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Normal Elliptic Integrals of the First and Second Kinds*

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This paper was submitted for publication in the open literature at least 6 months prior to the issuance date of this Microcard. Since the U.S.A.E.C. has no evidence that it has been published, the paper is being distributed in Microcard form as a preprint.

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Various phases of this investigation have been supported by the Ames Laboratory of the U. S. Atomic Energy Commission, by the Office of Naval Research under Contract Nonr-220(31) with the California Institute of Technology, and by the National Aeronautics and Space Administration under Grant NSG-293 to the Iowa State University of Science and Technology. I wish to thank the California Institute of Technology for giving me the opportunity to carry out a part of the work there, and I am especially grateful to Professor A. Erdélyi for encouragement and advice.

* Contribution No. 1304. This work was supported in part by the Ames Laboratory of the U. S. Atomic Energy Commission.

1. Introduction. Legendre's choice of normal elliptic integrals is ordinarily adopted in practical work, particularly in integral tables, numerical tables, and computer programs. Weierstrass' notation is seldom used in these contexts, in part because it is designed primarily for elliptic functions rather than integrals, in part because Legendre's integrals were first to be extensively tabulated, and perhaps because it is sometimes bothersome to express an integral of the second kind in terms of the Weierstrass zeta function. Unfortunately, certain transformation formulas that are symmetrical or even trivial in Weierstrass' notation are fairly clumsy in Legendre's. This is by no means a purely academic matter, for the linear transformations are used to put the modulus and amplitude within ranges that have been tabulated, and the quadratic transformations are used for direct calculation.

In this paper we describe a choice of normal elliptic integrals which appears to be very well suited for use in integral tables as well as for numerical work. The associated transformation formulas are simple, and the hypergeometric character of elliptic integrals is brought to the forefront. After a brief review of the hypergeometric R function in § 2, special cases of this function are chosen in § 3 to be the complete integrals $R_K(z_1, z_2)$ and $R_E(z_1, z_2)$ and the incomplete integrals $R_F(z_1, z_2, z_3)$ and $R_G(z_1, z_2, z_3)$. All four functions are homogeneous and symmetric in their arguments, and the symmetry is shown in § 4 to be the counterpart of the linear transformations of Legendre's integrals. The quadratic Landen and

Gauss transformations take a unified form, derived in § 5 as a special case of a new quadratic transformation of the R function. In § 6 a type of practical problem is considered for which the present choice of normal integrals offers a special advantage: geometrical and physical properties of ellipses and ellipsoids (for example, the surface area of an ellipsoid) can be represented without obscuring their symmetry in the semi-axes. Two definite integrals that are useful in this connection are evaluated in terms of the R function.

Normal elliptic integrals of the third kind can be defined in a similar way but are not considered in the present paper.

2. Properties of the R function. We shall summarize here some properties of the R function that will be needed later. Derivations can be found in [2] along with other properties, especially differential relations and additional integral representations.

The function R of n complex variables $z_1, \dots, z_n$ and n+1 complex parameters $a, b_1, \dots, b_n$ is defined by the following hypergeometric series if $|1 - z_i| < 1$ ($i = 1, \dots, n$) and by its analytic continuation if $|\arg z_i| < \pi$:

$$(2.1) \quad R(a; b_1, \dots, b_n; z_1, \dots, z_n) = \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a, m_1 + \dots + m_n)(b_1, m_1) \dots (b_n, m_n)}{(b_1 + \dots + b_n, m_1 + \dots + m_n) m_1! \dots m_n!} (1 - z_1)^{m_1} \dots (1 - z_n)^{m_n},$$

where $(a, m) = \Gamma(a + m)/\Gamma(a)$. The auxiliary parameters a' and c are defined by

$$(2.2) \quad a + a' = c = b_1 + \dots + b_n,$$

and we assume $c \neq 0, -1, -2, \dots$.

¶ The R function is clearly invariant under permutation of the subscripts $1, \dots, n$ (i. e., under permutation of the b's and z's together); and it is homogeneous of degree $-a$ in the variables $z_1, \dots, z_n$. The parameter a can be changed to a' by the Euler transformation

$$(2.3) \quad R(a; b_1, \dots, b_n; z_1, \dots, z_n) = \left(\prod_{i=1}^n z_i^{-b_i} \right) R(a'; b_1, \dots, b_n; z_1^{-1}, \dots, z_n^{-1}).$$

Also, if $n = 2$, the parameters a and a' can be interchanged with b_1 and b_2 :

$$(2.4) \quad z_2^a R(a; b_1, b_2; z_1, z_2) = z_2^{b_1} R(b_1; a, a'; z_1, z_2).$$

Two or more R functions are said to be associated if their corresponding parameters differ by integers. Any $n+1$ associated R functions of n variables are connected by a linear homogeneous relation with coefficients that are homogeneous polynomials in the z's. All such relations can be obtained from the following special relations, wherein variables and parameters common to all terms of an equation are suppressed for brevity:

$$(2.5) \quad (c-1) R(b_1-1) = (a'-1) R + a z_1 R(a+1), \quad (i=1, \dots, n),$$

$$(2.6) \quad c R = \sum_{i=1}^n b_i R(b_i+1),$$

$$(2.7) \quad c R(a-1) = \sum_{i=1}^n b_i z_i R(b_i+1).$$

An integral representation of R, valid when a and a' have positive real parts, is

$$(2.8) \quad B(a, a') R(a; b_1, \dots, b_n; z_1, \dots, z_n) = \int_0^\infty dt \, t^{a'-1} \prod_{i=1}^n (t + z_i)^{-b_i},$$

where $\arg t = 0$, $|\arg(t+z_1)| < \pi$, and B denotes the beta function. Provided that $\operatorname{Re}(a'-b_n) > 0$, it follows from (2.8) that

$$(2.9) \quad B(a, a') R(a; b_1, \dots, b_n; z_1, \dots, z_{n-1}, 0) \\ = B(a, a'-b_n) R(a; b_1, \dots, b_{n-1}; z_1, \dots, z_{n-1}) .$$

Two forms of (2.8) are of special interest in connection with elliptic integrals:

$$(2.10) \quad \int_0^{\phi} d\theta (\sin \theta)^{2a-1} (\sin^2 \phi - \sin^2 \theta)^{a'-1} (\cos \theta)^{1-2b_1} (1-k^2 \sin^2 \theta)^{-b_2} \\ = \frac{1}{2} B(a, a') (\sin \phi)^{2c-2} R(a; b_1, b_2, b_3; \cos^2 \phi, \Delta^2, 1) ,$$

where $\Delta^2 = 1-k^2 \sin^2 \phi$; and

$$(2.11) \quad \int_0^u dv [\wp(v) - \wp(u)]^{a'-1} \prod_{i=1}^3 [\wp(v) - e_i]^{1/2-b_i} \\ = \frac{1}{2} B(a, a') R[a; b_1, b_2, b_3; \wp(u) - e_1, \wp(u) - e_2, \wp(u) - e_3] ,$$

where $\wp(u)$ is the Weierstrass elliptic function and where the e 's, satisfying $e_1 + e_2 + e_3 = 0$, are the values of $\wp(u)$ at the zeros of $\wp'(u)$. Substitution of $\sin \theta = (1+t)^{-1/2} \sin \phi$ and $\wp(v) = t + \wp(u)$ reduces (2.10) and (2.11), respectively, to (2.8). In both cases, the various parameters are related by (2.2) with $n = 3$, and convergence requires the real parts of a and a' to be positive.

3. Normal elliptic integrals. A convenient choice of normal elliptic integrals (two complete and two incomplete since we are not considering integrals of the third kind) is the set of four functions defined as follows:

$$\begin{aligned}
 R_K(z_1, z_2) &= R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; z_1, z_2\right), \\
 R_E(z_1, z_2) &= R\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; z_1, z_2\right), \\
 R_F(z_1, z_2, z_3) &= R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; z_1, z_2, z_3\right), \\
 R_G(z_1, z_2, z_3) &= R\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; z_1, z_2, z_3\right).
 \end{aligned}
 \tag{3.1}$$

Although Legendre's normal elliptic integrals are in widespread use, the choice proposed here has an important advantage that ought to be stressed: each of the functions (3.1) is a symmetric function of its arguments, for its b-parameters are all equal and so are not changed by permutation of the b's and z's together. Some consequences of this symmetry will be discussed in § 4.

Each of the four functions, being homogeneous, depends in an essential way on only the ratios of its arguments; there is therefore no loss in taking one argument to be unity if this should be convenient for tabulation or other numerical work. When every argument is unity, the function itself has the value unity according to (2.1). Each complete integral is a multiple of an incomplete integral with one vanishing argument, as shown by (2.9):

$$\begin{aligned}
 R_F(z_1, z_2, 0) &= \frac{\pi}{2} R_K(z_1, z_2), \\
 R_G(z_1, z_2, 0) &= \frac{\pi}{4} R_E(z_1, z_2).
 \end{aligned}
 \tag{3.2}$$

Finally, as the notation suggests, only R_G among the four normal integrals differs widely from Legendre's choice. Using standard symbols for Legendre's integrals, we find from (2.10), (3.2), and (2.4) that

$$\begin{aligned}
 (3.3) \quad K(k) &= \frac{\pi}{2} R_K(1 - k^2, 1), \\
 E(k) &= \frac{\pi}{2} R_E(1 - k^2, 1), \\
 F(\phi, k) &= \sin \phi \ R_F(\cos^2 \phi, \Delta^2, 1), \\
 E(\phi, k) &= \sin \phi \ R\left(\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}; \cos^2 \phi, \Delta^2, 1\right).
 \end{aligned}$$

Regarded as a hypergeometric function, $E(\phi, k)$ has unequal b-parameters; consequently, with respect to permutations of its arguments, it has an asymmetry that sets it apart from the others. As we shall see in § 4, this asymmetry is responsible for its complicated behavior under the linear transformations that leave $F(\phi, k)$ essentially unchanged [4, vol. 2, p. 316]. In order to have a second incomplete integral that is symmetric, while compromising with conventional notation, we can define

$$(3.4) \quad G(\phi, k) = 2 \sin \phi \ R_G(\cos^2 \phi, \Delta^2, 1),$$

the factor 2 being included so that

$$(3.5) \quad G\left(\frac{\pi}{2}, k\right) = E(k).$$

Any elliptic integral that can be expressed in terms of F and E can alternatively be expressed in terms of F and G , as we shall now show. Between any four associated R functions of three variables, there exists

a linear relation with coefficients that are homogeneous polynomials in the variables. The R function representing $E(\phi, k)$ can therefore be expressed as a linear combination of R_F , R_G , and a third function which Euler's transformation (2.3) shows to be algebraic:

$$(3.6) \quad R\left(\frac{3}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; z_1, z_2, z_3\right) = (z_1 z_2 z_3)^{-1/2}.$$

Finding the actual coefficients of the linear combination requires no more than repeated applications of Eqs. (2.5) to (2.7); we shall omit the details and state the result:

$$(3.7) \quad (z_3 - z_1) R\left(\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}; z_1, z_2, z_3\right) \\ = -z_1 R_F(z_1, z_2, z_3) + 2 R_G(z_1, z_2, z_3) - (z_1 z_2 / z_3)^{1/2}.$$

From this follow at once the relations

$$(3.8) \quad E(\phi, k) = -\cos \phi \cot \phi R_F(\cos^2 \phi, \Delta^2, 1) + 2 \csc \phi R_G(\cos^2 \phi, \Delta^2, 1) - \Delta \cot \phi, \\ G(\phi, k) = \cos^2 \phi F(\phi, k) + \sin^2 \phi E(\phi, k) + \Delta \sin \phi \cos \phi.$$

¶ Although Legendre's normal integrals are generally used for practical problems, the normal form of Weierstrass is more convenient for theoretical investigations because of its symmetry [4, vol. 2, p. 304]. Some examples in § 6 will help to indicate why the hypergeometric normal integrals defined by (3.1) may be convenient in practice; in the meantime we shall show that their symmetry is the same as that of the Weierstrass integrals. Let α, β, γ

denote any permutation of 1, 2, 3 and define z_a by the first of the equations

$$\begin{aligned} z_a &= \wp(u) - e_a, \quad (a = 1, 2, 3), \\ \wp(u) &= (z_1 + z_2 + z_3)/3, \\ (3.9) \quad e_a &= (z_\beta + z_\gamma - 2z_a)/3, \\ \wp'(u) &= -2(z_1 z_2 z_3)^{1/2}. \end{aligned}$$

The analogs of (3.3) are then

$$\begin{aligned} \omega_a &= \frac{\pi}{2} R_K(e_a - e_\beta, e_a - e_\gamma), \\ \eta_a &= -\omega_a e_a + \frac{\pi}{2} R_E(e_a - e_\beta, e_a - e_\gamma), \\ (3.10) \quad u &= R_F(z_1, z_2, z_3), \\ \zeta(u) &= -u \wp(u) + 2 R_G(z_1, z_2, z_3). \end{aligned}$$

The first pair of equations follows from the second pair by virtue of the relations $e_a = \wp(\omega_a)$ and $\eta_a = \zeta(\omega_a)$. The symmetry of the first and second Weierstrass integrals (u and ζ) with respect to permutation of the e 's is clearly equivalent to the symmetry of R_F and R_G with respect to permutation of the z 's.

Although the expression for u is a special case of (2.11), that for ζ requires more effort. Using the addition theorems for $\zeta(u)$ and $\wp(u)$ [7, pp. 47, 40], we have

$$\begin{aligned} \zeta(u) + \frac{1}{2} \wp'(u)/[\wp(u) - e_1] &= \zeta(u + \omega_1) - \zeta(\omega_1) \\ &= - \int_0^u \wp(v + \omega_1) dv \\ &= -e_1 u - (e_1 - e_2)(e_1 - e_3) \int_0^u [\wp(v) - e_1]^{-1} dv. \end{aligned}$$

With the help of (2.11) and (3.9), this becomes

$$(3.11) \quad \zeta(u) = (z_2 z_3 / z_1)^{1/2} - e_1 u - \frac{1}{3}(z_1 - z_2)(z_1 - z_3) R\left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}; z_1, z_2, z_3\right).$$

The last term is related to R_F and R_G by an equation that is derived in the same way as (3.7):

$$(3.12) \quad \frac{1}{3}(z_1 - z_2)(z_1 - z_3) R\left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}; z_1, z_2, z_3\right) \\ = z_1 R_F(z_1, z_2, z_3) - 2 R_G(z_1, z_2, z_3) + (z_2 z_3 / z_1)^{1/2}.$$

Substitution in (3.11) now completes the proof of (3.10).

One other way of writing the second Weierstrass integral in terms of hypergeometric functions is rather interesting. From (2.5) and (3.10) we have

$$(3.13) \quad R\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; z_\alpha, z_\beta, z_\gamma\right) = 2 R_G(z_1, z_2, z_3) - z_\alpha R_F(z_1, z_2, z_3) \\ = \zeta(u) + u \wp(u) - z_\alpha u \\ = \zeta(u) + e_\alpha u.$$

By summing over cyclic permutations of α, β, γ and using the relation $e_1 + e_2 + e_3 = 0$, we obtain $3 \zeta(u)$ as a sum of three R functions that differ from one another only by permutations of their arguments.

4. Linear transformations. At the beginning of § 3 it was emphasized that the hypergeometric normal integrals defined by (3.1) are symmetric functions of their arguments. Because they are also homogeneous, we can make any one of the arguments equal to unity and use the remaining arguments to

determine the variables ϕ and k of (3.3). Comparison of the six ways of doing this establishes relations between elliptic integrals having different values of ϕ and k . A related procedure was used by Tricomi [7, pp. 208-211] to deduce the linear transformations of F and E from the symmetry of the Weierstrass functions; we shall instead show how the linear transformations of F and G are connected with the symmetry of R_F and R_G .

In order to derive the various transformations systematically and put them in a unified form, we start from the following set of relations connecting z_1, z_2, z_3 with ϕ and k , as well as an exactly similar set connecting z_1^*, z_2^*, z_3^* with ϕ^* and k^* :

$$(4.1) \quad \cos \phi = (z_1/z_3)^{1/2}, \quad \Delta = (1 - k^2 \sin^2 \phi)^{1/2} = (z_2/z_3)^{1/2},$$

$$k^2 = (z_3 - z_2)/(z_3 - z_1), \quad \sin^2 \phi = (z_3 - z_1)/z_3.$$

For reasons to be explained later, we are considering only functions that are even in both ϕ and k . From (3.3), (3.4), and the homogeneity of the R function, we obtain

$$(4.2) \quad F(\phi, k)/\sin \phi = z_3^{1/2} R_F(z_1, z_2, z_3),$$

$$G(\phi, k)/\sin \phi = 2 z_3^{-1/2} R_G(z_1, z_2, z_3),$$

along with a similar pair of equations in which unstarred variables are replaced by starred variables. If the two sets of z 's are related by

$$(4.3) \quad z_1^* = z_\alpha, \quad z_2^* = z_\beta, \quad z_3^* = z_\gamma,$$

where α, β, γ is any permutation of 1, 2, 3, the symmetry of R_F and R_G ensures that

$$(4.4) \quad \begin{aligned} R_F(z_1^*, z_2^*, z_3^*) &= R_F(z_1, z_2, z_3) , \\ R_G(z_1^*, z_2^*, z_3^*) &= R_G(z_1, z_2, z_3) . \end{aligned}$$

The transformation equations of F and G are therefore

$$(4.5) \quad \begin{aligned} F(\phi^*, k^*) / \sin \phi^* &= (z_\gamma / z_3)^{1/2} F(\phi, k) / \sin \phi , \\ G(\phi^*, k^*) / \sin \phi^* &= (z_3 / z_\gamma)^{1/2} G(\phi, k) / \sin \phi , \end{aligned}$$

where

$$(4.6) \quad \begin{aligned} \cos \phi^* &= (z_\alpha / z_\gamma)^{1/2}, & \Delta^* &= (z_\beta / z_\gamma)^{1/2}, \\ (k^*)^2 &= (z_\gamma - z_\beta) / (z_\gamma - z_\alpha), & \sin^2 \phi^* &= (z_\gamma - z_\alpha) / z_\gamma . \end{aligned}$$

It follows immediately that

$$(4.7) \quad F(\phi^*, k^*) G(\phi^*, k^*) / \sin^2 \phi^* = F(\phi, k) G(\phi, k) / \sin^2 \phi .$$

For each permutation in turn, the z 's that occur in (4.5) and (4.6) can be expressed in terms of ϕ and k by using (4.1); the results are listed in Table I. Since the elliptic integrals are even functions of their modulus and odd functions of their amplitude, the sign of k^* is irrelevant in the present context and the choice of sign of ϕ^* determines the signs of $F(\phi^*, k^*)$ and $G(\phi^*, k^*)$. For example, we find from the second transformation in the table that if $k^* = k$ and $\sin \phi^* = i \tan \phi$ (Jacobi's imaginary transformation), then

$$F(\phi^*, k^*) = i F(\phi, k), \quad G(\phi^*, k^*) = i \sec^2 \phi G(\phi, k) .$$

Table I. Linear transformations of $F(\phi, k)$ and $G(\phi, k)$.

α, β, γ	$(k^*)^2$	$\frac{\sin^2 \phi^*}{\sin^2 \phi}$	$\cos \phi^*$	$\frac{\sin \phi F(\phi, k)}{\sin \phi^* F(\phi^*, k)}$	$\frac{\sin \phi G(\phi, k)}{\sin \phi^* G(\phi^*, k)}$
1,3,2	$k^2/(k^2-1)$	$(1-k^2)/\Delta^2$	$(\cos \phi)/\Delta$	Δ	$1/\Delta$
3,2,1	$1 - k^2$	$-\sec^2 \phi$	$\sec \phi$	$\cos \phi$	$\sec \phi$
2,1,3	$1/k^2$	k^2	Δ	1	1
2,3,1	$1/(1-k^2)$	$(k^2-1)\sec^2 \phi$	$\Delta \sec \phi$	$\cos \phi$	$\sec \phi$
3,1,2	$(k^2-1)/k^2$	$-k^2/\Delta^2$	$1/\Delta$	Δ	$1/\Delta$

The main point is that both $F(\phi, k)$ and $G(\phi, k)$, as a result of the symmetry of R_F and R_G , are left unchanged except for a multiplicative factor by the five linear transformations. This invariance is not shared by $E(\phi, k)$ in any one of the five cases; under Jacobi's imaginary transformation, for instance, it follows from (3.8) that

$$E(\phi^*, k^*) = i [F(\phi, k) - E(\phi, k) + \Delta \tan \phi] .$$

However, even the relatively simple transformations of F and G are cumbersome in comparison with the symmetry property of R_F and R_G .

As a final remark, we observe that the transformations we have been discussing can be described in three different languages as the linear transformations of incomplete elliptic integrals, as the invariance of the R function under simultaneous permutations of its arguments and b -parameters, or as those Euler transformations of Appell's function F , which leave its a -parameter unchanged.

5. Quadratic transformations. Quadratic transformations of elliptic integrals are renowned for the extraordinary rate of convergence obtained when they are used recursively for numerical calculation. Two particular quadratic transformations are especially simple and useful. The Landen transformation connects elliptic integrals with different moduli and amplitudes in such a way that the smaller modulus goes with the larger amplitude; in the Gauss transformation the larger modulus goes with the larger amplitude. Although the two transformations are superficially quite different, Tricomi [7, p. 224] has emphasized that all such quadratic transformations are equivalent in the sense that any one can be obtained from any other by means of the linear transformations discussed in the preceding section. One may expect this equivalence to be conspicuous for the quadratic transformations of R_F and R_G , since their linear transformations reduce to a simple statement of symmetry. Moreover, we shall derive a new quadratic transformation of the R function with three variables, from which the quadratic transformations of R_F and R_G can be obtained by assigning special values to a free parameter.

Before doing so, we consider first the Landen transformation of R_K and R_E . A well-known quadratic transformation of the Gauss hypergeometric function [4, vol.1, p. 113, Eq. (31)] is equivalent to

$$(5.1) \quad R(a; b, b; z_1, z_2) = R[a; 2b-a, a-b+\frac{1}{2}; (\sqrt{z_1} + \sqrt{z_2})^2/4, \sqrt{z_1 z_2}] ,$$

where $|\arg z_1| < \pi$ ($1 = 1, 2$). It should be noted that the variables on the

right-hand side also have arguments that are less than π in magnitude. As a special case of (5.1) we have

$$(5.2) \quad R_K(z_1, z_2) = R_K[(\sqrt{z_1} + \sqrt{z_2})^2/4, \sqrt{z_1 z_2}] .$$

By (3.3) and the homogeneity of R , this leads at once to the familiar Landen transformation of K , but it lies even closer to the elegant formulation due to Gauss. For $n = 0, 1, 2, \dots$, let the arithmetic and geometric means of α_n and β_n be denoted by α_{n+1} and β_{n+1} , respectively; as n tends to infinity, α_n and β_n tend rapidly to a common limit M , usually called Gauss' arithmetic-geometric mean. According to (5.2), we have

$$(5.3) \quad R_K(\alpha_0^2, \beta_0^2) = R_K(\alpha_1^2, \beta_1^2) = R_K(\alpha_n^2, \beta_n^2) = R_K(M^2, M^2) = 1/M .$$

Alternatively, (5.3) can be proved in an elementary way from the integral representation (2.8) of $R_K(\alpha_0^2, \beta_0^2)$ by substituting $t = x(x + \beta_1^2)/(x + \alpha_1^2)$.

A second special case of (5.1) is

$$(5.4) \quad R_E(z_1, z_2) = R[-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}; (\sqrt{z_1} + \sqrt{z_2})^2/4, \sqrt{z_1 z_2}] .$$

From the relations between associated R functions, we find

$$(5.5) \quad R(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}; z_1, z_2) = 2 R_E(z_1, z_2) - z_2 R_K(z_1, z_2) .$$

Hence the Landen transformation of R_E is

$$(5.6) \quad R_E(z_1, z_2) = 2 R_E[(\sqrt{z_1} + \sqrt{z_2})^2/4, \sqrt{z_1 z_2}] - \sqrt{z_1 z_2} R_K(z_1, z_2) ,$$

$$(5.7) \quad R_E(\alpha_0^2, \beta_0^2) = 2 R_E(\alpha_1^2, \beta_1^2) - \beta_1^2/M .$$

We turn now to R functions of three variables and derive a quadratic transformation that is analogous to (5.1) except for having only one free parameter. In (2.8) we substitute $t = x(x+f)/(x+g)$, where f and g remain to be chosen but are supposed different from zero and from each other. The result is

$$B(a, a') R(a; b_1, \dots, b_n; z_1, \dots, z_n) \\ = \int_0^\infty dx [x(x+f)]^{a'-1} (x+g)^{a-1} (x+u)(x+v) \prod_{i=1}^n [(x+u_i)(x+v_i)]^{-b_i},$$

where

$$\begin{aligned} u + v &= 2g, & uv &= fg, \\ u_1 + v_1 &= f + z_1, & u_1 v_1 &= z_1 g, \quad (i = 1, \dots, n), \\ (u - u_i)(v - v_i) &= -(u - v_i)(v - u_i). \end{aligned}$$

Because of the last equation, it is possible to choose f and g so that $u_1 = v_1 = u$ and $u_2 = v_2 = v$. The number of distinct factors in the integrand is thereby reduced to yield

$$(5.8) \quad R(a; b_1, \dots, b_n; z_1, \dots, z_n) \\ = R(a; 1-a', 1-a, 2b_1-1, 2b_2-1, b_3, b_3, \dots, b_n, b_n; f, g, u, v, u_3, v_3, \dots, u_n, v_n),$$

where

$$(5.9) \quad f = \sqrt{z_1 z_2}, \quad g = (\sqrt{z_1} + \sqrt{z_2})^2 / 4.$$

The remaining arguments are determined from f and g by the definitions given above.

The number of arguments on the right side of (5.8) can be further reduced by choosing special values of the parameters. The case of greatest interest is $n = 3$, $a' = 1$, $b_1 = b_2 = 1/2$:

$$(5.10) \quad R(a; \frac{1}{2}, \frac{1}{2}, a; z_1, z_2, z_3) = R(a; 1 - a, a, a; w_1, w_2, w_3) ,$$

$$(5.11) \quad \begin{aligned} 2 w_1 &= \sqrt{(z_1 z_2)} + (z_1 + z_2)/2 , \\ 2 w_2 &= \sqrt{(z_1 z_2)} + z_3 + \sqrt{(z_1 - z_3)} \sqrt{(z_2 - z_3)} , \\ 2 w_3 &= \sqrt{(z_1 z_2)} + z_3 - \sqrt{(z_1 - z_3)} \sqrt{(z_2 - z_3)} , \end{aligned}$$

$$(5.12) \quad \begin{aligned} z_1 &= [w_1 + \sqrt{(w_1 - w_2)} \sqrt{(w_1 - w_3)}]^2 / w_1 , \\ z_2 &= [w_1 - \sqrt{(w_1 - w_2)} \sqrt{(w_1 - w_3)}]^2 / w_1 , \\ z_3 &= w_2 w_3 / w_1 . \end{aligned}$$

The sign of $\sqrt{(z_1 z_2)}$ is determined by the requirement $|\arg z_i| < \pi$ ($i=1,2,3$); the signs of the remaining square roots are immaterial since both signs occur symmetrically.

Equation (5.10) is equivalent to a previously unknown quadratic transformation of Appell's function F_1 into itself. In addition to applying the Euler transformation (2.3) to (5.10), we can get other quadratic transformations from (5.8) by choosing $a = a' = 1$ as well as $b_1 = 1/2$ (for $n = 3$) or $b_1 = b_2 = 1/2$ (for $n = 4$).

As predicted earlier, the Landen and Gauss transformations of R_F both have the same form:

$$(5.13) \quad R_F(z_1, z_2, z_3) = R_F(w_1, w_2, w_3) .$$

A distinction arises only because the arguments on each side are symmetric in just two of the arguments on the other side. Referring to (3.3), we can choose $w_1 = \cos^2 \phi$ to get the usual form of the Landen transformation, $w_1 = 1$ to get the Gauss transformation, or $w_1 = \Delta^2$ to get a less useful complex transformation [1, Eq. (165.07)]; $z_3 = 1$ and $z_3 = \cos^2 \phi$ give the inverse Landen and Gauss transformations, respectively.

To find the quadratic transformation of R_G , we set $a = -1/2$ in (5.10) and use the following relations between associated R functions:

$$(5.14) \quad R(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}; z_1, z_2, z_3) = 2 R_G(z_1, z_2, z_3) - z_3 R_F(z_1, z_2, z_3) ,$$

$$(5.15) \quad R(-\frac{1}{2}; \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}; w_1, w_2, w_3) \\ = 4 R_G(w_1, w_2, w_3) - (w_2 + w_3) R_F(w_1, w_2, w_3) - (w_2 w_3 / w_1)^{1/2} .$$

With the help of (5.11), (5.12), and (5.13), it follows that

$$(5.16) \quad 2 R_G(z_1, z_2, z_3) = 4 R_G(w_1, w_2, w_3) - \sqrt{(z_1 z_2)} R_F(z_1, z_2, z_3) - \sqrt{z_3} .$$

If desired, the Landen and Gauss transformations of $G(\phi, k)$ can be deduced from this as indicated in the last paragraph; for numerical work, however, it appears most convenient to use (5.11) and (5.16) directly. When z_3 or w_3 vanishes, (5.13) and (5.16) reduce to the transformations (5.2) and (5.6) of the complete integrals.

6. Some integrals related to ellipses and ellipsoids. To illustrate the practical convenience of hypergeometric normal integrals, we evaluate some definite integrals related to the geometry of ellipses and ellipsoids. Since integrals of this sort occur in a variety of physical problems, their evaluation in terms of hypergeometric functions may be of practical interest, especially because some important cases are not listed in present integral tables. In addition, certain curious geometrical relations emerge from the results. More important for present purposes, the relation of R_F and R_G to the geometry of the ellipsoid is found to be closely analogous to the relation of R_K and R_E to the geometry of the ellipse; in particular, if α, β, γ are lengths of semi-axes, then $R_G(\alpha^{-2}, \beta^{-2}, \gamma^{-2})$ is one-third the ratio of surface area to volume for an ellipsoid, just as $R_E(\alpha^{-2}, \beta^{-2})$ is one-half the ratio of perimeter to area for an ellipse. As in these two examples, the present notation reflects faithfully whatever symmetry a geometrical or physical quantity may have with respect to permutation of the semi-axes.

In 1907 R. Hargreaves [5] discussed a class of integrals related to the ellipsoid, which would be denoted in present notation by

$$R\left(\frac{1}{2} - p; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \alpha^2, \beta^2, \gamma^2\right) \text{ and } R\left(\frac{3}{2} - p; \frac{3}{2}, \frac{1}{2}, \frac{1}{2}; \alpha^2, \beta^2, \gamma^2\right)$$

with p an integer. He paid particular attention to the cases $p = 0$ and $p = 1$, and singled out as "the principal integrals of ellipsoidal type" those which we have called $R_F(\alpha^2, \beta^2, \gamma^2)$ and $R_G(\alpha^2, \beta^2, \gamma^2)$. Having evaluated these in terms of Legendre's normal integrals, he worked back from

Landen's transformation of the latter (their linear and Gauss transformations are not discussed) to obtain the algebraic form of Landen's transformation given in the preceding section.

We define a' as usual by (2.2) and introduce the direction cosines $l = \sin \theta \cos \phi$, $m = \sin \theta \sin \phi$, $n = \cos \theta$. The integrals of present interest are

$$(6.1) \int_0^{\pi/2} (\cos \theta)^{2b_1-1} (\sin \theta)^{2b_2-1} (z_1 \cos^2 \theta + z_2 \sin^2 \theta)^{-a} (z_1' \cos^2 \theta + z_2' \sin^2 \theta)^{-a'} d\theta \\ = \frac{1}{2} B(b_1, b_2) (z_1')^{-b_1} (z_2')^{-b_2} R(a; b_1, b_2; z_1/z_1', z_2/z_2'),$$

$$(6.2) \int_0^{\pi/2} \int_0^{\pi/2} l^{2b_1-1} m^{2b_2-1} n^{2b_3-1} (z_1 l^2 + z_2 m^2 + z_3 n^2)^{-a} (z_1' l^2 + z_2' m^2 + z_3' n^2)^{-a'} \\ \times \sin \theta d\theta d\phi \\ = \frac{1}{4} B(b_1, b_2, b_3) \left[\prod_{i=1}^3 (z_i')^{-b_i} \right] R(a; b_1, b_2, b_3; z_1/z_1', z_2/z_2', z_3/z_3').$$

Convergence requires all b 's to have positive real parts, and $B(b_1, \dots, b_n)$ stands for $\Gamma(b_1) \dots \Gamma(b_n) / \Gamma(b_1 + \dots + b_n)$. We assume here that every z and every z' has positive real part, although a less stringent condition is given at the end of this section. Noteworthy special cases, including those discussed by Hargreaves, occur when every z' is unity. The right sides of both equations can be rewritten by means of the Euler transformation (2.3), and the special transformation (2.4) is applicable to the first equation. Integrals more general than (6.1) can be evaluated by recourse to (2.8), but we confine ourselves here to the direct analog of the double integral (6.2).

Postponing the proof of these equations, we consider first some applications. Let a point on the surface of an ellipsoid be denoted by (x, y, z) , its distance from the center by r , and the perpendicular distance from the center to the tangent plane at (x, y, z) by p (the "support function"):

$$\frac{x^2}{a^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1, \quad r^2 = x^2 + y^2 + z^2, \quad \frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{\beta^4} + \frac{z^2}{\gamma^4}.$$

In polar coordinates, the quantities x, y, z, r, p are given as functions of θ and ϕ by

$$\begin{aligned} (x, y, z) &= (r\ell, rm, rn), \\ r &= (a^{-2}\ell^2 + \beta^{-2}m^2 + \gamma^{-2}n^2)^{-1/2}, \\ rp &= (a^{-4}\ell^2 + \beta^{-4}m^2 + \gamma^{-4}n^2)^{-1/2}. \end{aligned}$$

The average of a function $f(\theta, \phi)$ over the first octant will be denoted by

$$(6.3) \quad \langle f \rangle = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} f(\theta, \phi) \sin \theta \, d\theta \, d\phi.$$

For an ellipse, obvious similar relations will be understood, such as $x = r \cos \theta$ and $y = r \sin \theta$; no confusion should arise from using the same symbol in this case for an average over the first quadrant:

$$(6.4) \quad \langle f \rangle = \frac{2}{\pi} \int_0^{\pi/2} f(\theta) \, d\theta.$$

For an ellipse, we find at once from (6.1) that

$$\begin{aligned} (6.5) \quad a^{-2b_1} \beta^{-2b_2} \langle x^{2b_1-1} y^{2b_2-1} r^{2-2a} \rangle &= \langle x^{2b_1-1} y^{2b_2-1} r^{2-2a} \rangle \\ &= \pi^{-1} B(b_1, b_2) R(a; b_1, b_2; a^{-2}, \beta^{-2}). \end{aligned}$$

Similarly, for an ellipsoid, (6.2) gives

$$\begin{aligned}
 (6.6) \quad & a^{-2b_1} \beta^{-2b_2} \gamma^{-2b_3} \langle x^{2b_1-1} y^{2b_2-1} z^{2b_3-1} r^3 p^{2a} \rangle \\
 & = \langle x^{2b_1-1} y^{2b_2-1} z^{2b_3-1} r^{3-2a} \rangle \\
 & = (2\pi)^{-1} B(b_1, b_2, b_3) R(a; b_1, b_2, b_3; a^{-2}, \beta^{-2}, \gamma^{-2}) .
 \end{aligned}$$

The equalities connecting average values are not at all obvious from the geometrical meanings of r and p ; their origin will be illustrated by calculating in two ways the surface area S of an ellipsoid. References to three derivations of S are given by Watson [8]; a fourth, closely related to the second one given here, is due to Jacobi [6, pp. 110-112]. Firstly, a volume element with base dS and apex at the center of the ellipsoid has height p and slant height r . From two ways of calculating its volume, we find the relation $p dS = r(r^2 \sin \theta d\theta d\phi)$ and hence

$$(6.7) \quad S = 4\pi \langle r^3 p^{-1} \rangle = 4\pi a \beta \gamma R_G(a^{-2}, \beta^{-2}, \gamma^{-2}) .$$

On the other hand, a scale transformation which deforms the unit sphere into the ellipsoid clearly carries the vector surface element with components $(\ell, m, n) \sin \theta d\theta d\phi$ into a surface element $(\beta \gamma \ell, \gamma a m, a \beta n) \sin \theta d\theta d\phi$ of the ellipsoid. When the magnitude of this surface element is integrated over the original angles θ and ϕ , which are not the polar coordinates of a point on the ellipsoid, the total area S is seen to be identical with $4\pi a \beta \gamma \langle r^{-1} \rangle$, in agreement with (6.6) and (6.7). One may therefore say

that identities such as $\langle r^3 p^{-1} \rangle = a\beta\gamma \langle r^{-1} \rangle$ arise from the possibility of assigning two geometrical meanings to the angles of integration.

We list now a few special cases of (6.5) to show the relation of the complete elliptic integrals to the geometry of the ellipse. Remarks entirely similar to those of the last paragraph apply to its perimeter L ; denoting its area by $A = \pi \langle r^2 \rangle = \pi a\beta$, we have

$$\begin{aligned}
 (6.8) \quad \langle r^{-1} \rangle &= R_E(a^{-2}, \beta^{-2}) = L/2A, \\
 \langle r \rangle &= R_K(a^{-2}, \beta^{-2}) = a\beta R_K(a^2, \beta^2), \\
 \langle r^3 \rangle &= a\beta R_E(a^2, \beta^2).
 \end{aligned}$$

A corresponding list relates the incomplete integrals to the ellipsoid. In addition to the surface area S and the volume $V = (4\pi/3) \langle r^3 \rangle = (4\pi/3) a\beta\gamma$, we introduce the electric capacity C :

$$\begin{aligned}
 (6.9) \quad \langle r^{-1} \rangle &= R_G(a^{-2}, \beta^{-2}, \gamma^{-2}) = S/3V, \\
 \langle r \rangle &= R_F(a^{-2}, \beta^{-2}, \gamma^{-2}), \\
 \langle r^2 \rangle &= a\beta\gamma R_F(a^2, \beta^2, \gamma^2) = a\beta\gamma / C, \\
 \langle r^4 \rangle &= a\beta\gamma R_G(a^2, \beta^2, \gamma^2).
 \end{aligned}$$

Besides determining the capacity of a conducting ellipsoid, $\langle r^2 \rangle$ occurs as a factor in the Coulomb energy of a wide class of ellipsoidal charge distributions [3]. It is connected also with the depolarization factors L_a, L_β, L_γ

of a dielectric or magnetic ellipsoid:

$$(6.10) \quad L_a = \langle x^2 \rangle / a^2 = \frac{1}{3} a \beta \gamma R\left(\frac{3}{2}; \frac{3}{2}, \frac{1}{2}, \frac{1}{2}; a^2, \beta^2, \gamma^2\right),$$

$$\sum_{\alpha} L_{\alpha} = 1, \quad \sum_{\alpha} a^2 L_{\alpha} = \langle r^2 \rangle.$$

We turn finally to a proof of (6.1) and (6.2), starting from an integral representation [2] of the R function:

$$(6.11) \quad B(b_1, \dots, b_n) R(a; b_1, \dots, b_n; z_1, \dots, z_n)$$

$$= \int_0^1 \dots \int_0^1 du_1 \dots du_n \delta\left(1 - \sum_{i=1}^n u_i\right) \left(\prod_{i=1}^n u_i^{b_i-1}\right) \left(\sum_{i=1}^n u_i z_i\right)^{-a},$$

where all b 's have positive real parts and δ is the Dirac delta function. It is assumed that all the z 's lie to one side of some fixed line passing through the origin of the complex plane, and that the negative real numbers do not lie to the same side. (The sum of the z 's, weighted by the u 's, likewise lies to one side of this line and is thus kept from encircling the origin.) Integration over u_1 removes the delta function but imposes the restriction $0 \leq u_2 + \dots + u_n \leq 1$ on the remaining variables. Taking any positive numbers $t_1, \dots, t_n$, we let

$$u_i = v_i t_i V^{-1}, \quad (i = 2, \dots, n),$$

$$V = (1 - v_2 - \dots - v_n) t_1 + v_2 t_2 + \dots + v_n t_n.$$

The inverse relations are

$$v_i = u_i t_i^{-1} U^{-1}, \quad (i = 2, \dots, n),$$

$$U = (1 - u_2 - \dots - u_n) t_1^{-1} + u_2 t_2^{-1} + \dots + u_n t_n^{-1} = V^{-1}.$$

Since U is positive, the range of the v 's is given by $0 \leq v_1 \leq 1$ and

$0 \leq v_2 + \dots + v_n \leq 1$; the Jacobian of the transformation is

$$\frac{\partial(u_2, \dots, u_n)}{\partial(v_2, \dots, v_n)} = t_1 \dots t_n v^{-n}.$$

Carrying out this change of variables, we find

$$\begin{aligned} B(b_1, \dots, b_n) R(a; b_1, \dots, b_n; z_1, \dots, z_n) &= \left(\prod_{i=1}^n t_i^{b_i} \right) \int_0^1 \dots \int_0^1 dv_1 \dots dv_n \\ &\times \delta\left(1 - \sum_{i=1}^n v_i\right) \left(\prod_{i=1}^n v_i^{b_i-1} \right) \left(\sum_{i=1}^n v_i t_i z_i \right)^{-a} \left(\sum_{i=1}^n v_i t_i \right)^{-a'}. \end{aligned}$$

Although the t 's have so far been positive, we can now relax this requirement by applying the principle of analytic continuation to the last equation. If t_i and $t_i z_i$ ($i = 1, \dots, n$) lie to one side of some fixed line passing through the origin of the complex plane, then the weighted sums which occur in the integrand cannot encircle the origin. Furthermore, since the ratio $t_i z_i / t_i = z_i$ cannot reach the negative real axis, only the principal branch of the R function comes into play if we take principal values of all quantities on the right side of the equation. With minor changes of notation, we conclude that

$$\begin{aligned} (6.12) \quad &\int_0^1 \dots \int_0^1 du_1 \dots du_n \delta\left(1 - \sum_{i=1}^n u_i\right) \left(\prod_{i=1}^n u_i^{b_i-1} \right) \left(\sum_{i=1}^n u_i z_i \right)^{-a} \left(\sum_{i=1}^n u_i z_i' \right)^{-a'} \\ &= B(b_1, \dots, b_n) \left[\prod_{i=1}^n (z_i')^{-b_i} \right] R(a; b_1, \dots, b_n; z_1/z_1', \dots, z_n/z_n'), \end{aligned}$$

where, for $i = 1, \dots, n$ and for any fixed angle ψ such that $|\psi| \leq \pi/2$, we assume $\operatorname{Re} b_i > 0$, $|\arg z_i - \psi| < \pi/2$, and $|\arg z_i' - \psi| < \pi/2$. The parameter a' is defined by (2.2). For $n = 2$, the substitution $u_1 = \cos^2 \theta$ leads at once to (6.1); for $n = 3$, we take $u_1 = \ell^2$ and $u_2 = \pi^2$ to get (6.2); for higher n , if desired, analogous equations can be obtained in the same way.

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