r_m) \cos(p_{ks} - k \psi_m) \\
& + \sum_{n \neq m}^M \sum_s T_{sn} \sum_k K_k s + k (v r_{mn}) I_k (v r_m) \sin(p_{ks} - k \psi_m) \dots (71)
\end{aligned}$$

Since A_s , N_s , C_s and Q_s are expressed in terms of B_{sm} , P_{sm} , E_{sm} and T_{sm} in equations (52), (53), (54) and (55), the above pair of equations for X and Y involve only the latter four sets of constants. There remain however, only two available sets of rod boundary conditions, equations (34) and (35). Approximations are therefore needed to achieve a compatible critical condition.

7. SIMPLIFICATIONS AND APPROXIMATIONS.

It is evident that the insertion of expressions for A_s and N_s , C_s and Q_s into the above forms for $X(r_m)$ and $Y(r_m)$, followed by formal application of rod boundary conditions will yield a very unwieldy relation. This multiply-infinite set of equations cannot be handled except by chopping the series of coefficients off at low order. It is important to ascertain the truncation that causes the least loss of physical content of the problem. The following approximation is proposed:

The flux distribution that is associated with the sink terms is essentially symmetric (independent of $\bar{\psi}_m$) at each rod surface $r_m = a$.

The physical logic that makes this reasonable is that if the rod were immersed in an infinite reactor, the assumption would be exact; when superposed on a basic $J_s(\mu r)$ or $I_s(\nu r)$ function that is slowly varying over the core, the assumption is still accurate. The consequence of this approximation is that B_{sm} is zero for $s \neq 0$, and similarly for P_{sm} , E_{sm} and T_{sm} . This causes A_s , N_s , C_s and Q_s to be sharply abbreviated. Furthermore, it allows us to neglect all coefficients of $\text{sink } \bar{\psi}_m$ and $\text{cos } k \bar{\psi}_m$ in (70) and (71) except when $k = 0$, and since $\text{sink } \bar{\psi}_m$ is automatically zero for $k = 0$, this leaves only coefficients of $\text{cos } 0 \bar{\psi}_m = 1$. The details of the reduction and combination of equations (70)

and (71) are too tedious to reproduce. The final forms for $X(r_m)$ and $Y(r_m)$ only are given.

$$X(r_m) \simeq -J_0(\mu r_m) \sum_{n=0}^M B_{0n} \sum_{s=0}^{\infty} \frac{Y_s(\mu R)}{J_s(\mu R)} J_s(\mu d_m) J_s(\mu d_n) \cos s(\beta_m - \beta_n) \\ + J_0(\mu r_m) \sum_{n \neq m}^M B_{0n} Y_0(\mu r_m) + B_{0m} Y_0(\mu r_m) \dots (72)$$

$$Y(r_m) \simeq -I_0(\nu r_m) \sum_{n=0}^M E_{0n} \sum_{s=0}^{\infty} \frac{K_s(\nu R)}{I_s(\nu R)} I_s(\nu d_m) I_s(\nu d_n) \cos s(\beta_m - \beta_n) \\ + I_0(\nu r_m) \sum_{n \neq m}^M E_{0n} K_0(\nu r_m) + E_{0m} K_0(\nu r_m) \dots (73)$$

Approximations in $Y(r_m)$

Examination of characteristic functions of $Y(r_m)$ in (73) shows typical ratios of the type $\frac{K_s(\nu R)}{I_s(\nu R)} I_s(\nu d_m) I_s(\nu d_n)$. From equation (7), ν^2 is of order $\frac{1}{L^2} + \frac{1}{\tau}$, since $\mu^2 L^2 \ll 1$ and $\mu^2 \tau \ll 1$ for a practical reactor with small fast and thermal leakage. In the case of a water moderated core with $L^2 \ll \tau$, then $\nu \simeq \frac{1}{L}$, or for a graphite or heavy water core with $L^2 \simeq \tau$, $\nu \simeq \frac{\sqrt{2}}{L}$. Invoking the leakage conditions again, for simplicity using a spherical core model, $\left(\frac{\pi}{R}\right)^2 L^2 \ll 1$ or $\frac{R}{L} \gg \pi$. This is equivalent to saying

$$\nu R \gg \pi \quad \text{or} \quad \nu R \gg \frac{\pi}{\sqrt{2}}$$

This puts the K_s and I_s functions in the region of large argument. For example with $\nu = 0.5$ and $R = 20$ cm as for a typical water moderated core $\nu R = 10$.

Consider for instance the form of $K_0(x)$ and $I_0(x)$ for large x .

$$K_0(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x}$$

$$I_0(x) \simeq \frac{e^x}{\sqrt{2\pi x}}$$

The ratio is of order e^{-2x} or e^{-20} . Now so long as d_m and d_n are two or three diffusion lengths less in magnitude than the core radius R , their contribution of such terms as $I_s(vd_m)$ and $I_s(vd_n)$ to raise this ratio is not serious. Arbitrary coefficients E_{0n} are comparable in size with B_{0n} , which allows the summation expressions over modified Bessel functions to be neglected. Also, the interaction terms $K_0(vr_{mn})$ are small if the rods are separated by as much as two or three diffusion lengths. This leaves only

$$Y(r_m) \simeq E_{0m} K_0(vr_m) \quad \dots\dots(74)$$

where $K_0(vr_m)$ is normally not negligible since r_m becomes the rod radius a .

Now abbreviate in $X(r_m)$ in (72) the sum over s by $\textcircled{H}_{mn}$.

$$\textcircled{H}_{mn} = \sum_s \frac{Y_s(\mu R)}{J_s(\mu R)} J_s(\mu d_m) J_s(\mu d_n) \cos s(\beta_m - \beta_n) \quad \dots\dots(75)$$

Regrouping terms somewhat,

$$X(r_m) = J_0(\mu r_m) \left\{ \sum_{n \neq m}^M B_{0n} Y_0(\mu r_{mn}) - \sum_{n=0}^M B_{0n} \textcircled{H}_{mn} \right\} + Y_0(\mu r_m) B_{0n} \quad \dots\dots(76)$$

Apply the rod boundary conditions (34) and (35) at $r_m = a_m$,

noting that the quantities in braces are independent of the variable r_m .

$$\left[\frac{J_0(\mu a_m)}{d_{1m}} + \mu J_1(\mu a_m) \right] \left\{ \sum_{n \neq m}^M B_{0n} Y_0(\mu r_{mn}) - \sum_{n=0}^M B_{0n} \textcircled{H}_{mn} \right\} + \left[\frac{Y_0(\mu a_m)}{d_{1m}} + \mu Y_1(\mu a_m) \right] B_{0m} + \left[\frac{K_0(\nu a_m)}{d_{1m}} + \nu K_1(\nu a_m) \right] E_{0m} = 0 \quad \text{.....(77)}$$

$$S_1 \left[\frac{J_0(\mu a_m)}{d_{2m}} + \mu J_1(\mu a_m) \right] \left\{ \sum_{n \neq m}^M B_{0n} Y_0(\mu r_{mn}) - \sum_{n=0}^M B_{0n} \textcircled{H}_{mn} \right\} + S_1 \left[\frac{Y_0(\mu a_m)}{d_{2m}} + \mu Y_1(\mu a_m) \right] B_{0m} + S_2 \left[\frac{K_0(\nu a_m)}{d_{2m}} + \nu K_1(\nu a_m) \right] E_{0m} = 0 \quad \text{.....(78)}$$

Introduce the abbreviations for certain characteristic constants of the m^{th} rod,

$$a_{31}^m = \frac{J_0(\mu a_m)}{d_{1m}} + \mu J_1(\mu a_m) \quad \text{.....(79)}$$

$$a_{32}^m = \frac{Y_0(\mu a_m)}{d_{1m}} + \mu Y_1(\mu a_m) \quad \text{.....(80)}$$

$$a_{34}^m = \frac{K_0(\nu a_m)}{d_{1m}} + \nu K_1(\nu a_m) \quad \text{.....(81)}$$

$$a_{41}^m = S_1 \left[\frac{J_0(\mu a_m)}{d_{2m}} + \mu J_1(\mu a_m) \right] \quad \text{.....(82)}$$

$$a_{42}^m = S_1 \left[\frac{Y_0(\mu a_m)}{d_{2m}} + \mu Y_1(\mu a_m) \right] \quad \text{.....(83)}$$

$$a_{44}^m = S_2 \left[\frac{K_0(\nu a_m)}{d_{2m}} + \nu K_1(\nu a_m) \right] \quad \text{.....(84)}$$

Now multiplying equation (77) through by a_{44}^m and equation (78)

8. GENERAL CRITICAL DETERMINANT.

If this set of homogeneous equations is to be non-trivial, the determinant of the coefficients of the B_{0m} is zero. The resultant critical determinant, of order $M + 1$ may be abbreviated by

$$\Delta = 0 \quad \text{.....(88)}$$

where the elements of Δ are (m = row, n = column)

$$\Delta_{mn} = \textcircled{H}_{mn} - Y_0(\mu r_{mn}) \quad (m \neq n) \quad \text{.....(89)}$$

$$\Delta_{mn} = \textcircled{H}_{mn} - \frac{1}{Y_{am}} \quad \text{.....(90)}$$

Recalling the definition of $\textcircled{H}_{mn}$ in equation (75), the following properties are of interest

$$\textcircled{H}_{mm} = \sum_s \frac{Y_s(\mu R)}{J_s(\mu R)} J_s^2(\mu d_m) \quad \text{.....(91)}$$

$\cos s(\beta_m - \beta_n)$ being identically 1. Also,

$$\textcircled{H}_{mn} = \textcircled{H}_{nm} \quad \text{.....(92)}$$

since $\cos s(\beta_m - \beta_n) = \cos s(\beta_n - \beta_m)$. Equations (89) and (90) represent the most compact statement of the critical condition, but do not fully express the symmetries that exist. The size (or order) of the critical determinant clearly corresponds directly to the number of rods in the reactor.

$$\Delta = \begin{vmatrix} \textcircled{H}_{00} - \frac{1}{Y_{a0}} & \textcircled{H}_{01} - Y_0(\mu r_{01}) & \textcircled{H}_{02} - Y_0(\mu r_{02}) & \text{---} \\ \textcircled{H}_{01} - Y_0(\mu r_{01}) & \textcircled{H}_{11} - \frac{1}{Y_{a1}} & \textcircled{H}_{12} - Y_0(\mu r_{12}) & \text{---} \\ \textcircled{H}_{02} - Y_0(\mu r_{02}) & \textcircled{H}_{12} - Y_0(\mu r_{21}) & \textcircled{H}_{22} - \frac{1}{Y_{a2}} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{vmatrix} = 0 \quad \text{.....(93)}$$

through by a_{34}^m and subtracting, the constant E_{0m} can be eliminated.

In terms of a new composite rod parameter

$$\Psi_{am} = \frac{a_{31}^m a_{44}^m - a_{41}^m a_{34}^m}{a_{32}^m a_{44}^m - a_{42}^m a_{34}^m} \quad \dots\dots(85)$$

The resultant equation is

$$- \sum_{n \neq m}^M B_{0n} Y_0(\mu r_{mn}) + \sum_{n=0}^M B_{0n} \textcircled{H}_{mn} - \frac{B_{0m}}{\Psi_{am}} = 0 \quad \dots\dots(86)$$

For each rod m in the system an individual equation is developed.

The first several are written out. In order, they refer to rod 0,

rod 1, ---.

$$\begin{aligned} B_{00} \left[\textcircled{H}_{00} - \frac{1}{\Psi_{a0}} \right] + B_{01} \left[\textcircled{H}_{01} - Y_0(\mu r_{01}) \right] + B_{02} \left[\textcircled{H}_{02} - Y_0(\mu r_{02}) \right] \\ + \dots = 0 \\ B_{00} \left[\textcircled{H}_{10} - Y_0(\mu r_{10}) \right] + B_{01} \left[\textcircled{H}_{11} - \frac{1}{\Psi_{a1}} \right] + B_{02} \left[\textcircled{H}_{12} - Y_0(\mu r_{12}) \right] \\ + \dots = 0 \\ B_{00} \left[\textcircled{H}_{20} - Y_0(\mu r_{20}) \right] + B_{01} \left[\textcircled{H}_{21} - Y_0(\mu r_{21}) \right] + B_{02} \left[\textcircled{H}_{22} - \frac{1}{\Psi_{a2}} \right] \\ + \dots = 0 \quad \dots\dots(87) \end{aligned}$$

Note the symmetry about the main diagonal, by use of equation (92) and the evident fact that $Y_0(\mu r_{mn}) = Y_0(\mu r_{nm})$.

Terms along the main diagonal refer to the core size and material through μ and R , to the location of each rod in the core through d_m and B_m , and to properties of the rod through Ψ_{am} . Terms off the diagonal describe the mutual interaction, through d_m, d_n, B_m, B_n and $Y_0(\mu r_{mn})$.

The correlation between the perturbation theory flux-squared weighting and the terms $J_s^2(\mu d_m)$ in $\mathbb{H}_{mm}$ is to be noted. The appearance of the factors $\cos s(B_m - B_n)$ and $J_s(\mu d_m)J_s(\mu d_n)$ in $\mathbb{H}_{mn}$, although lacking obvious physical meaning, is not unexpected.

9. REDUCTION OF CRITICAL CONDITION FOR SYMMETRIC ARRAYS.

An internal check of the method is available by applying it to a number of special cases that have already been solved, using the symmetric properties and identity of rods in composition and size. Since these special arrays are encountered, it is not always necessary to employ the full generality of an $M + 1$ determinant. We assume throughout the following development that distances d_m of eccentric rods are the same, and that the angles β_m are equal multiples of some fraction of a full circle, $\beta_m = (m - 1)(2\pi/M)$, where rod 1 serves to define the reference direction. Also Ψ_{am} is unique, of value Ψ_a .

Case 1. Central Rod.

The critical determinant has only one element Δ_{00} , hence (93) reduces to

$$\Delta_{00} = \textcircled{H}_{00} - \frac{1}{\Psi_a} = 0$$

Now $J_s(\mu d)$ with $d = 0$ in (91) reduces to $J_s(0)$ which is identically zero except for $s = 0$, when it is unity. Thus

$$\textcircled{H}_{00} = \frac{Y_0(\mu R)}{J_0(\mu R)}$$

and

$$\Psi_a = \frac{J_0(\mu R)}{Y_0(\mu R)} \quad \text{.....(94)}$$

which is the standard critical condition. (8)

Case 2. Single Eccentric Rod.

Now, with no center rod, Δ_{00} , Δ_{01} , Δ_{10} , etc. do not appear, and the determinant (93) is

$$\Delta_{11} = \textcircled{H}_{11} - \frac{1}{\Psi_a} = 0$$

and since from (91)

$$\textcircled{H}_{11} = \int_s \frac{Y_s(\mu R)}{J_s(\mu R)} J_s^2(\mu d),$$

$$\Psi_a = \left[\int_s \frac{Y_s(\mu R)}{J_s(\mu R)} J_s^2(\mu d) \right]^{-1} \dots\dots(95)$$

again a known solution. (8)

Case 3. Two Opposing Rods.

The determinant is now of order two,

$$\Delta = \begin{vmatrix} \textcircled{H}_{11} - \frac{1}{\Psi_{a1}} & \textcircled{H}_{12} - Y_0(\mu r_{12}) \\ \textcircled{H}_{12} - Y_0(\mu r_{12}) & \textcircled{H}_{22} - \frac{1}{\Psi_{a2}} \end{vmatrix} = 0$$

However, $\frac{1}{\Psi_{a1}} = \frac{1}{\Psi_{a2}} = \frac{1}{\Psi_a}$ from the properties of these functions.

The angle $\beta_1 - \beta_2$ is π , and $\cos s(\beta_1 - \beta_2) = \cos s\pi = (-1)^s$.

Thus multiplying out and taking the square root,

$$(\textcircled{H}_{11} - \frac{1}{\Psi_a}) = \pm [\textcircled{H}_{12} - Y_0(\mu r_{12})]$$

Examination of the nature of the functions reveals that the only meaningful sign is negative, hence,

$$\textcircled{H}_{11} + \textcircled{H}_{12} - Y_0(\mu r_{12}) = \frac{1}{\Psi_a}$$

The sum of the $\textcircled{H}$'s is

$$\sum_s \frac{Y_s(\mu R)}{J_s(\mu R)} J_s^2(\mu d) [1 + (-1)^s]$$

The quantity in brackets is equal to 2 if s is 0, 2, 4, ---, and zero if s is 1, 3, 5, ---. Thus, we find

$$\Psi_a = \left[\frac{2\pi}{s} \frac{Y_{2s}(\mu R)}{J_{2s}(\mu R)} J_{2s}^2(\mu d) - Y_0(\mu r_{12}) \right]^{-1} \dots (96)$$

This has been obtained previously. (9)

Case 4. Three Rods in a Triangle.

The angles between rods are now all $120^\circ = 2\pi/3$, and $\cos s(\theta_m - \theta_n)$ takes on values $\cos s \frac{2\pi}{3}$ ($m = 1, n = 2$ or $m = 2, n = 3$) and $\cos s \frac{4\pi}{3}$ ($m = 1, n = 3$). However, $\cos s \frac{2\pi}{3} = \cos s \frac{4\pi}{3}$ and hence $\textcircled{H}_{21} = \textcircled{H}_{31} = \textcircled{H}_{32}$. Also $\textcircled{H}_{11} = \textcircled{H}_{22} = \textcircled{H}_{33}$ and $Y_0(\mu r_{12}) = Y_0(\mu r_{13}) = Y_0(\mu r_{23}) = Y_0(\mu d)$. The determinant reduces to the form

$$\Delta = \begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = (a - b)^2 (a + 2b) = 0$$

The relevant factor to set equal to zero is $a + 2b$, which gives

$$\textcircled{H}_{11} - \frac{1}{\Psi_a} + 2 \left[\textcircled{H}_{21} - Y_0(\mu r_{12}) \right] = 0$$

We invoke the fact that $\cos s \frac{2\pi}{3} = -\frac{1}{2}$ for $s = 1, 2, 4, 5, \dots$
 $= 1$ for $s = 0, 3, 6, \dots$

Hence the sum

$$\textcircled{H}_{11} + 2 \textcircled{H}_{21} = \sum_s \frac{Y_s(\mu R)}{J_s(\mu R)} J_s^2(\mu d) \left[1 + 2 \cos s \frac{2\pi}{3} \right]$$

reduces to

$$\textcircled{H}_{11} + 2\textcircled{H}_{21} = 3\sum_s \frac{Y_{3s}(\mu R)}{J_{3s}(\mu R)} J_{3s}^2(\mu d)$$

and

$$\Psi_a = \left[3\sum_s \frac{Y_{3s}(\mu R)}{J_{3s}(\mu R)} J_{3s}^2(\mu d) - 2Y_0(\mu r_{12}) \right]^{-1} \dots (97)$$

as known. (9)

Case 5. M Rods on a Ring.

For higher numbers of equally spaced identical rods, the algebraic reduction of the critical determinant becomes involved. The reduction of the 4 x 4 determinant corresponding to 4 rods 90° apart has been carried out and found to verify the previously known (9) general formula,

$$\Psi_a = \left[M\sum_s \frac{Y_{Ms}(\mu R)}{J_{Ms}(\mu R)} J_{Ms}^2(\mu d) - \sum_{m=2}^M Y_0(\mu r_{1m}) \right]^{-1} \dots (98)$$

Case 6. Central Plus Eccentric Rod.

The determinant is

$$\Delta = \begin{vmatrix} \textcircled{H}_{00} - \frac{1}{\Psi_a} & \textcircled{H}_{01} - Y_0(\mu r_{01}) \\ \textcircled{H}_{01} - Y_0(\mu r_{01}) & \textcircled{H}_{11} - \frac{1}{\Psi_a} \end{vmatrix} = 0$$

Since $Y_0(\mu r_{01}) = Y_0(\mu d)$, we obtain (9)

$$\left(\textcircled{H}_{00} - \frac{1}{\Psi_a} \right) \left(\textcircled{H}_{11} - \frac{1}{\Psi_a} \right) - \left[\textcircled{H}_{01} - Y_0(\mu d) \right]^2 \dots (99)$$

The $\textcircled{H}$ functions take on a special form as noted in previous cases.

From Case 1, $\textcircled{H}_{00} = \frac{Y_0(\mu R)}{J_0(\mu R)}$

From Case 2, $\textcircled{H}_{11} = \frac{Y_s(\mu R)}{J_s(\mu R)} J_s^2(\mu d)$

Finally,

$$\textcircled{H}_{01} = \frac{Y_0(\mu R)}{J_0(\mu R)} J_0(\mu d)$$

With these relations, the critical condition may be computed.

Case 7. Ring and Central Rod.

The reduction of the determinant for specific numbers of M equally-spaced identical eccentric rods plus a central rod is tedious, but leads to an independently-obtained solution⁽¹¹⁾, namely

$$\left[\frac{1}{Y_a} - \frac{Y_0(\mu R)}{J_0(\mu R)} \right] \left[\frac{1}{Y_a} - \frac{M \sum_s \frac{Y_{Ms}(\mu R)}{J_{Ms}(\mu R)} J_{Ms}^2(\mu d) + \sum_{m=2}^M Y_0(\mu r_{1m})}{Y_a} \right] - M \left[\frac{Y_0(\mu R)}{J_0(\mu R)} J_0(\mu d) - Y_0(\mu d) \right]^2 \dots\dots(100)$$

10. COMPARISON WITH EXPERIMENT.

The foregoing theory was tested against a set of experiments in which reactivities of symmetric and asymmetric arrays were measured. Critical mass studies for the Army Package Power Reactor⁽¹⁴⁾ made at Oak Ridge National Laboratory were available. The APPR-1 mockup was an enriched U^{235} , water moderated reactor with stainless steel plates and uranium foil. Control elements were square, water filled black absorbers. As discussed by A. M. Perry⁽¹⁵⁾, several configurations as shown in Figure 7 were made critical.

Modified two-group calculations of the bare equivalent versions of these assemblies and others were made using the symmetric reduced formulas (94) through (100) and the general determinant, equation (93), for the one asymmetric case. The physical data and relevant modified two-group constants are listed:

$R = 34.37$ cm	bare equivalent core
$H = 68.24$ cm	
$a = 4.392$ cm	cylinder of equal perimeter to square
$\tau_c = 35.5$ cm ²	
$D_{1c} = 1.21$ cm	
$D_{2c} = 0.166$ cm	
$d = 14.56$ cm	
$d_1 = 11.41$ cm	
$d_2 = 0.3685$ cm	
$K_0 = 1.256$	

The results are compared in the following table.

FIGURE 7. EXPERIMENTAL ARRAYS

Configuration	δk_e Experimental	δk_e Theoretical
no rods	0	0
X	0.027	0.0287
O	0.063	0.0665
XX	-----	0.0693
OX	0.089	0.0868
XXX	-----	0.1218
OOX	-----	0.1330
OOXX	0.166	0.1600*
XXXX	0.173	0.1744
OOXXX	0.229	0.2460

* Prior to the development of the method for asymmetric systems, the case OXXX was computed as if the eccentric rods were in an equilateral triangle at 120° from each other in the ring. The resultant calculated reactivity was 0.190, which is higher, as expected because of the more favorable location of the rods.

The calculated values of δk_e are also shown in Figures 8, 9 and 10 in a manner intended to group related systems. Figure 8 shows the three symmetric rings, i.e. eccentric rods at equal angles. Figure 9 gives symmetric rings with a central rod present in each case. The various asymmetric configurations--single eccentric rod, central plus eccentric, and a central with three eccentric at 90° bear no simple geometric relation to each other, but good agreement between theory and experiment is seen, Figure 10.

Considering the fact that the theoretical method is based on a first-order approximation and that the bare equivalent model is used, it is believed that the agreement with experiment is entirely adequate.

FIGURE 8. δk_e OF SYMMETRIC RINGS

FIGURE 9. δk_e OF SYMMETRIC RING PLUS CENTRAL ROD

FIGURE 10. δk_e OF ASYMMETRIC SYSTEMS

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12. APPENDIX: USE OF TRANSFORMED COORDINATE AXES.

A proof is given of the principle that solutions of the two-group equations may be expressed in terms of coordinates r_m . Reference is made to the addition theorems of Watson (see Section 5), and the transformations employed for applying rod surface conditions (see Section 6). By analogy with equations (58) and (59), but starting with equations (38) and (39), we may write

$$Y_s(\mu r) \cos s\theta = \sum_k (-1)^k J_s + k(\mu d_m) Y_k(\mu r_m) \cos (s\beta_m - k\psi_m)$$

$$Y_s(\mu r) \sin s\theta = \sum_k (-1)^k J_s + k(\mu d_m) Y_k(\mu r_m) \sin (s\beta_m - k\psi_m)$$

Now form the sums appearing in equation (22)

$$\begin{aligned} & \sum_s (c_{s1} Y_s(\mu r) \cos s\theta + d_{s1} Y_s(\mu r) \sin s\theta) \\ &= \sum_s \left\{ \sum_k (-1)^k Y_k(\mu r_m) J_s + k(\mu d_m) [c_{s1} (\cos s\beta_m \cos k\psi_m \right. \\ & \quad \left. + \sin s\beta_m \sin k\psi_m) - d_{s1} (\sin s\beta_m \cos k\psi_m - \cos s\beta_m \sin k\psi_m)] \right\} \\ &= \sum_k (-1)^k Y_k(\mu r_m) J_s + k(\mu d_m) (c_{s1} \cos s\beta_m - d_{s1} \sin s\beta_m) \cos k\psi_m \\ & \quad + \sum_k (-1)^k Y_k(\mu r_m) J_s + k(\mu d_m) (c_{s1} \sin s\beta_m + d_{s1} \cos s\beta_m) \sin k\psi_m \end{aligned}$$

Defining a new set of arbitrary coefficients

$$c'_{ki} = (-1)^k \sum_s J_s + k(\mu d_m) (c_{s1} \cos s\beta_m - d_{s1} \sin s\beta_m)$$

$$d'_{ki} = (-1)^k \sum_s J_s + k(\mu d_m) (c_{s1} \sin s\beta_m + d_{s1} \cos s\beta_m)$$

The sum becomes

$$\sum_k \frac{E_c^i}{k k_1 k} Y_k(\mu r_m) \cos k \Psi_m + \sum_k \frac{E_d^i}{k k_1 k} Y_k(\mu r_m) \sin k \Psi_m$$

A similar argument can be used for the functions $K_s(vr)$. The above process was applied to transform functions of r into functions of r_m where m was a given rod. It is clear that the approach can be generalized to apply for all rods, and finally the complete solution is the sum of the expressions for all M rods.

END