MDDC - 460

Pile Theory Notes for M.I.T. S[redacted]

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1. Review of Relevant Concepts from Nuclear Physics

A great many properties of nuclei have been discussed in previous lectures. Fortunately of these only a few are of importance in describing a nuclear chain reacting system. The following is a list of the more important properties:

1. The number of neutrons released on fission. Undoubtedly this quantity is not a constant. However, we are only interested in its average value. This value is clearly greater than 1. Otherwise, as we shall see, it would be impossible to run the Hanford piles or explode a bomb. According to the Smyth Report its value is somewhere between 1 and 3. We shall denote it by the symbol η . Most of the neutrons appear at once, but some are delayed.

2. The cross section for the production of fission by the capture of neutron in a fissionable nucleus, such as U^{235} or Pu^{239}. This cross section, which we shall denote by the symbol σ_f , is a function of the energy of the neutrons which are captured.

3. The cross sections for capture of neutrons in other nuclei. These are also functions of energy and will be denoted by σ_c .

4. Cross sections for scattering by all the nuclei present, σ_s .

5. The average logarithmic energy loss suffered by a high energy neutron in colliding with a nucleus, ξ .

In order to keep our ideas more definite we shall occasionally use the cross section for absorption in the resonance region, by which we will only mean the cross section of absorption evaluated in a certain range of energies not far above the energy at thermal equilibrium. It will be denoted σ_a.

Even though we have eliminated such constants as the magnetic and quadrupole moments, this list is still too long. Consequently, we shall develop

the theory of chain reacting piles by practicing on a simpler model in which only a few of the essential quantities are involved and in which the relevant cross sections are important at only a single energy.

Although we can hardly consider the numbers of nuclei of various types as properties of the nuclei, the proportions in which we mix up our nuclei and the total numbers present are always important data.

II. The Homogeneous Closed Box with Monokinetic Neutrons

The imposing title of this section is merely a technical description for a simple model. In this model all energies are those of thermal equilibrium. The neutrons released by fission come out in thermal equilibrium with the medium and they remain in equilibrium until their death either by capture to create another fission or by capture in non-fissionable nuclei.

We shall now define thermal utilization, f; this is the fraction of all neutrons captured which are captured in fissionable nuclei. It is clear that ηf is then the number of neutrons produced per neutron absorbed. If no neutrons escape from our box and if

$$\eta f = 1$$

the total number of neutrons inside will remain constant. Nevertheless, fissions will go on and energy will be released. The rate of fission is determined by

$$n v \sigma_f N_f V \qquad (1)$$

where n is the neutron density, v is the velocity of the neutrons in thermal equilibrium, N_f is the density of fissionable nuclei, and V is the volume of the box. We need only multiply this expression by the average energy released per fission to find the power level at which our reactor is running. (It should be remarked that $v\sigma_f$ is approximately constant over a reasonable temperature range. It may also be useful to know that 3×10^{10} fissions per second is approximately

1 watt.)

If $\eta f < 1$

the box progressively empties while if $\eta f > 1$

the box fills up with neutrons. In both cases the power is proportional to the density of neutrons.

It only remains to calculate f, the thermal utilization, from the known values in our list of fundamental constants. Just as (1) gives the rate of fission

$$n v \left[\sigma_f N_f + \sigma_c N_c\right] V \qquad (2)$$

gives the rate of capture of neutrons as a result of all processes. Consequently,

$$f = \frac{\sigma_f N_f}{\sigma_f N_f + \sigma_c N_c} \qquad (3)$$

It is interesting to note that $\sigma_s N_s$ does not appear anywhere in the theory.

III. Time Dependence of the Neutron Density

The rate at which the box empties or fills with neutrons depends on something in addition to the value of ηf. Some sort of characteristic time is required; for example, if we could state that a neutron lives just $\bar{t}$ long, then the neutron density in the box would change by the factor ηf each $\bar{t}$, and we could write

$$n = n_0 e^{\ln(\eta f) t/\bar{t}} \qquad (4)$$

We might tentatively identify $\bar{t}$ with $\frac{1}{v\left[\sigma_f N_f + \sigma_c N_c\right]}$, which must be a mean life time for the neutrons, assuming that none are delayed.

In order to get the time dependence more accurately we note that, when there are no delays, $(\eta f - 1) n v \left[\sigma_f N_f + \sigma_c N_c\right] V = \frac{\partial n}{\partial t} V$ (5)

is the rate of increase of neutrons present, and consequently the neutron density

is given by

$$n = n_0 e^{(\eta f - 1)\frac{t}{\tau_0}} \tag{6}$$

$$1/\tau_0 = v[\sigma_f N_f + \sigma_c N_c] \tag{7}$$

Our first guess, given by formula (4), is not too bad as long as $|\eta f - 1| << 1$.

IV. What Has Been Left Out

We have committed many sins of omission in the above analysis. Improving our theory in the future we must take account of the following facts:

1. Neutrons released in the fission process are released at high energies rather than in thermal equilibrium.

2. Not all of the neutrons associated with the fission process are released immediately upon fission.

3. In order to contain all of the neutrons in our box we have been forced to paint it with an ideal nuclear paint which reflects all neutrons. Unfortunately there is no such paint.

We shall now generalize our model through successive steps so as to bring it into better accord with these facts. First we shall consider the effect of the delayed neutron; then we shall consider the effects introduced by allowing the energy of fission neutrons to take its proper value; and finally we shall rub off the ideal nuclear paint.

V. Time Dependence of the Neutron Density When Delayed Neutrons Are Considered

When we cause a large number of fissions at a given instant and observe the neutron emission for some time thereafter we find that most of the neutrons appear at once. However, some neutrons continue to show up for a long time afterwards. The frequency with which these delayed neutrons appear can be analyzed as a function of time into a sum of descending exponentials. This analysis

has been presented in an earlier lecture. If we call the neutrons associated with a given descending exponential, let us say the i^{th} exponential, neutrons of type i, we are confronted with a half dozen different types of delayed neutrons. Since in this section we are at best considering a simplified model, we can get a good idea of the effect of delayed neutrons on the time variation of the neutron density by considering only one type of delayed neutron. This fictitious type decays away with the mean life τ_d, and on the average the fraction β of all the neutrons released in fission are of this type.

Formula (1) still gives the rate of fission. However, only $(1-\beta)$ of all the neutrons produced per fission appear in our box immediately. These we call the prompt neutrons; there are now $\eta f (1-\beta)$ of them per absorption instead of ηf as was the case in our previous model. The other β arrive gradually. Before they appear, we may call them latent neutrons; and at any time, aside from the ordinary free neutrons, we should find a certain number of latent neutrons in the pile. If we attempt to rewrite (5), we will have to take into consideration not only the decrease in prompt neutrons by the factor $(1-\beta)$ but also the rate at which latent neutrons are transformed into free neutrons. This rate is given by

$$\frac{c}{\tau_d} \qquad (8)$$

where c is the density of latent neutrons. (8) can easily be varified by setting up the equation for the decay of latent neutrons after a single fission.

$$\frac{c}{\tau_d} + \frac{\partial c}{\partial t} = 0 \qquad (8)$$

The solution of this equation exhibits characteristic exponential decay which we are attempting to introduce into the theory. We are now in a position to improve equation (5).

$$\frac{n[\eta f (1-\beta)-1]}{\tau} + \frac{c}{\tau_d} = \frac{\partial n}{\partial t} \qquad (5a)$$

equation (5a)
is our improved result. Despite the improvement we are unable to solve because of the introduction of the unknown c. We therefore look around for another equation which will determine c. The rate of creation of latent neutrons is

$$n v [\sigma_f N_f + \sigma_c N_c] V \eta f \beta$$

and the rate at which they turn into free neutrons is given by (8). Consequently we may write

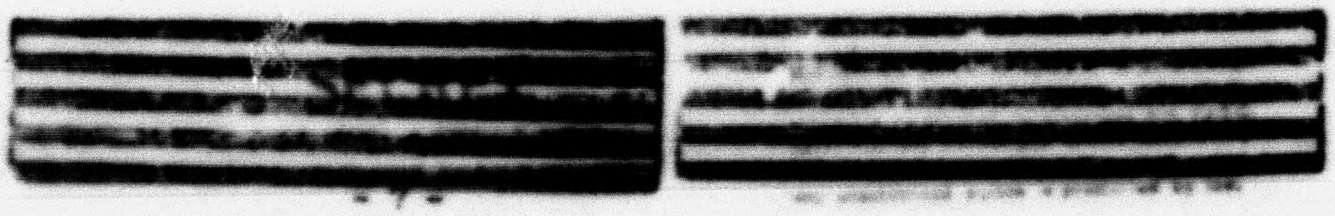
$$\frac{n \eta f \beta}{\tau_0} - \frac{c}{\tau_d} = \frac{dc}{dt} \qquad (9b)$$

Both (5a) and (5b) obtain simultaneously, and our problem is to solve the pair simultaneously. Before investigating the general solution let us ask what happens when $(\eta f = 1)$. By adding the two equations we discover that

$$n + c = \text{constant} \qquad (10)$$

By substituting for c in (5a) it is possible to determine the variation of n. As expected one of the answers, of which there are two, is that n is a constant. The question arises, What is the significance of the other answer? In order to see the significance, let us consider what happens in our box when we start it off by injecting the first batch of free neutrons. At this moment there are no latent neutrons, but latent neutrons are created as the free neutrons are absorbed. Since the total number of latent and free neutrons remains constant, the number of free neutrons must decrease as the number of latent neutrons increases. Eventually, as is shown by the existence of the infinite period associated with n equals constant, an equilibrium division between n and c will be obtained (more exactly such an equilibrium is approached exponentially as time goes on).

We shall now look at some of the features of the general solution of (5a) and (5b). As a result of previous experience, we may expect the time dependence



to be given by an exponential, say $e^{t/\tau}$. Assuming

$$n = n_{o1} e^{t/\tau_1} + n_{o2} e^{t/\tau_2} \tag{11}$$

$$c = c_{o1} e^{t/\tau_1} + c_{o2} e^{t/\tau_2} \tag{12}$$

and substituting in equations (5a) and (5b) we obtain

$$n_{oj} \frac{\eta f(1-\beta)-1}{\tau_0} + \frac{c_{oj}}{\tau_d} = \frac{n_{oj}}{\tau_j} \tag{5a'}$$

$$j = 1,2$$

$$n_{oj} \frac{\eta f \beta}{\tau_0} - \frac{c_{oj}}{\tau_d} = \frac{c_{oj}}{\tau_j} \tag{5b'}$$

The solution of these equations gives the following formula for τ_j .

$$\frac{1}{\tau_j} = \frac{1}{2}\left\{\omega(-) \pm \sqrt{\omega^2(+) + \frac{4\eta f \beta}{\tau_0 \tau_d}}\right\} \tag{13}$$

where $$\omega(\pm) = \left[\frac{\eta f(1-\beta)-1}{\tau_0} \pm \frac{1}{\tau_d}\right]$$

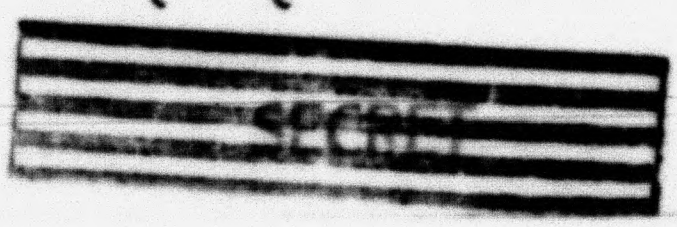
or its equivalent

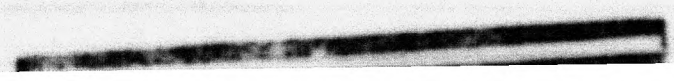
$$\frac{1}{\tau_j} = \frac{1}{2}\left\{\omega(-) \pm \sqrt{\omega^2(-) + \frac{4(\eta f - 1)}{\tau_0 \tau_d}}\right\} \tag{13'}$$

and also determines the associated ratio

$$\frac{n_{oj}}{c_{oj}} = \frac{\tau_0}{2\eta f \beta}\left\{\omega(+) \pm \sqrt{\omega^2(+) + \frac{4\eta f \beta}{\tau_0 \tau_d}}\right\} \tag{14}$$

or $$= \frac{\tau_0}{2\eta f \beta}\left\{\omega(+) \pm \sqrt{\omega^2(-) + \frac{4(\eta f - 1)}{\tau_0 \tau_d}}\right\}$$





If we call the ratios

$$\frac{n_{oj}}{c_{oj}} \equiv a_j \tag{14'}$$

we are enabled to rewrite equation (11)

$$n = c_{01} a_1 e^{t/\tau_1} + c_{02} a_2 e^{t/\tau_2} \tag{11'}$$

In (11') and (12) only c_{01} and c_{02} are not already determined from our fundamental data. The values of these constants will be fixed by the initial conditions. That is to say, if n(0) and c(0) are the values of n and c and t = 0:

$$\begin{aligned} n(0) &= c_{01} a_1 + c_{02} a_2 \\ c(0) &= c_{01} + c_{02} \end{aligned} \tag{15}$$

Suppose, for example, that there are no neutrons present in our box at time zero but that for some reason a single fission then takes place. If we want to find the future behavior of the neutron densities, we can start out by writing the particular forms of equations (15) for this case. They are

$$\begin{aligned} \eta(1-\beta)V &= c_{01} a_1 + c_{02} a_2 \\ \eta\beta/V &= c_{01} + c_{02} \end{aligned} \tag{16}$$

By a small amount of algebra we can find the following equations for c_{01} and c_{02}:

$$\begin{aligned} c_{01} &= \frac{\eta}{V} \frac{(1-\beta) - \beta a_2}{a_1 - a_2} \\ c_{02} &= \frac{\eta}{V} \frac{(1-\beta) - \beta a_1}{a_2 - a_1} \end{aligned} \tag{16a}$$

In order to finish up our problem we introduce the additional assumption that

$$\eta t = 1$$

Whenever $\eta f = 1$, we shall have

$$a_1 = -1 \ , \quad a_2 = \frac{\tau_0}{\beta \tau_d}$$

$$\frac{1}{\tau_1} = -\frac{\beta}{\tau_0} + \frac{1}{\tau_d} \ , \quad \frac{1}{\tau_2} = 0 \tag{17}$$

and applying these results to the present example

$$\eta \frac{V}{\eta} = \frac{1-\beta-\eta/\tau_d}{1+\tau_0/[\beta\tau_d]} e^{-(\frac{\beta}{\tau_0}+\frac{1}{\tau_d})t} + \frac{\tau_0/[\beta\tau_d]}{1+\tau_0/[\beta\tau_d]}$$

(18)

$$c\frac{V}{\eta} = \frac{\tau_0/\tau_d - (1-\beta)}{1+\tau_0/(\beta\tau_d)} e^{-(\frac{\beta}{\tau_0}+\frac{1}{\tau_d})t} + \frac{1}{1+\tau_0/[\beta\tau_d]}$$

which gives the desired complete description for the time dependence.

We have just worked out (13) and (14) for the case $\eta f = 1$.
It is also easy to write down the answers if $\beta = 0$ or if $\eta f = 0$.

For $\beta = 0$ we have

$$\frac{1}{\tau_1} = \frac{\eta f - 1}{\tau_0} \ , \quad a_1 = \infty$$

$$\frac{1}{\tau_2} = -\frac{1}{\tau_d} \ , \quad a_2 = \frac{\tau_0}{(\eta f - 1)\tau_d + \tau_0} \tag{19}$$

Now for $\eta f = 0$ we have

$$\frac{1}{\tau_1} = -\frac{1}{\tau_0} \ , \quad a_1 = \infty$$

$$\frac{1}{\tau_2} = -\frac{1}{\tau_d} \ , \quad a_2 = \frac{\tau_0}{\tau_d - \tau_0} \tag{20}$$

These results are straight forward, and their significance is obvious. A far more important result can be found by considering what happens when

$$\eta f(1-\beta) - 1 \gg + \frac{\tau_0}{\tau_d}$$

In that case

$$\frac{1}{\tau_1} \approx \frac{\eta f (1-\beta) - 1}{\tau_0} \gg \frac{1}{\tau_d} \qquad (21)$$

From (21) it follows that after a short time the neutron density and the power level must be rapidly rising. Since τ_0 / τ_d is usually extremely small, we should attempt to build reactors in which we can be sure that at all times

$$\eta f (1-\beta) < 1 \qquad (22)$$

Looking back on the material on delayed neutrons presented in an earlier lecture, we see that $\beta \approx .05$

Consequently we find that it is wise to keep $\eta f < 1.05$ (22a)

On the other hand, in a practical reacting system we want to make $\eta f > 1$ at some times in order to get our neutron density to rise to the level of standard operations. We are therefore interested in finding the values of τ when

$$|\eta f - 1| << 1, \quad \eta f (1-\beta) - 1 < 0 \qquad (23)$$

and (of necessity) $\beta << 1$.

For example, let us assume that

$$|\eta f (1-\beta) - 1| \geq |\eta f - 1| \qquad (24)$$

Then using (13') we find as a good approximation for the periods

$$\frac{1}{\tau_1} \approx \frac{\eta f (1-\beta) - 1}{\tau_0} - \frac{1}{\tau_d} \qquad (25)$$

$$\frac{1}{\tau_2} \approx \frac{\eta f - 1}{\tau_0 - [\eta f (1-\beta) - 1] \tau_d}$$

[redacted]

From (24) and (25) it follows that the periods are safe; that is to say, no rising period for the reaction is permitted which is faster than τ_d .

VI. Measurement of $\eta f - 1$

Occasionally throughout these lectures it will be interesting to discuss not merely the problem of designing a chain reacting unit from a known set of nuclear constants but also a problem, which in some sense is the inverse problem, that of determining some of the nuclear constants from the behavior of a reacting unit or from an independent experiment. The quantity $\eta f - 1$ can hardly be called a nuclear constant but in our procedure so far we have derived it from known nuclear properties. Let us now assume that we have constructed a reacting unit on the basis of approximate values of the relevant nuclear constants and set out to determine $\eta f - 1$ from the behavior of our reactor.

It is easy to rework formula (13') into

$$\eta f - 1 = \frac{\tau_j}{\tau_j} + \frac{\eta f \beta \tau_d}{\tau_j + \tau_d} \qquad (13a)$$

This formula is particularly useful since, as we just showed the safe rising periods for a reactor are sufficiently long that the first term on the right can be neglected by comparison with the second. For reasons which we shall come to in a later section the same thing can often be said for safe falling periods. In any case for a practical reactor we should be able to put ourselves in this situation. It then is possible to write

$$\frac{\eta f - 1}{\eta f} \approx \beta \frac{\tau_d}{\tau_j + \tau_d} \qquad (13a')$$

which will be valid to a very good approximation. From (13a') we are able to determine $\eta f - 1$ by measuring the actual time behavior of our reactor. Only the properties of the delayed neutrons need be known in addition.

Of course, (13a) can be solved explicitly for ηf , but when a long period is employed (13a') is quite adequate. More important when we introduce the consideration of finite size by rubbing off the ideal nuclear paint, the simple solution for ηf is no longer maintained while the analogue of (13') is still valid.

VII. The Slowing Down of Neutrons without Resonance Capture

Earlier we promised to bring our crude model into better accord with reality by allowing the fission neutrons to be released at their rightful energy. (This energy is of the order of a million electron volts.) In order to perform a necessary analysis of the new situation it will be necessary to introduce some more of the fundamental nuclear properties from our list of Section I. We shall proceed in two steps: first, assuming that there is no resonance absorber present, and later, introducing modifications which arise when resonance absorber must be considered.

If it were true that both σ_f and σ_c were proportional to $1/v$ throughout the whole energy spectrum, changing the energy of the neutrons released in fission would have no effect whatsoever on our analysis. However, if it were true that σ_f, σ_c suddenly dropped to zero above a certain critical velocity v_c, the time taken by the neutrons to slow down from the velocity at which they are emitted in fission to the critical velocity must be added in some way to the characteristic time which they live before absorption in the $1/v$ region. Let us investigate the manner in which this slowing down time should be added to the mean life against "thermal" capture by returning to a model in which we forget that any of the neutrons are delayed. Suppose that we knew the frequency distribution of neutrons entering the $1/v$ region as a function of time after their release in fission. Let us call this distribution $K(\theta)$, and assume that we have normalized $K(\theta)$ so that

$$\int_0^\infty K(\theta)\,d\theta = 1 \tag{26}$$

If we define q(t) the number of neutrons slowing into the 1/v region per unit time and unit volume, we can write for q(t)

$$q(t) = \eta f\ v(\sigma_c N_c + \sigma_f N_f)\int_0^t n(t')K(t-t')dt' \tag{27}$$

while the analogue of equation (5) becomes

$$q(t) - nv(\sigma_c N_c + \sigma_f N_f) = \frac{\partial n}{\partial t} \tag{5a}$$

Combining these two equations, we obtain

$$\frac{\eta f}{\tau_0}\int_0^t n(t')K(t-t')dt' - \frac{n}{\tau_0} = \frac{\partial n}{\partial t} \tag{5a'}$$

From our previous experience we are led to seek a solution of (5a') of the form

$$n = n_0 e^{t/\tau}$$

and in turn we obtain a new characteristic equation

$$\frac{\eta f}{\tau_0}\int_0^t e^{-\theta/\tau}K(\theta)d\theta - \frac{1}{\tau_0} = \frac{1}{\tau} \tag{28}$$

This equation may be solved under various assumptions as to the form of K(θ).

For example, if $K(\theta) = \delta(\theta - \theta_0)$ (28')

we obtain $$\frac{\eta f\, e^{-\theta_0/\tau} - 1}{\tau_0} = \frac{1}{\tau} \quad \text{for } \tau > \theta_0$$

More generally we may expand $e^{-\theta/\tau}$ under the integral and obtain

$$\frac{\eta f\left(1 - \frac{\bar\theta}{\tau} + \frac{1}{2}\frac{\bar\theta^2}{\tau^2} \cdots\right) - 1}{\tau_0} = \frac{1}{\tau}, \quad \bar\theta^n \equiv \int_0^\infty \theta^n K(\theta)d\theta \tag{28a}$$

for all values of t sufficiently large that K(t) is essentially zero. This procedure is a practical one since all neutrons slow down in a very short time

after their release in the fission process (or by turning from latent into free neutrons). For the same reason the series $1 - \frac{\bar{\theta}}{\tau} + \frac{1}{2}\frac{\bar{\theta}^2}{\tau^2} - \cdots$ will converge rapidly for all values of τ of practical interest. It is therefore possible to make an accurate approximation for τ by substituting $\eta + \bar{\theta} + \tau_0$ for τ_0, in formula (6).

We have been claiming that $\bar{\theta}$ is small, for example, compared to τ_0. Let us now attempt to calculate $\bar{\theta}$ in terms of more fundamental nuclear constants. In order to make this calculation we shall assume that as the neutrons slow down they do so by making elastic collisions with the scattering nuclei in the box. The law of scattering will be such that the scattering is isotropic in the center of gravity system of neutron and nucleus. When this is the case, the neutrons bounce down the energy scale losing on the average the same fraction of their energy at each collision. The quantity f measures this fractional loss. It is defined by $f = \left(\frac{E_{\text{after}}}{E_{\text{before}}}\right)_{\text{1 collision}}$ where E is the neutron energy, and in accordance with the assumptions we have just made can be shown to be given by

$$f = 1 - \frac{(A-1)^2}{2A} \ln \frac{A+1}{A-1} \qquad (29)$$

where A is the mass of the recoiling nucleus in units of neutron mass.

The average time between collisions at a given energy is given by

$$\Delta t = \frac{1}{\sigma_s N_s v}$$

and the change in energy in the collision is given by

$$\Delta E = f E$$

Consequently

$$\frac{\Delta E}{E \Delta t} = f \sigma_s N_s v \qquad (30)$$

Upon adding up all the collisions we obtain rather closely

$$\bar{\theta} = \int_{v_c}^{v_{\text{initial}}} \frac{dE}{f \sigma_s N_s v E} = 2 \int_{v_c}^{v_{\text{initial}}} \frac{dv}{f \sigma_s N_s v^2} \qquad (30a)$$

In order to get an approximate evaluation of this integral we shall assume that σ_s is a constant. Since the largest contribution to the integral arises in the region where v is near v_c, we may pick the value of σ_s at v_c without making a substantial error. With this assumption we get

$$\bar{\theta} \approx \frac{2}{f \sigma_s N_s v_c} \tag{30a'}$$

and obtain for $\bar{\theta}/\tau_o$ the approximate formula

$$\frac{\bar{\theta}}{\tau_o} \approx \frac{2 v(\sigma_c N_c + \sigma_f N_f)}{f \sigma_s N_s v_c} \tag{31}$$

In this formula v_c may be realistically chosen to be several times v, the mean value of the velocity for neutrons in thermal equilibrium. For most of the materials which do the major part of the slowing down in chain reacting systems f is greater than its value for carbon, while for carbon a simple calculation shows that f is .158. $\bar{\theta}/\tau_o$ is then determined to be

$$\frac{\bar{\theta}}{\tau_o} \leq \frac{\sigma_c N_c + \sigma_f N_f}{\sigma_s N_s}$$

Again substituting the values of our fundamental nuclear constants and the proportions found in practical reacting systems, we find that $\bar{\theta}/\tau_o << 1$ as previously assumed. In fact we may go back to using τ_o alone rather than $\tau_o + \eta + \bar{\theta}$ in the formulae for the time variation of neutron densities. At the end of the next section we shall make further remarks on the realistic nature of the assumptions employed in this section.

VIII. The Slowing Down of Neutrons Resonance Capture Considered

The presence of resonance capture in the energy region above thermal energies will decrease the number of neutrons which become thermal as compared with the number of neutrons which start the slowing down process after fission. Let us call the fraction of those neutrons which start slowing down actually

[redacted]

arriving at the top of the 1/v region, p, the resonance escape probability. To find the slowing down density q(t) at the top of the 1/v region we must multiply the right-hand side of equation (27) by the factor p. This essentially replaces ηf by $\eta f p$ in all the formulae where ηf occurs. For example, when $\eta f p = 1$, $\gamma\tau = 0$.

A review of our work will show that in computing the characteristics of our chain reacting boxes the combination ηf always occurs together; neither η nor f occurs by itself. In the future the product $\eta f p$ will occur and no one of η, f, and p will occur by itself. We therefore define

$$k \equiv \eta f p \tag{32}$$

and in the future any further analogues of $\eta f p$ we shall also call k. k is the reproduction constant; it tells us the number of neutrons which return to fission energies (through the process of fission) for each neutron which starts out at that energy. This number is counted over one cycle: that is, slowing down, thermal diffusion, and capture to create the new neutrons by means of the fission process.

In order to calculate p we will have to make use of our knowledge of the cross sections for resonance capture and of the density of resonance absorbing nuclei present in the box. We also need the scattering cross section and the density of scattering nuclei; in particular, we want the value of the scattering cross section in the energy region of the resonance absorption. In this energy region the chance that a neutron will survive a single collision is given by

$$1 - \frac{\sigma_a N_a}{\sigma_a N_a + \sigma_s N_s} \tag{33}$$

This is also the probability that the neutron will be slowed down as a result of the collision, and if this occurs, the neutron loses on the average the amount of energy given by $\Delta E = \xi E$

It follows that the probability of surviving two collisions is

$$\left(1 - \frac{\sigma_a(E) N_a}{\sigma_s(E) N_s + \sigma_a(E) N_a}\right)\left(1 - \frac{\sigma_a(E(1-\xi)) N_a}{\sigma_s(E(1-\xi)) N_s + \sigma_a(E(1-\xi)) N_a}\right) \quad (33')$$

Since the neutron again loses energy, we see that by repeating our multiplications a sufficient number of times we will compute the probability of escaping resonance capture on slowing down through the whole resonance region. Actually the neutrons do not bounce along hitting only the average energy values; and in considering a large group of neutrons, we may substitute for the spread in energy the concept of a fractional collision which reduced the energy by $\Delta E = \frac{\xi E}{n}$ while giving a survival probability of $\left(1 - \frac{\sigma_s N_s}{\sigma_a N_a + \sigma_s N_s}\right)^{\frac{1}{n}}$

As n becomes large, the neutrons will hit all energies in better correspondence with the actual facts. Also as n becomes large, the probability of survival may be rewritten as

$$e^{-\frac{\sigma_s N_s}{\sigma_a N_a + \sigma_s N_s} \frac{1}{n}} \quad (33a)$$

It follows that

$$p = e^{-\int \frac{\sigma_s(E) N_s}{\sigma_a(E) N_a + \sigma_s(E) N_s} \frac{dE}{\xi E}} \quad (34)$$

if the integral is taken all over the energy region of resonance absorption.

At the end of the last section we promised to make some remarks justifying the assumptions implied. The main point to note is that in a region of high resonance absorption very few neutrons will be absorbed in the small fission cross section which may extend to the same energy. The neutrons which go around the cycle are almost exclusively those which get into the 1/v region, and

consequently our calculation of the cycle time $\tau_0 + k\theta$ is essentially correct. If for some reason the fission cross section competes favorably with the resonance absorption (for example, if resonance absorption is absent) the neutrons which cycle at this higher energy region do so in a shorter time than we have assumed, and the cycle time is less lengthened by the introduction of the slowing down than predicted by our calculation. There is one chance for trouble. If the 1/v region does not extend to the energy at which the chemical binding freezes our scattering nuclei, our calculation of θ is inapplicable. In this case a much more complicated calculation must be indulged in, but the result, that θ is small compared to τ_0, is maintained.

We should also say a word about the validity of formula (34). Our arguments in deriving this formula were plausible rather than rigorous. For example, if σ_a were infinite over a range of energies greater than the possible energy loss of a neutron as a result of a single collision, p should be zero. However, formula (34) predicts an appreciable survival. Formula (34) can be shown by rigorous arguments to be correct when $\sigma_a N_a << \sigma_s N_s$ or when the width of the resonance becomes very small compared to the average energy jump. In most practical work (34) will give an accurate answer.

II. Measurement of p, the Resonance Escape Probability

We have just seen that the introduction of resonance absorber to our ideal chain reacting box changes ηf everywhere in the theory to $\eta f p$. Mentally at least, we are able to measure p and consequently $\int \frac{\sigma_a N_a}{\sigma_s N_s + \sigma_a N_a} \frac{dE}{E}$ by introducing resonance absorber into the box. To do this measurement we observe the changes in the time behavior of the neutron density and from them we infer the change in k. The ratio of k before and after the introduction of resonance absorber is exactly p.

In practice using a nuclear chain reaction may be a bit expensive and wasteful, particularly if the introduction of resonance absorber decreases k so much that the reactor is no longer workable. We shall, therefore, describe a somewhat cheaper method for measuring p.

In this method we employ three new instruments: a neutron source which emits neutrons at some energy not far above that at which resonance absorption takes place, an ideal resonance detector which becomes radioactive as the result of its own ability to capture neutrons in a very low energy resonance, and a block of scattering material. The ideal resonance detector must have its resonance absorption at an energy which is lower than the resonance absorption of the material for which we are attempting to measure $\int \frac{\sigma_a N_a}{\sigma_s N_s + \sigma_a N_a} \frac{dE}{E}$ With these instruments we make two measurements. First, we measure the activation of the detector inside the block of scattering material at a short distance from the source. Then we introduce some of the resonance absorber whose effect we want to measure and repeat the experiment. The ratio of activations in the first and second experiments gives p for the scattering material employed and the concentration of resonance absorber introduced. A more detailed discussion of this method will be given later.

It is interesting to note that since the introduction of slowing down problems in Section VIII the scattering properties of our reactor have entered the theory. They come in both the calculation of the slowing down time and the resonance escape probability. They enter because the rate of slowing down depends not only on f but also on the frequency of collisions. In a later more realistic version of the theory the scattering properties will enter for a different reason and in that case they will not always enter in combination with f as they do here.

In describing the measurement of p the scattering properties enter; and the density of scattering nuclei is a relevant quantity. If, however, p has been measured for one density of scattering nuclei and one density resonance absorbing nuclei and f is known for the scattering nuclei, by use of formula (34) the p's can be computed for other properties and densities of scattering and resonance absorbing nuclei. In fact, it can be computed for other scattering substances for which the values of f and σ_s are known.

X. Measurement of f, the Thermal Utilization

In order to measure f, we introduce two new pieces of equipment: the ideal thermal detector and the standard box. The ideal thermal detector is an instrument which tells, usually by radioactivation, the density of thermal neutrons in which it had been placed. It is sensitive to nothing but thermal neutrons. The standard box is a more complicated affair. It contains no fissionable material and no resonance absorber, but otherwise it exactly resembles the box which we have been employing to produce our chain reaction. It contains exactly the same density of the same scattering nuclei and the same density of the same thermally absorbing nuclei.

It is useful in the laboratory since it provides, for example, a method for the intercomparison of neutron sources and a convenient place in which to "keep" a standard thermal neutron density. In this section we will confine our interest in the standard box to the measurement of f.

Inside the box we will place a neutron source, either the one used in the last experiment for the measurement of p or more conveniently, let us say, a Ra-Be source. We will measure the activation both of the ideal resonance detector and of the ideal thermal detector. Since all the neutrons from the source slow down through the energy region in which the ideal resonance detector is sensitive

in just the same way as they do in the chain reacting box, the activation of the ideal resonance detector is proportional to q(t), the number of neutrons becoming thermal per unit volume and unit time with the same proportionality constant which holds between the activation and q(t) if we place the same detector in the chain reacting box. The thermal neutron detector will, of course, be given an activation proportional to the density of thermal neutrons in the standard box or the reacting box, depending only on which box it is introduced into.

In the reacting box the densities measured by the two ideal detectors are independent of the position of the detectors in the box, while in the standard box the densities measured are functions of position. In order to get rid of the space dependence we shall move the detectors around inside the standard box and compute the average activations which in turn reflect the average densities. Now in the reacting box where no source is introduced equation (5c) obtains. On the other hand, in the standard box

$$\bar{j} = \bar{n} v \sigma_c N_c \tag{35}$$

where j and $\bar{j}$ are the space averages. Let us define r by

$$R = \left(\eta - \frac{\partial_n}{\partial t}\right)\bar{n} \tag{36}$$

From (5c) and (35) it follows that

$$R = \frac{\sigma_c N_c}{\sigma_c N_c + \sigma_f N_f} = 1 - f \tag{37}$$

Now if we are fortunate or foresighted, we will have adjusted the constants of the reacting box in such a way that $\frac{\partial n}{\partial t} = 0$
(In practice this adjustment, as we will see in a later section, is possible without changing either η or f.) When the adjustment has been made, (37)

becomes

$$R = \frac{(\bar{q}/n)}{(q/n)} = 1 - f \qquad (37')$$

As we have seen, $\frac{\bar{I}}{q} = \frac{\bar{A}_\infty}{A_\infty}$

where $\bar{A}_\infty$ is the activation of the ideal resonance detector in the standard box averaged all over the volume of the box, and A_∞ is the activation of the same detector in the reacting box. If we also define $\bar{A}_{th}$ and A_{th} in the analogous manner, we find the following formula for f, in terms of the activations which we measure with the detectors: $f = 1 - \frac{\bar{A}_\infty A_{th}}{A_\infty \bar{A}_{th}}$ (38)

XI. Controlling the Chain Reacting Box

Most of the time so far we have been talking about chain reacting boxes whose properties are fixed. In such a box a rising neutron density will continue to rise indefinitely or a falling one will continue to fall. Our boxes would clearly be much more useful if we were able to control the time behavior of the neutron density to suit ourselves. In order to do this, it is necessary to have at least one of the properties at our disposal. Let us suppose that the property which we can control is the thermal utilization. As long as ηp is not too small: that is, as long as $\eta p > 1$, we shall have the ability to cause neutron density to rise or fall.

We can easily keep for ourselves the power of controlling f by leaving space inside the reacting box, into which we can introduce or from which we can take out thermal absorber. (Of course in order to keep our simple theory the introduction or removal must be done uniformly throughout the box.)

XII. Homogeneous Thermal Neutron Pile

We shall now return to the homogeneous closed box with monokinetic neutrons. We shall open up the box by rubbing off the ideal nuclear paint.

Neutrons which formerly hit the ideal nuclear paint and bounced back into the box now leak out. The neutron density at the edge of the box falls down. In an earlier lecture it has been shown that at the edge of the box we have the following condition on the neutron density:

$$\frac{\partial \log n}{\partial s} = \frac{1}{d} = \frac{\sigma_s N_s}{.11} \qquad (39)$$

where $\frac{\partial \log n}{\partial s}$ means the derivative in the direction of the inward normal to the surface. The outflow from the box will decrease the number of neutrons which go the full distance around the cycle from fission to fission. We expect, therefore, that the reproduction constant k will be changed from ηf to $\eta f \ell$, where ℓ is the probability that the neutron has not leaked out in the course of the cycle. ℓ should be computed for all the neutrons which start out from fission no matter what position in the box fission takes place. We may, therefore, expect that ℓ will be a function of the size and shape of the box.

It has also been shown in an earlier lecture that the flux of neutrons through a unit area, ϕ, is given by

$$\phi = D \frac{\partial n}{\partial s} \qquad (40)$$

where $\frac{\partial n}{\partial s}$ is the derivative along the normal to the surface in the direction of the flux and D is approximately

$$\frac{v}{3\sigma_s N_s} \qquad (41)$$

Under these conditions the equation of continuity

$$-\operatorname{div} \vec{\phi} + (\text{source density}) = \frac{\partial n}{\partial t} \qquad (42)$$

becomes

$$\Delta n + \frac{\eta f - 1}{D\tau_0} n = \frac{\partial n}{D \partial t} \qquad (42')$$

where $\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

in the absence of delayed neutrons, or

$$\Delta n + \frac{\eta f (1-\beta) - 1}{D \tau_s} + \frac{c}{D \tau_d} = \frac{\partial n}{D \partial t} \qquad (42a)$$

when a single type of delayed neutron is considered.

Equation (42a) takes the place of equation (5a), and equation (5b) remains valid without change when we investigate time dependence of the neutron density in the pile. For the time being, however, we shall restrict ourselves to investigating the information supplied by (42').

We start out by investigating (42') when $\frac{\partial n}{\partial t} = 0$.

Calling $$K_0^2 \equiv \frac{\eta f - 1}{D \tau_s} \quad , \qquad (43)$$

the equation is seen to be $$\Delta n + K_0^2 n = 0 \qquad (44)$$

The mathematical treatment of equation (44) is well known for many geometries. For example, with "cubic" symmetry the solution of (44) is given by

$$n = n_0 \sin \gamma x \sin \gamma y \sin \gamma z \quad , 3\gamma^2 = K_0^2 \qquad (45)$$

The length of the side of the cubic lattice cell is

$$a = \frac{\pi}{\gamma} = \frac{\pi \sqrt{3}}{K_0} \qquad (46)$$

On the other hand, we know from (39) that the neutron density in our pile extrapolates to zero a very short distance (closely d) beyond the surface . It follows that a pile which is cubic in shape will be in the stationary state

($\frac{\partial n}{\partial c} = 0$) as we have assumed if the length of its sides is given by

$$a_c = a - 2d \qquad (47)$$

Since we can re-define our pile to include the extrapolation region around its boundaries (this is the region between the boundary of the pile and a fictitious boundary, a distance d beyond the physical boundary of the pile), we shall always confine ourselves to determining a and its analogues: that is to say, to determining the extrapolated boundaries of the pile. Before building any piles, however, it is better to take into consideration the difference between a and a_c.

If we build our pile, and pile means extrapolated pile, with dimensions bigger than a, the average leakage around the cycle will be decreased. Extra neutrons will remain inside on each cycle and the neutron density will rise; if we build it smaller, the neutron density will fall. In order to calculate the time behavior of these piles, either smaller or larger than the so-called critical size, we must return again to (42').

In equation (42') let us assume that $n = ST$

where S is a function of the coordinates only; T is a function of time only. (42') then separates, and we find that $T = T_0 e^{t/\tau}$

where τ is a constant and the equation for S is

$$\Delta S + K^2 S = 0 \;, \quad K^2 = K_0^2 - \frac{1}{D\tau} \qquad (44a)$$

Equation (44a) has solutions which are formally the same as those of equation (44). For a cubic pile, therefore, we may write

$$n = n_0 \sin\gamma x \sin\beta y \sin\beta z\, e^{t/\tau} \;, \quad 3\beta^2 = K_0^2 - \frac{1}{D\tau} \qquad (45a)$$

If we have a pile of this type in which the length of the sides is

r,
$$3\left(\frac{\pi}{r}\right)^2 = K_0^2 - \frac{1}{D\tau}$$

or
$$\frac{1}{\tau} = \frac{\eta f - 1 - 3\frac{\pi^2}{r^2} D\tau_0}{\tau_0} \qquad (48)$$

Not only does equation (47) give us the period for the exponential behavior of the neutron density as a function of time, but it also shows us how to calculate ℓ. It is
$$\ell = \frac{\eta f - K^2 D\tau_0}{\eta f}$$

The quantity $\eta f \ell - 1$ is called the excess multiplication constant. It is given by k for the closed box times the correction for leakage and minus the neutron with which the cycle is started.

In the formula which relates the critical size to the nuclear properties formula (43), the combination $D\tau_0$ is the source of the dimension of length. Of course, $D\tau_0$ has the dimensions $[L^2]$. Let us call it L^2. If we substitute for D and for τ_0 the values given in (41) and in (7), we find that L^2 is
$$L^2 = D\tau_0 = \frac{1}{3\sigma_s N_s (\sigma_a N_a + \sigma_f N_f)} \qquad (49)$$

where v is the velocity of the neutrons.

Before discussing the appropriate value to be used for v, let us find out what L^2 means. If we consider a point source of thermal neutrons located in an infinite region in which $\eta = 0$ but all the other properties are the same as in the pile. we find that
$$L^2 = \frac{\bar{r}^2}{6} \qquad (50)$$

λ^2 is the mean square distance away from the point source at which neutrons are absorbed. In fact, the equation for the diffusion of thermal neutrons may be written

$$\Delta n - \frac{n}{L^2} = 0$$

and (49) is easily verified from the point source solution of this equation.

Since both σ_c and σ_f vary as 1/v and since the dimension of the pile is proportional to L, in order to find the real physical dimensions we must understand how to compute v. This sensitivity to v is a new phenomenon in our theory and is characteristic of the finite size. Crudely speaking, v is the velocity of neutrons in thermal equilibrium. This crude concept is good enough to establish a temperature dependence of the critical dimension. v rises with the square root of the temperature, and the critical dimension rises with the square root of v. The critical size is consequently proportional to the three-fourths power of the absolute temperature.

For a more exact determination of the velocity v of the thermal neutrons we must take into account the fact that the neutrons have a velocity distribution which is given by the well known formula due to Maxwell. The velocities, therefore, must be averaged over the Maxwell distribution. The question of what kind of average to perform may be settled by going back to equation (42), in which the only place that the velocity appears is in the first term. (For a 1/v absorber the second term which contains $v\sigma_c v\sigma_f$ is, of course, independent of the velocity of the neutrons.) Hence, the average value of v appropriate to our formulae is

$$\bar{v} = \frac{\int v M(v)\, dv}{\int M(v)\, dv}$$

where $M(v)$ is the Maxwellian distribution function giving the number of neutrons per unit velocity interval. The above considerations assume that the distribution of neutrons after collision remains always Maxwellian.

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In this section we have picked as an example only a cubic pile. The results, however, are easily modified to take account of other geometric shapes. For example, formula (46) for the critical dimension always turns up in the form

$$a = \frac{G}{k_\infty} \qquad (46a)$$

where G depends only on the geometry. In particular for the critical radius of a sphere we obtain

$$a = \frac{\pi}{k_\infty} \qquad (46a')$$

As might be expected, the volume of the critical sphere is less than the volume of any other pile with the same nuclear properties and densities. For example, the ratio of the volume of the cube to the volume of the sphere when each is critical is given by

$$\frac{Vol_c}{Vol_s} = \frac{3\sqrt{3}}{4\pi} \approx 1.24$$

Unfortunately spheres are not easily engineered. Later, however, we shall discover other methods of approaching the minimum amounts of the valuable materials used in the construction of piles.

Aside from constructing spherical piles, in order to minimize the amount of material used we should clearly increase the density of the materials employed. If the proportions of the constituent materials remain the same, we see from formula (48) that L^3 is inversely proportional to the cube of the density. Since the volume is proportional to L^3, the total mass is inversely proportional to the square of the density.

XIII. The Slowing Down of Neutrons in a Pile

In Section VII we discussed the slowing down of neutrons. There we were interested in the length of time it took the neutrons to go from fission energies

to the top of the 1/v region. In Section VIII the resonance capture was considered. The arguments and the results of both these sections remain valid for those neutrons which remain inside of a pile. Since the neutrons that leak out are of no further interest to us as far as the computation of critical size or time dependence is concerned, we can take over the results of Sections VII and VIII bodily. There is, however, another very important aspect of the slowing down process which concerns us when we allow the neutrons in the pile to be released on fission with their real energies. Before they become thermal, these neutrons may have traveled quite a long distance, and the space distribution of the neutrons as they slow down is a primary concern in finding the critical size.

We may look at the travels of the neutrons in slowing down as a diffusion and merely add the mean square distance which the neutrons diffuse in slowing down to the mean square distance which they travel while in thermal equilibrium. This way of looking at things is a little bit crude, but should give us a first approximation to the real situation. Our first problem then is to calculate the mean square distance.

In order to calculate the mean square distance let us look back on formula (48). If we multiply on the right side of equation (48) both numerator and denominator by $\sigma_s N_s$, we see that we can write for L^2

$$L^2 = \frac{N}{3(\sigma_s N_s)^2} \qquad (49')$$

where N is the number of collisions before capture. The mean square distance is given by

$$\overline{r^2} = 6L^2 = \frac{2N}{(\sigma_s N_s)^2}$$

Similarly we may assume that the mean square distance in slowing down is given by

$$\bar{\lambda}_f^2 = \frac{\frac{2}{3}\ln\left(\frac{E_{fiss}}{E_{th}}\right)}{(\sigma_s N_s)^2_f}$$

where we have computed the number of collisions by dividing logarithmic energy loss into the total logarithmic energy span to be covered and have assumed that the scattering cross section is not a function of energy. The analogue of L^2 for the slowing down process we may call L_f^2. It is given by

$$L_f^2 = \frac{\ln\left(\frac{E_{fiss}}{E_{th}}\right)}{3(\sigma_s N_s)^2_f} \tag{51}$$

A more rigorous calculation considering the energy dependence of σ_s gives

$$L_f^2 = \int_{E_{th}}^{E_{fiss}} \frac{dE}{3(\sigma_s N_s)^2_f E} \tag{51}$$

Let us define M^2 as the sum of L^2 and L_f^2. If we substitute M^2 for L^2 in (42') or in (43), we essentially substitute M for L in all the formulae for the critical dimensions. This procedure cannot be completely accurate since in general the spacial distribution of neutrons captured at thermal energy around a point source of fast neutrons is not of the same form as that around a thermal neutron source. In order to refine the treatment we need to know more about the space distribution of q, the slowing down density around a point source.

Such spacial distributions can be determined experimentally for various sources including fission sources. In general, the shape of the spacial distribution near the source is found to be roughly Gaussian. The details depend both on the source and on the slowing down material. Hydrogenous materials are the greatest exception to our qualitative statement. The region which can be considered to be Gaussian is very small, and it might be said that an exponential would give a better fit to the data. Typical shapes for the slowing down distribution are shown in Figure 1. The experiments for measuring space distribution

of the slowing down density can be performed by using the ideal resonance detector in a large block of the scattering material. The fission source can be created by running an intense beam of thermal neutrons through the large block, in the center of which a small amount of fissionable isotope has been placed.

If we integrate the slowing down distribution over all space, we must find all neutrons except those which disappear in resonance capture. We shall now define q to be the slowing down density that would be present if we forgot about resonance capture. The real source of thermal neutrons is then pq. With this definition a unit source of neutrons will lead to a normalized distribution

$$1 = \int g(x,y,z)\,dx\,dy\,dz \tag{52}$$

In the case of a Gaussian the normalization condition gives us

$$g = \frac{e^{-\frac{x^2+y^2+z^2}{4L_f^2}}}{(4\pi)^{3/2} L_f^3} = \frac{e^{-\left(\frac{r}{2L_f}\right)^2}}{(2\sqrt{\pi}\ L_f)^3} \tag{53}$$

This is the standard form for the slowing down distribution which is to be employed here to investigate the influence of slowing down more accurately, but it should be remembered that this form is not universally applicable. In graphite, for example, the slowing down distribution is reasonably well represented by a superposition of three such standard distributions, each with a different value for L_f^2.

There are theoretical reasons for expecting the Gaussian form in the slowing down distribution and also for expecting the exceptional slowing down distribution obtained in hydrogenous materials. If the number of collisions which the neutron undergoes in slowing down is large, and the free path is never large compared with L_f, it follows from the theory of random flights that the slowing down distribution must be Gaussian. However, we would only expect a single Gaussian to apply to those neutrons which started out at exactly the same energy

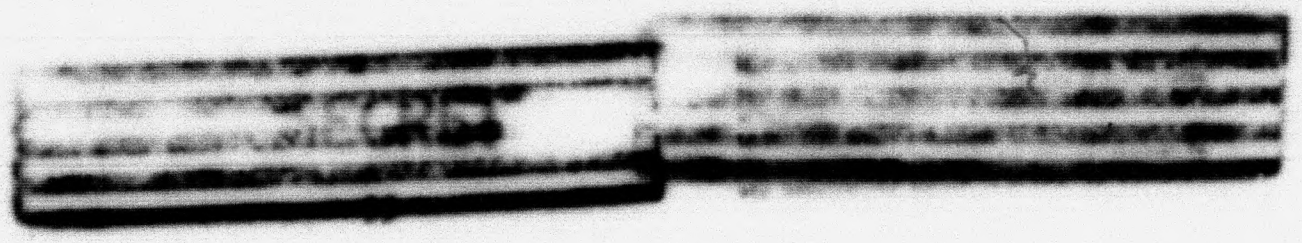
and underwent exactly the same number of collisions. If the number of collisions is large, the correlation between energy and number of collisions is very good, but most sources, or a particular fission sources, do not emit monokinetic neutrons. Consequently, it is not surprising to find that a superposition of Gaussians is a better representation than a single Gaussian.

In hydrogen a neutron can lose essentially all of its energy in a single collision. Also the cross section of hydrogen rises steeply as the energy of the neutron falls. Consequently, the largest fraction of the distance travelled by neutrons in hydrogenous media is likely to be travelled in the first or the first and second flights. The restrictions under which the Gaussian distribution for the slowing down is to be expected are all violated. On the other hand, the fact that only the first couple collisions made by the neutron have much importance in determining its final position implies that some sort of exponential distribution will obtain at any considerable distance from the source.

If we look at L_f as a function of the lower limit of the integral in (51), (53) gives the slowing down distribution as a function of both space and energy around a point source. We should recognize that (53) is a point source solution of the differential equation

$$\Delta q = \frac{\partial q}{\partial (L_f^2)} \qquad (54)$$

This differential equation which is derivable either from the theory of random flights or by a simple argument which we will give in the next paragraph is formally identical with the equation of heat conduction. L_f^2 in our equation plays the same role that time does in the heat equation. For this reason L_f^2 has been called by Fermi the age, and we shall call it the Fermi age.



To derive equation (54) we consider the equation of continuity in four dimensions, x, y, z, and $l_n E$. The density of neutrons per unit volume in this space is called n. The net number of neutrons entering a volume element by translation in space x, y, z is given by $D \Delta n \, dx dy dz \, d l_n E$

The net number entering the volume by translation in log energy is given by

$$\frac{\partial (n v f \sigma_s N_s)}{\partial l_n E} dx dy dz \, d l_n E$$

Consequently, the equation of continuity reads

$$D \Delta n + \frac{\partial (n v f \sigma_s N_s)}{\partial l_n E} = \frac{\partial n}{\partial t} \qquad (55)$$

On identifying $n v f \sigma_s N_s$ with q, we can rewrite (55)

$$\frac{D \Delta q}{v f \sigma_s N_s} + \frac{\partial q}{\partial l_n E} = \frac{\partial n}{v f \sigma_s N_s \partial t} \qquad (55a)$$

from which (54) follows when $\frac{\partial q}{\partial t} = 0$

by using the differential of (51).

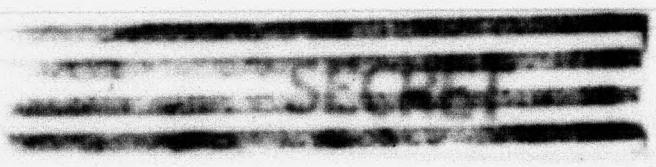
In determining the characteristic size of a pile we shall employ equation (54) or its solution (53) to give the description of the slowing down process.

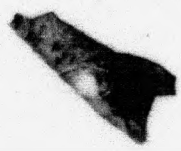
XIV. The Critical Size of a Pile

The problem of finding the critical size of the chain reacting unit has now been reduced to the problem of finding the simultaneous solution of two differential equations. These two equations are equation (54) for the slowing down of neutrons and the thermal diffusion equation given below:

$$D \Delta n + \rho q - \frac{n}{\tau_0} = 0 \qquad (56)$$

The two equations are interconnected: that is to say, the slowing down density





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enters the thermal diffusion equation, and the thermal neutron density is connected with the initial condition for the slowing down equation by the following relation:

$$\frac{n}{\zeta}\,\gamma^t = q\,(L_f^2 = 0) \tag{57}$$

In attempting to solve these equations simultaneously we pay no attention to the delayed neutrons; we need to pay no attention to them if we are only looking for the critical size. In this case the rates at which neutrons feed in and out of the latent type are always exactly equal. Of course, when the pile is not in a stationary state (that is, when it is not exactly of critical dimension), it is necessary to modify the equations in order to bring in the effect of the delayed neutrons on the time behavior. For the moment we shall only handle the case of exact criticality.

We shall attempt to find a solution in the form

$$q = S(x,y,z)\,H(L_f^2)$$
$$n = S(x,y,z)$$

where

$$\Delta S + K_0^2 S = 0 \tag{44}$$

In this case equation (54) yields us

$$-K_0^2 H = \frac{\partial H}{\partial (L_f^2)} \tag{54'}$$

which has the solution

$$H = H_0\, e^{-K_0^2 L_f^2}$$

Upon substituting into (56) and (57), we obtain

$$D K_0^2 + p H_0\, e^{-K_0^2 L_f^2} - \frac{1}{\zeta} = 0 \tag{56'}$$

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$$\frac{nf}{\tau_0} = H_0 \qquad (57')$$

and combining these two relations our final result is

$$(1 + K_0^2 L^2) = k\, e^{-K_0^2 L_f^2} \qquad (58)$$

This equation is often called the characteristic equation determining the critical size. It is so called because it determines K_0^2, and at the same time equation (44) relates K_0^2 to the actual critical dimension.

When $K_0^2 (L^2 + L_f^2) << 1$,

the characteristic equation can be simplified to read

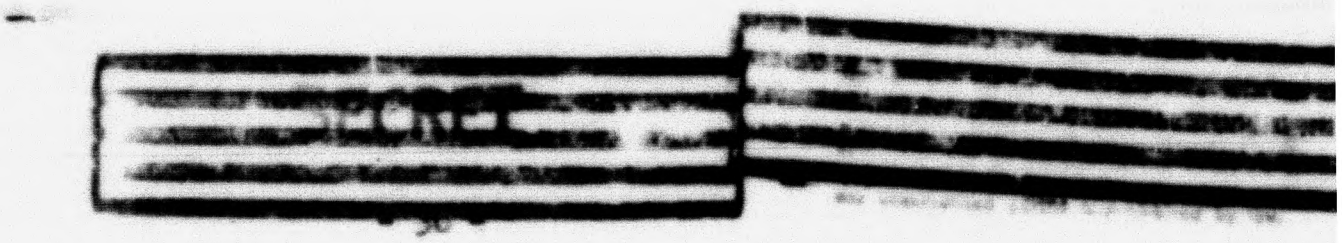
$$k = (1 + K_0^2 L^2)(1 + K_0^2 L_f^2 + \cdots)$$

and it then follows that K_0^2 is given by

$$K_0^2 = \frac{k-1}{L^2 + L_f^2} = \frac{k-1}{M^2} \qquad (59)$$

This is the result for K_0^2 that was obtained by adding together the mean squares in our very crude treatment in the last section. We now see that this result is obtained here whenever $|k-1| << 1$. The details which we promised to consider, namely the details of the Gaussian shape, have all been put into the new characteristic equation, (58).

Suppose now that instead of the Gaussian we use as a description for the slowing down a diffusion, not dissimilar from the thermal diffusion. Such a diffusion will give a certain type of exponential as a point source solution and may then be appropriate for hydrogenous substances. In order to write down the equations we assume that there is not only a thermal density n but also a fictitious density for fast neutrons which we symbolize by n_f. We also invent a mean life for fast neutrons which we symbolize by τ_f and a diffusion coefficient



D_f which is just what you would suppose. Let us proceed without defining τ_f too closely. After we have proceeded a short way, it will become clear that we do not need to know the exact value for τ_f but only for the combination $D_f\ \tau_f$ which is given by the usual number L_f^2. The two diffusion equations which we use are

$$D_f \Delta n_f + \frac{k}{p}\ \eta/\tau_0 - \frac{n_f}{\tau_f} = 0$$
$$D \Delta n + p\ n_f/\tau_f - \eta/\tau_0 = 0 \qquad (60)$$

Using the same S as before, we try to find a solution of (60) in the form

$$n_f = A_f S \qquad n = S$$

On substituting in (60) we obtain the algebraic equations

$$(K_0^2 D_f + \frac{1}{\tau_f})\, A_f = \frac{k}{p}\,\frac{1}{\tau_0} \qquad (60')$$
$$(K_0^2 D + \frac{1}{\tau}) = p\,\frac{1}{\tau_f}\, A_f$$

If these two equations are multiplied together, we obtain another characteristic equation for the determination of the critical size.

$$(1 + K_0^2 L^2)(1 + K_0^2 L_f^2) = k \qquad (61)$$

Again we should note that under the same conditions as in our last case we are able to obtain the simple form

$$K_0^2 \approx \frac{k-1}{M^2} \qquad (59)$$

Of course, it is no accident that this simple form always arises. Let us consider a general description of the slowing down, that is one which may be determined purely empirically. For this purpose we define $K(\varsigma)$, the slowing down density from a point source at a distance ς. We assume that $K(\varsigma)$ has

been normalized so that it will give over all space just one neutron per unit time. The total number of neutrons which are slowing down is given by Q where Q is the source strength in neutrons per unit time. We are now in a position to modify the equation for thermal diffusion of neutrons in such a way that it becomes an integral equation

$$D\Delta n + \frac{k}{\tau_0}\int n(x',y',z')\,K\left(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)dx'dy'dz' - \frac{n}{\tau_0} = 0$$

The integral term in this equation takes account of the source density of thermal neutrons. It already contains the interconnection between the absorption and the source strength for fast neutrons.

In order to solve the integral equation we make an expansion for $n(x', y', z')$ as it appears under the integral

$$n(x',y',z') = n(x,y,z) + x\frac{\partial n}{\partial x} + y\frac{\partial n}{\partial y} + z\frac{\partial n}{\partial z} + \frac{x^2}{2}\frac{\partial^2 n}{\partial x^2} + xy\frac{\partial^2 n}{\partial x\partial y} + \cdots\cdots$$

On integrating over this expansion term by term and assuming at the same time that the space dependence of the neutron density is given by S, we obtain a generalized expression for the characteristic equation in the form

$$(1 + K_0^2 L^2) = k(1 - K_0^2 L_f^2 + \cdots) \qquad (62)$$

where

$$L_f^2 = \bar{\rho}^2/6 \quad \text{and} \quad \bar{\rho}^2 = 4\pi\int \rho^2 K(\rho)\rho^2 d\rho$$

It is clear that as k approaches one, the neutron density flattens out. Only the first few terms of the Taylor expansion are necessary because the first few terms give a sufficiently good approximation for n(x',y',z') at all values of K(ρ), which are substantially different from zero. Consequently, we again find from our general form for the character'stic equation that K_0^2 is

given by

$$K_0 \approx \frac{k-1}{M^2} \qquad (59)$$

Since our formulations for the problem of critical size differ essentially only in the description applied to the slowing down process, we may be able to summarize them in a simple form by writing

$$1 + K_0^2 L^2 = k l_f \qquad (63)$$

We may call this equation the general characteristic equation, just as the one before, and in this equation we are permitted to calculate l_f by any method we choose. l_f always means the fraction of neutrons which stay in the pile during the slowing down process. It includes those which undergo resonance capture.

XV. The Time Behavior of Non-critical Piles

Any one of the three treatments of the last section may be extended to include the time behavior when the pile is not critical. For the sake of simplicity we shall stick here to a single one of the three treatments. We will take for our example the Gaussian case.

In order to treat the Gaussian case it is necessary for us to return to the section before last and to extend a derivation of equation (54) to include an actual time variation as well as an energy and a space variation. We did actually include such a time variation in our derivation. However, we threw it away at the last minute. When we keep it, we may write equation

$$\Delta f = \frac{\partial f}{\partial (L_f)^2} + \frac{\partial f}{D \partial t} \qquad (54a)$$

In conformity with our treatment of the last section we shall attempt the solution of this equation of the form

$$g = TSH, \quad \Delta S + K^2 S = 0 \tag{64}$$

As a result of our previous experience, we may immediately write

$$T = T_0 e^{t/\tau}$$

Under these circumstances we obtain for H the following equation

$$-[K^2 + \frac{1}{D\tau}] H = \frac{\partial H}{\partial (L_f^2)} \tag{54a'}$$

and $$H = H_0 e^{-K^2 L_f^2} e^{-B^2/\tau}$$

The time dependent thermal diffusion equation can be written

$$(1 + K^2 L^2 + \frac{\tau}{\tau}) = \frac{gP}{n} \tau_0 \tag{56a}$$

where we have already employed equation (44) to determine the form of the left hand side. The equation for the interconnection of the initial condition of the slowing down density with the thermal density now also must include the delayed neutrons; that is to say, we must also include the rate at which latent neutrons of various types become free. When this is done, the interconnection equation reads

$$g(L_f^2 = 0) = \frac{k}{p} \frac{(1-\beta)n}{\tau_0} + \sum \frac{c_i}{\tau_i} \tag{57a}$$

Since the latent neutron density has entered our last equation, it is necessary in order to have a complete set of equations to write down the equations which connect the latent neutron densities with the thermal neutron density. They are

$$c_i \left(\frac{1}{\tau_i} + \frac{1}{\tau}\right) = \frac{k}{p} \frac{n\beta_i}{\tau_0} \tag{65}$$

Upon combining all these equations we can eliminate all the densities and obtain

a new characteristic equation which is

$$(1+K^2L^2+\frac{\tau_i}{\tau}) = k\left[1-\beta+\sum \frac{\beta_i \tau}{\tau_i+\tau}\right] e^{-K^2L_f^2} e^{-\tilde{\theta}/\tau} \qquad (13c)$$

With a little algebraic manipulation this equation can be reworked to read

$$k l_f e^{-\tilde{\theta}/\tau} - K^2L^2 - 1 = \frac{\tau}{\tau_0} + k l_f e^{-\tilde{\theta}/\tau} \sum \frac{\beta_i \tau_i}{\tau+\tau_i} \qquad (13d)$$

From either of these equations it is possible to determine the period when the size is known and the shape is given. On the other hand, if it is known that the period is infinite, then (13d) reduces to

$$k l_f = K^2L^2 + 1 \qquad (63)$$

which is the characteristic equation for the determination of the critical size.

We have already pointed out that $\tilde{\theta}/\tau$ is very small compared to one. It is, therefore, possible to set $e^{-\tilde{\theta}/\tau} \approx 1$

and equation (13d) becomes

$$\frac{k l_f - K^2L^2 - 1}{k l_f} = \frac{\tau_0}{k l_f \tau} + \sum \frac{\beta_i \tau_i}{\tau+\tau_i} \qquad (13d')$$

Equation (8d') is often called the inhour equation. It is given by Nordheim as

$$\frac{k l_f - K^2L^2 - 1}{k l_f} = C(\text{Inhour}) = C\left[\frac{54}{\tau} + \frac{33}{\tau+27} + \frac{1139}{\tau+6.5} + \frac{1793}{\tau+54} + \frac{575}{\tau+93}\right]$$

where the period is to be measured in seconds, and c is a constant.

When the periods have been found from any of these equations, it is then possible to go back to the equations which connect the densities and find the ratios for the various densities which are to be associated with the given periods. When this is done using the inhour formula to determine the period, we find that just enough ratios are determined to give all the ratios of the

latent neutron densities and the free neutron density, but there are not enough ratios to distinguish between those neutrons which are slowing down and those which are thermal. This is not surprising because we have made the time for slowing down so short that neutrons during the slowing down process are already to be considered as thermal neutrons. If we want to make further distinction, it is necessary to go back to the more exact form of our characteristic equation and to keep the factor $e^{-\tau/t}$. We then obtain an infinite sequence of solutions by means of which we hope to represent the initial condition applying to the slowing down densities as well as to all the other densities.

XVI. Generalization to Include Other Harmonics

The equation

$$\Delta S + K^2 S \tag{64}$$

can be solved in the standard geometries for any value of K^2. So far we have used but a single value of K^2, and we have assumed that whenever S equals zero we have located one of the boundaries of the pile, or possibly of one of the images. This procedure gives a unique solution for a pile of a given dimension.

Let us now assume that

$$S = \sum B_i S_i \tag{64a}$$

where

$$\Delta S_i + K_i^2 S_i = 0$$

and a set of K_i^2, S_i are so determined that the S_i are all zero at any point on the extrapolated boundaries of the pile. It is now possible, however, to have

other zeros of the S_i which are not located on the pile boundaries.

As an example, let us consider a cubic pile. We may then write

$$S = \sum B_{ijk} \sin \gamma_i x \sin \gamma_j y \sin \gamma_k z$$

where the γ's are determined by (45a)

$$\gamma_\ell = \frac{m_\ell \bar{a}}{\nu}, \quad \ell = i, j, k \quad , \quad m_\ell = \text{integers}$$

and

$$K^2_{ijk} = \frac{\Pi^2}{\nu^2}\left(m_i^2 + m_j^2 + m_k^2\right)$$

This is a multiple Fourier expansion of S. To each value of K^2_{ijk} there corresponds a characteristic equation which determines the period associated with the harmonic (ijk). With each of these there is also associated a set of equations which may then be solved for the corresponding densities.

Upon referring to the inhour equation and to Fig. 2, we see that as K^2_{ijk} increases each of the periods decreases. Consequently, as time goes on, the higher harmonics disappear with respect to the lower ones, and finally only the lowest harmonic is of significance in a pile. It is this lowest harmonic which has been taken by us as a unique solution. There are, however, cases, particularly when independent sources are present, in which the higher harmonics maintain a certain significance. For example, when the reproduction constant is less than one, the lower harmonics should always be taken into account if any permanent sources of neutrons are present.

XVII. The Approach to Critical

In practice when building a pile it is unwise to construct the whole chain reacting system first, and later sit down to see what is going on. Instead

the pile is built bit by bit in order that we may observe the approach to the critical condition. To observe this approach to the critical condition we must first make a few computations to see how the neutron density acts as we increase the size of the pile.

Suppose that we place a source in the center of a very small cubic pile. This pile is smaller than the critical dimension, but we shall gradually increase its size until it runs. As we build the pile, we shall observe the neutron density in the stationary state at every size.

For simplicity we shall choose our neutron source to be a source of thermal neutrons. If the source strength is given by Q, in a pile which is cubic and has a size of length r the multiple Fourier analysis of this source is given by

$$\sum_{ijk} Q \sin \frac{m_i \pi x}{r} \sin \frac{m_j \pi y}{r} \sin \frac{m_k \pi z}{r} \qquad (66)$$

Upon analyzing the thermal neutron density into the same multiple Fourier analysis, we find that the thermal diffusion equation may be written

$$n_{ijk}/\tau_0 \left[1 + L^2 K_{ijk}^2 - k e^{-K_{ijk}^2 L_f^2}\right] = Q \qquad (56a')$$

where n is given by

$$n = \sum_{ijk} n_{ijk} \sin \frac{m_i \pi x}{r} \sin \frac{m_j \pi y}{r} \sin \frac{m_k \pi z}{r} \qquad (45a')$$

The coefficient n_{ijk} may be evaluated from the thermal diffusion equation and a new expression

$$n = \sum Q \tau_0 \frac{\sin \frac{m_i \pi x}{r} \sin \frac{m_j \pi y}{r} \sin \frac{m_k \pi z}{r}}{L^2 K_{ijk}^2 + 1 - k e^{-K_{ijk}^2 L_f^2}} \qquad (45a'')$$

relates the thermal neutron density to the source strength. In this expression the smallest denominator is always that one associated with the (111) harmonic.

As r increases, all the K^2_{ijk} decrease until finally K^2_{111} will reduce the first denominator to zero (provided k is greater than one). When r has reached this value, the pile rides up on the first harmonic. The other harmonics are still, however, tied to the source strength and are consequently left behind. We have reached the critical condition (r = a).

Below critical, but near it, only the first harmonic is of major importance. We may then write the density of thermal neutrons at the center of the pile

$$n = \frac{Q \tau_0}{L^2 K^2_{111} + 1 - k e^{-K^2_{111} L^2_f}}$$

The difference between K_{111} and K^2_0 is small. It is given by

$$K^2_{111} - K^2_0 = 3\pi^2\left(\frac{1}{r^2} - \frac{1}{a^2}\right) = 6\pi^2 \frac{a-r}{a^3}$$

Upon substituting for K_{111} in the last equation, we now obtain for the thermal neutron density at the center

$$n = \frac{Q \tau_0 a^3}{6\pi^2(L^2 + L^2_f e^{-K^2_0 L^2_f})(a-r)} = \frac{const}{a-r}$$

and we see that the difference between a and r; that is, the distance that we are away from critical; is measured by a constant divided by the thermal neutron density. If we then construct a graph, such as in Fig. 3, we are able to extrapolate to the critical dimension sometime before we actually arrive at it.

It should be mentioned that this type of procedure can be applied whether or not we build up our pile in a completely symmetric fashion. It is in fact more usual in practice to build one dimension almost complete and then extend the pile only in another direction.

END