Lauricella's Hypergeometric Function F_D

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1. Introduction. In 1880 Appell defined four hypergeometric series in two variables, which were generalized to n variables in a straightforward way by Lauricella in 1893 [2],[7] . One of Lauricella's series, which includes Appell's function F_1 as the case n = 2 (and, of course, Gauss's hypergeometric function as the case $n = 1$), is

(1.1) $$F_D(a; b_1, \dots, b_n; c; z_1, \dots z_n)$$
$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a, m_1 + \cdots + m_n)(b_1, m_1) \cdots (b_n, m_n)}{(c, m_1 + \cdots + m_n) \quad m_1! \cdots m_n!} z_1^{m_1} \cdots z_n^{m_n},$$

where $(a,m) = \Gamma(a+m)/\Gamma(a)$. The F_D function has special importance for applied mathematics and mathematical physics because elliptic integrals are hypergeometric functions of type F_D [3] .

In § 2 of this paper we shall define a hypergeometric function $R(a; b_1, \dots, b_n; z_1, \dots, z_n)$ which is the same as F_D except for small but important modifications. Since R is homogeneous in the variables $z_1, \dots, z_n$, it depends in a nontrivial way on only n-1 ratios of these variables; indeed, every R function with n variables is expressible in terms of an F_D function with n-1 variables, and conversely. Although R and F_D are therefore equivalent, R turns out to be more convenient for both theory and application.

The choice of R was initially suggested by the observation that the Euler transformations of F_D would be greatly simplified by introducing homogeneous variables. However, it will be seen in § 6 that this simplicity is not superficial and that the R function is the natural outcome of one procedure for generalizing the Gauss hypergeometric function. Instead of starting from the hypergeometric series, as Appell and Lauricella did, we can start from the remark that a solution of the Euler-Poisson equation,

$$(1.2) \qquad (z_1 - z_2)\frac{\partial^2 u}{\partial z_1 \partial z_2} + b_1 \frac{\partial u}{\partial z_2} - b_2 \frac{\partial u}{\partial z_1} = 0 ,$$

if it is also a homogeneous function of z_1 and z_2, is a Gauss hypergeometric function [4, p. 57]. If we ask, more generally, for a homogeneous function of $z_1,\dots,z_n$ which satisfies an Euler-Poisson equation in each pair of variables, the answer is the R function. The system of Euler-Poisson equations admits two groups of transformations that are directly related to the Euler transformations of F_D.

Although some properties of R given in this paper are equivalent to known properties of F_D, others represent substantial extensions of previous results and some are entirely new. We mention especially (1) a set of $n+3$ relations between contiguous R functions of n variables, (2) a proof that there exists a linear relation between $n+1$ associated R functions, (3) two ways of representing an R function as the integral of an R function with fewer variables, and (4) an integral representation of the product of two R functions.

2. The function R of n variables. We shall consider the properties of a Lauricella function F_D with parameters restricted by $c = b_1 + \cdots + b_n \neq 0, -1, -2, \dots$. Let a function R of n complex variables $z_1,\dots,z_n$ and

$n+1$ complex parameters $a, b_1, \ldots, b_n$ be defined by the following power series if $|1 - z_i| < 1$ $(i = 1, \ldots, n)$ and by its analytic continuation if $|\arg z_i| < \pi$:

$$(2.1)\quad R(a; b_1, \ldots, b_n; z_1, \ldots, z_n) = F_D(a; b_1, \ldots, b_n; b_1 + \cdots + b_n; 1-z_1, \ldots, 1-z_n)$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a, m_1 + \cdots + m_n)(b_1, m_1) \cdots (b_n, m_n)}{(b_1 + \cdots + b_n, m_1 + \cdots + m_n)\, m_1! \cdots m_n!} (1-z_1)^{m_1} \cdots (1-z_n)^{m_n}.$$

This series has two important properties, symmetry and homogeneity, which are expressible by functional relations and hence are valid for its analytic continuation as well. Symmetry is obvious:

(2.2) $R(a; b_1, \ldots, b_n; z_1, \ldots, z_n)$ is invariant under permutation of the subscripts $1, \ldots, n$ (i. e., under permutation of the b's and z's together).

To show that R is a homogeneous function of degree $-a$ in the variables $z_1, \ldots, z_n$, let Σ designate the n-fold summation in (2.1) and let the general term of the series be $u(m_1, \ldots, m_n)$. Then, if D_i stands for $\partial / \partial z_i$, we have

$$z_i D_i R = \sum \left[1 - (1 - z_i)^{-1}\right] m_i \, u(m_1, \ldots, m_n)$$

$$= \sum \left[m_i \, u(m_1, \ldots, m_n) - (1 - z_i)^{-1}(m_i + 1)\, u(m_1, \ldots, m_i + 1, \ldots, m_n)\right]$$

$$= \sum \left[m_i - \frac{a + m_1 + \cdots + m_n}{b_1 + m_1 + \cdots + b_n + m_n} (b_i + m_i)\right] u(m_1, \ldots, m_n) .$$

Hence R satisfies Euler's relation,

$$(2.3)\quad \sum_{i=1}^{n} z_i D_i R = -aR ,$$

which implies homogeneity:

$$(2.4) \qquad R(a; b_1,\dots,b_n; \tau z_1,\dots,\tau z_n) = \tau^{-a}\, R(a; b_1,\dots,b_n; z_1,\dots,z_n).$$

It is clear from (2.1) that a function F_D of $n - 1$ variables, with no relations required between the parameters, can always be expressed in terms of a function R of n variables:

$$(2.5) \qquad F_D(a; b_1,\dots,b_{n-1}; c; z_1,\dots,z_{n-1}) = R(a; b_1,\dots,b_n; 1-z_1,\dots,1-z_{n-1}, 1),$$

where b_n is defined by

$$(2.6) \qquad c = b_1 + \cdots + b_n \; .$$

Conversely, given a function R of n variables, we can use its homogeneity to make one variable equal to unity and thus express it in terms of a function F_D of $n - 1$ variables:

$$(2.7) \qquad R(a; b_1,\dots,b_n; z_1,\dots,z_n) = z_n^{-a}\, F_D\left(a; b_1,\dots,b_{n-1}; c; 1-\frac{z_1}{z_n},\dots,1-\frac{z_{n-1}}{z_n}\right),$$

where Eq. (2.6) now defines c .

Equation (2.7) is noteworthy. It shows first that R can be regarded as the result of introducing homogeneous variables in F_D . Considered as a candidate for the defining equation of R, it makes homogeneity evident but leaves permutation symmetry to be proven, in contrast to (2.1). Secondly, it shows one of the advantages of working with R and one more variable rather than with F_D ; for the symmetry of R, which is not at all apparent on the right side of (2.7), expresses the behavior of F_D under certain Euler transformations. The transformations in question are those that leave a unchanged;

for instance, the transformation of Appell's function [2, p. 30]

$$F_1(a; b, b'; c; x, y) = (1-x)^{-a} F_1\left(a, c-b-b', b'; c; \frac{x}{x-1}, \frac{x-y}{x-1}\right)$$

is equivalent to the statement that $R(a; b_1, b_2, b_3; z_1, z_2, z_3)$ is invariant under transposition of the subscripts 1 and 3.

The remaining Euler transformations of F_D [2, p. 116], in which the value of a is changed, result from combining the symmetry property (2.2) with a single transformation that will be derived in § 7 :

$$(2.8)\qquad R(a; b_1, \cdots, b_n; z_1, \cdots, z_n) = \left(\prod_{i=1}^{n} z_i^{-b_i}\right) R(a'; b_1, \cdots, b_n; z_1^{-1}, \cdots, z_n^{-1}),$$

where a' is defined by

$$(2.9)\qquad a + a' = c = b_1 + \cdots + b_n \neq 0, -1, -2, \cdots .$$

The notation of (2.9) will be used throughout this paper.

3. Special values of parameters and variables. The function R reduces to another function of the same type with one less variable if one of its parameters b_i vanishes, or if two of its variables are equal, or (with certain qualifications) if one of its variables vanishes or becomes infinite. We shall consider these cases in turn, with appropriate generalizations. Because of the symmetry relation (2.2), it does not matter which parameter or variable is taken to have a special value.

By comparing (2.1) and (2.7), we obtain a generalization of a known reduction formula for F_1 [2, p. 24]:

(3.1) $$F_D(a; b_1, \cdots, b_n; b_1 + \cdots + b_n; z_1, \cdots, z_n)$$
$$= (1-z_n)^{-a} F_D\left(a; b_1, \cdots, b_{n-1}; b_1 + \cdots + b_n; \frac{z_1 - z_n}{1-z_n}, \cdots, \frac{z_{n-1}-z_n}{1-z_n}\right).$$

When expressed in terms of R by use of (2.5), this result takes an equivalent but simpler form that is immediately obvious from (2.1):

(3.2) $$R(a; b_1, \cdots, b_n, 0; z_1, \cdots, z_{n+1}) = R(a; b_1, \cdots, b_n; z_1, \cdots, z_n).$$

Another obvious consequence of (2.1) is

(3.3) $$R(a; b_1, \cdots, b_n; 1, \cdots, 1) = 1 \quad ,$$

from which it follows by homogeneity that

(3.4) $$R(a; b_1, \cdots, b_n; z, \cdots, z) = z^{-a}$$

and, in particular,

(3.5) $$z^{-a} = R(a; b; z).$$

For any number of equal arguments, we find from (2.7) the relation

$$R(a; b_1, \cdots, b_n; z_1, \cdots, z_k, z, \cdots, z)$$
$$= z^{-a} F_D\left(a; b_1, \cdots, b_k; b_1 + \cdots + b_n; 1 - \frac{z_1}{z}, \cdots, 1 - \frac{z_k}{z}\right)$$

and hence, by (2.5),

(3.6) $$R(a; b_1, \cdots, b_n; z_1, \cdots, z_k, z, \cdots, z)$$
$$= R(a; b_1, \cdots, b_k, b_{k+1} + \cdots + b_n; z_1, \cdots, z_k, z).$$

As z tends to zero in (3.6), we obtain

$$(3.7)\quad R(a;b_1,\dots,b_n;z_1,\dots,z_k,0,\dots,0) = R(a;b_1,\dots,b_k,b;z_1,\dots,z_k,0)$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a,M+s)(b_1,m_1)\cdots(b_k,m_k)(b,s)}{(c,M+s)\;m_1!\cdots m_k!\;s!}(1-z_1)^{m_1}\cdots(1-z_k)^{m_k},$$

provided that this series converges. For brevity we have introduced $M = m_1+\dots+m_k$ and $b = b_{k+1}+\dots+b_n$. If $\mathrm{Re}(c-b-a)>0$, then the sum

$$f(M) = \sum_{s=0}^{\infty}\left|\frac{(a,M+s)\;(b,s)}{(c,M+s)\quad s!}\right|$$

exists for every nonnegative integral M [6, p. 47, Eq. (4)] ; further, one can show that positive constants K and M_0 exist such that

$$f(M) < K\,M^{|b|+\mathrm{Re}(a-c)}, \qquad (M > M_0).$$

By using this result, the series (3.7) is found to be absolutely convergent if $\mathrm{Re}(c-b-a)>0$ and $|1-z_i|<1$ ($i = 1, \dots, k$). The sum on s can then be carried out by means of a well-known theorem [9] for the Gauss hypergeometric function with unit argument:

$$\sum_{s=0}^{\infty}\frac{(a,M+s)\;(b,s)}{(c,M+s)\;s!} = \frac{(a,M)}{(c,M)}\,F(a+M,b;c+M;1)$$

$$= \frac{B(a,c-a-b)}{B(a,c-a)}\,\frac{(a,M)}{(c-b,M)},$$

where B is the beta function. We have, finally, if $\mathrm{Re}(a'-b)>0$,

$$(3.8)\quad B(a,a')\,R(a;b_1,\dots,b_n;z_1,\dots,z_k,0,\dots,0) = B(a,a'-b)\,R(a;b_1,\dots,b_k;z_1,\dots,z_k),$$

where a' is defined by (2.9) and

$$(3.9)\qquad b = b_{k+1}+\dots+b_n .$$

A logarithmic singularity may occur if Re b = Re a'; on the other hand, if $\mathrm{Re}(b - a') > 0$, we can use (2.8) and (2.4) to obtain

$$\mathcal{R}(a;\, b_1,\dots,b_n;\, z_1,\dots,z_k,\, x z_{k+1},\dots,\, x z_n)$$

$$= x^{a'-b} \left(\prod_{i=1}^{n} z_i^{-b_i} \right) \mathcal{R}(a';\, b_1,\dots,b_n;\, x z_1^{-1},\dots, x z_k^{-1},\, z_{k+1}^{-1},\dots,\, z_n^{-1}).$$

Equations (3.8) and (2.8) then give

$$\text{(3.10)} \qquad \lim_{x \to 0} \; x^{b-a'} \; \mathcal{R}(a,\, b_1,\dots,b_n,\, z_1,\dots,z_k,\, x z_{k+1},\dots,\, x z_n)$$

$$= \frac{\mathcal{B}(b-a',\, a')}{\mathcal{B}(a,\, a')} \left(\prod_{i=1}^{k} z_i^{-b_i} \right) \mathcal{R}(b-a',\, b_{k+1},\dots,\, b_n,\, z_{k+1},\dots,z_n).$$

From (3.8) and the homogeneity of R, it is easily deduced that, if $\mathrm{Re}(b-a) > 0$,

$$\text{(3.11)} \qquad \lim_{|x| \to \infty} \; x^{a} \; \mathcal{R}(a,\, b_1,\dots,b_n,\, z_1,\dots,z_k,\, x z_{k+1},\dots,\, x z_n)$$

$$= \frac{\mathcal{B}(a,\, a'-b)}{\mathcal{B}(a,\, a')} \; \mathcal{R}(a,\, b_{k+1},\dots,\, b_n,\, z_{k+1},\dots,\, z_n).$$

Similarly, if $\mathrm{Re}(a - b) > 0$, it follows from (3.10) that

$$\text{(3.12)} \qquad \lim_{|x| \to \infty} \; x^{-b} \; \mathcal{R}(a,\, b_1,\dots,b_n,\, z_1,\dots,z_k,\, x z_{k+1},\dots,\, x z_n)$$

$$= \frac{\mathcal{B}(a-b,\, a')}{\mathcal{B}(a,\, a')} \left(\prod_{k+1}^{n} z_i^{-b_i} \right) \mathcal{R}(a-b,\, b_1,\dots,b_k;\, z_1,\dots,z_k).$$

These limits will be used in § 7 to determine conditions for the convergence of certain integrals.

4. <u>Relations between associated functions.</u> The $2n+2$ functions $R(a \pm 1;\, b_1,\dots,b_n;\, z_1,\dots,z_n)$ and $R(a;\, b_1,\dots,b_i \pm 1,\dots,b_n;\, z_1,\dots,z_n)$ $(i = 1, \dots, n)$, denoted for brevity by $R(a \pm 1)$ and $R(b_i \pm 1)$, will be called contiguous to $R = R(a;\, b_1,\dots,b_n;\, z_1,\dots,z_n)$. One should note that this

definition of contiguity differs slightly from the one conventionally used for F_D ; for instance, if b_1 is increased by unity on the left side of (2.7), both b_1 and c must be increased by unity on the right side. The present definition simplifies the form of known relations for n = 2 or 3 and allows their extension without difficulty to any value of n.

Between R and any n functions contiguous to it there is a linear relation with coefficients that are homogeneous polynomials in the z's. All $\binom{2n+2}{n}$ such relations can be obtained from the following n+3 linearly independent relations:

$$(c - 1)\, R(b_i - 1) = (a' - 1)\, R + a\, z_i\, R(a+1), \quad (i = 1, \ldots, n), \tag{4.1}$$

$$c\, R = \sum_{i=1}^{n} b_i\, R(b_i + 1) \;, \tag{4.2}$$

$$c\, R(a - 1) = \sum_{i=1}^{n} b_i\, z_i\, R(b_i + 1) \;, \tag{4.3}$$

$$a\, c\, R(a + 1) = \sum_{i=1}^{n} b_i\, z_i^{-1} \left[c\, R - a'\, R(b_i + 1) \right] \;, \tag{4.4}$$

where a' and c are defined by (2.9).

Although (4.1) and (4.2) can be obtained with least effort from the integral representations (7.2) and (7.10), respectively, it is more satisfactory to avoid unnecessary restrictions on the parameters by starting from the power series (2.1), the general term of which we again denote by $u(m_1, \ldots, m_n)$. If $M = m_1 + \cdots + m_n$, the general term of the left side of (4.1) becomes

$$(c-1)\, u(m_1, \ldots, m_n)\, \frac{(c, M)\,(b_i - 1,\, m_i)}{(c-1, M)\,(b_i,\, m_i)}$$

$$= u(m_1, \ldots, m_n)\, \frac{(c + M - 1)\,(b_i - 1)}{b_i + m_i - 1} \;.$$

But, since $z_i = 1 - (1 - z_i)$, the general term of the right side of (4.1) is

$$\mu(m_1, \dots, m_n) \left[a' - 1 + \frac{a\,(a+1, M)}{(a, M)} - \frac{m_i\,(c, M)\,(b_i, m_i - 1)}{(c, M-1)\,(b_i, m_i)} \right]$$

$$= \mu(m_1, \dots, m_n)\ (c + M - 1) \left(1 - \frac{m_i}{b_i + m_i - 1} \right) ,$$

which completes the proof of (4.1). Likewise, the general term of the right side of (4.2) is

$$\mu(m_1, \dots, m_n) \frac{(c, M)}{(c+1, M)} \sum_{i=1}^{n} \frac{b_i\,(b_i + 1, m_i)}{(b_i, m_i)}$$

$$= c\ \mu(m_1, \dots, m_n) .$$

Equation (4.3) can be derived in the same fashion or, more simply, from (4.1) and (4.2). We replace a by a - 1 and b_i by $b_i + 1$ in (4.1), multiply by b_i, and sum i from 1 to n :

$$c^2\, R(a - 1) = (a' + 1) \sum_{i=1}^{n} b_i\, R(a - 1, b_i + 1) + (a - 1) \sum_{i=1}^{n} b_i\, z_i\, R(b_i + 1).$$

The first term on the right side is $(a' + 1)\, c\, R(a - 1)$, according to (4.2), and can be combined with the left side to yield (4.3). Similarly, (4.1) with b_i replaced by $b_i + 1$ becomes

$$a\, z_i\, R(a + 1, b_i + 1) = c\, R - a'\, R(b_i + 1).$$

We multiply by $b_i\, z_i^{-1}$, sum i from 1 to n, and use (4.2) to obtain (4.4).

Two R functions with parameters a, $b_1, \dots, b_n$ and $a + p$, $b_1 + q_1, \dots, b_n + q_n$, where p and q_i are integers, are called associated functions. (We exclude functions for which the sum of the b parameters is zero or a negative integer.) For n = 2 it is known that any three associated functions are related by a linear homogeneous equation with polynomial coefficients. To show this, the three functions are connected by a chain of intermediate functions, each contiguous to the next, and the intermediate functions are then eliminated from the equations relating each set of three contiguous links of the chain. To establish a similar result for $n > 2$, this procedure must be modified because $n + 1$ functions, each contiguous to the next, do not in general include any single function to which all the rest are contiguous.

We shall begin by proving a special case: any $n + 1$ associated functions having the same parameters $b_1, \ldots, b_n$ are connected by a linear homogeneous relation. Here and throughout this paper, a linear relation between R functions will be understood to mean a relation with coefficients that are polynomials in the z's. Furthermore, these polynomials may be assumed homogeneous in the z's; if they were not, the homogeneity of the R's could be used to deduce from a given relation two or more relations with homogeneous coefficients.

It will be convenient to replace R temporarily by

$$S(a; b_1,\ldots,b_n; z_1,\ldots,z_n) = B(a,a')\, R(a; b_1,\ldots,b_n; z_1,\ldots,z_n) , \tag{4.5}$$

where B is the beta function and (2.9) defines a'. Equations (4.1) and (4.2) become

$$S(b_i - 1) = S + z_i\, S(a + 1) , \quad (i = 1, \ldots, n) , \tag{4.6}$$

$$a'\, S = \sum_{i=1}^{n} b_i\, S(b_i + 1) . \tag{4.7}$$

Of course S is not defined if a or a' is zero or a negative integer, but even then we shall understand (4.6) and (4.7) to represent meaningful equations between R functions after division of all terms by $\Gamma(a)$ or $\Gamma(a')$. We rewrite (4.6) with b_i replaced by $b_i + 1$ and obtain by iteration

$$S(b_i+1) = S - z_i\, S(a+1, b_i+1) \tag{4.8}$$
$$= \sum_{p=0}^{m-1} (-1)^p z_i^p\, S(a+p) \quad + (-1)^m z_i^m\, S(a+m, b_i+1) .$$

The further replacement of a by a - 1 gives

$$S(b_i+1) = z_i^{-1}\, S(a-1) - z_i^{-1}\, S(a-1, b_i+1) \tag{4.9}$$
$$= \sum_{p=1}^{m} (-1)^{p-1} z_i^{-p}\, S(a-p) \; + (-1)^m z_i^{-m}\, S(a-m, b_i+1) .$$

We multiply (4.8) by $b_i z_i^{-m}$ and (4.9) by $b_i z_i^{m}$, sum i from 1 to n, and use (4.7):

$$\sum_{i=1}^{n} b_i z_i^{-m} S(b_i+1) = \sum_{p=0}^{m-1} (-1)^p S(a+p) \sum_{i=1}^{n} b_i z_i^{p-m} \tag{4.10}$$
$$+ (-1)^m (a'-m)\, S(a+m) ,$$

$$\sum_{i=1}^{n} b_i z_i^{m} S(b_i+1) = \sum_{p=1}^{m-1} (-1)^{p-1} S(a-p) \sum_{i=1}^{n} b_i z_i^{m-p} \tag{4.11}$$
$$+ (-1)^{m-1} (a-m)\, S(a-m) .$$

These equations, valid for any positive integral m if the sum in (4.11) is understood to vanish for $m = 1$, are useful generalizations of (4.3) and (4.4), which they include as the special case $m = 1$. By solving n of these equations, say (4.11) with $m = 1, \ldots, n$, one can obtain $S(b_1 + 1), \ldots, S(b_n + 1)$ as linear combinations of $S(a - 1), \ldots, S(a - n)$. Substitution in (4.7) then establishes a linear relation between $S, S(a-1), \ldots, S(a-n)$.

Systematic application of this relation according to the chain procedure mentioned earlier completes the proof that at most n of the functions S, $S(a \pm 1)$, $S(a \pm 2)$, ... are linearly independent.

It is now easy to deal with a set of $n + 1$ associated functions which may have different b parameters. For each value of i, let B_i be the largest value of b_i occurring in the set. By successive applications of (4.6), each function in the set can be expressed as a linear combination of the functions $S(a + p; B_1,\ldots,B_n; z_1,\ldots,z_n)$, $p = 0, \pm 1, \pm 2, \ldots$, and therefore as a linear combination of at most n of them. By elimination of these n from the $n + 1$ linear combinations, we reach the desired result: any $n + 1$ associated R functions are connected by a linear homogeneous relation with coefficients that are homogeneous polynomials in $z_1,\ldots,z_n$.

5. Differential relations. Returning to the derivation of Euler's relation, we observe from the equation preceding (2.3) that

$$(z_i D_i + b_i) R = a' \sum \frac{b_i + m_i}{c + M} \mu(m_1, \ldots, m_n),$$

where $M = m_1 + \cdots + m_n$ and where a' and c are defined as usual by (2.9). By comparison with

$$R(b_i + 1) = \sum \mu(m_1, \ldots, m_n) \frac{(c, M)(b_i + 1, m_i)}{(c+1, M)(b_i, m_i)},$$

it is apparent that

(5.1) $\quad a' b_i R(b_i + 1) = c(z_i D_i + b_i) R, \quad (i = 1, \ldots, n).$

Equation (4.1), with b_i increased by unity, combines with (5.1) to give

(5.2) $\quad a b_i R(a + 1, b_i + 1) = - c D_i R, \quad (i = 1, \ldots, n).$

Inspection of (5.1) and (5.2) leads to the following three equations, valid for all $i, j = 1, \ldots, n$:

$$(5.3) \qquad b_i\, D_j\, R(b_i + 1) = b_j\, D_i\, R(b_j + 1) \quad ,$$

$$(5.4) \qquad D_i\, (z_j\, D_j + b_j)\, R = D_j\, (z_i\, D_i + b_i)\, R \quad ,$$

$$(5.5) \qquad (z_i\, D_i + b_i)\, D_j\, R = (z_j\, D_j + b_j)\, D_i\, R \quad .$$

The last two of these are equivalent; we rewrite them in still a third form below since they constitute, together with Euler's relation (2.3), the fundamental system of differential equations satisfied by R :

$$(5.6) \qquad \left[(z_i - z_j)\, D_i\, D_j + b_i\, D_j - b_j\, D_i\right] R = 0 \ , \ (i,j = 1, \ldots, n) \ ,$$

$$(5.7) \qquad \sum_{i=1}^{n} z_i\, D_i\, R = - a\, R \quad .$$

Additional integrals of this system will be discussed in § 6 .

Equations of the form of (5.6) are sometimes called Euler-Poisson equations [4, Chs. III, IV] . They occur in Lauricella's theory of F_D as useful auxiliary equations that are not sufficient to form the foundation of the theory; all that is lacking for a sufficient foundation, however, is the homogeneity requirement (5.7), for with its help we can derive the system of differential equations customarily associated with F_D [2, p. 117] . We need only sum j from 1 to n in (5.6) for any $i = 1, \ldots, n - 1$, eliminate $D_n R$ by means of (5.7), take $z_n = 1$, and replace each remaining z by $1 - z$ in accordance with (2.5).

Although Appell [1] and Darboux [4, p. 57] observed that the homogeneous solutions of a single Euler-Poisson equation ($n = 2$) are essentially solutions of the hypergeometric equation, their observation does not appear to have been put to use in the theory of hypergeometric functions. It is now evident that one natural way of generalizing the Gauss function to any number of variables is to ask for the homogeneous solutions of a system of Euler-Poisson equations. This approach leads to the R function, whereas the historical route of generalizing the hypergeometric power series led more naturally to F_D .

We shall postpone further discussion of Euler-Poisson equations to §6 and consider some additional differential relations satisfied by R. An obvious consequence of (5.7) is

$$\sum_{j=1}^{n} z_j D_j (D_i R) = -(a+1) D_i R \quad ,$$

and substitution of $z_j D_i D_j R$ or $D_i D_j R$ from (5.6) gives two equations, each valid for $i = 1, \ldots, n$:

$$\text{(5.8)} \qquad \left[(z_i D_i + b_i) \sum_{j=1}^{n} D_j + (1 - a') D_i \right] R = 0 \quad ,$$

$$\text{(5.9)} \qquad z_i D_i^2 R = \sum_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{z_j}{z_i - z_j} (b_i D_j - b_j D_i) R - (a+1) D_i R \; .$$

Equations (5.6) and (5.9) exhibit all second derivatives of R as linear functions of its first derivatives.

If a is increased by unity in (4.2), substitution of (5.2) leads to

$$\text{(5.10)} \qquad a\, R(a+1) = - \sum_{i=1}^{n} D_i R \; .$$

Aside from a multiplicative function of a, the R function therefore satisfies a natural generalization of Truesdell's F-equation [8] :

$$F = e^{i a \pi}\, \Gamma(a)\, R \;, \tag{5.11}$$

$$\sum_{i=1}^{n} D_i F = F(a+1) \;. \tag{5.12}$$

The remaining functions contiguous to R can be expressed in terms of derivatives of R by substituting (5.1) in (4.3) and (5.10) in (4.1):

$$a' \, R(a-1) = \sum_{i=1}^{n} z_i \, (z_i D_i + b_i) \, R \;, \tag{5.13}$$

$$(1-c)\, R(b_i - 1) = z_i^{a'} \sum_{j=1}^{n} D_j \, z_i^{1-a'} \, R \;, \quad (i = 1, \ldots, n) \;. \tag{5.14}$$

Equations (5.1) and (5.2) are easily generalized by repeated application. If m is any positive integer, (5.2) clearly implies

$$(a,m)(b_i,m)\, R(a+m, b_i+m) = (-1)^m (c,m)\, D_i^{\,m} R \;, \quad (i = 1, \ldots, n) \;. \tag{5.15}$$

Repeated use of (5.1) is simplified by observing that, if $D = \partial/\partial z$,

$$(z\,D + b)\, f = z^{1-b} D\, z^b f \;,$$

$$(z\,D+b)(z\,D+b+1)\cdots(z\,D+b+m-1)\, f = z^{1-b} D^m z^{b+m-1} f \;.$$

Since the various differential operators in parentheses commute with each other, we obtain

$$(a',m)(b_i,m)\, R(b_i+m) = (c,m)\, z_i^{1-b_i} D_i^{\,m} z_i^{b_i+m-1} R \;, \quad (i=1, \ldots, n). \tag{5.16}$$

6. Euler-Poisson equations. The function R is placed in a better perspective by considering a system of Euler-Poisson equations without necessarily requiring its solutions to be homogeneous:

$$(6.1) \qquad \left[(z_i - z_j) D_i D_j + b_i D_j - b_j D_i\right] u = 0 , \quad (i,j = 1, \ldots, n) .$$

Introducing the convenient abbreviation

$$(6.2) \qquad \begin{aligned} E_{ij} = - E_{ji} &= (z_i - z_j) D_i D_j + b_i D_j - b_j D_i \\ &= (z_i D_i + b_i) D_j - (z_j D_j + b_j) D_i \\ &= D_j (z_i D_i + b_i) - D_i (z_j D_j + b_j) , \end{aligned}$$

we have the operator equations

$$(6.3) \qquad \sum_{cyc} E_{ij} D_k = E_{ij} D_k + E_{jk} D_i + E_{ki} D_j = 0 ,$$

$$(6.4) \qquad \sum_{cyc} D_i E_{jk} = 0 ,$$

$$(6.5) \qquad \sum_{cyc} E_{ij} (z_k D_k + b_k) = 0 ,$$

$$(6.6) \qquad \sum_{cyc} (z_i D_i + b_i) E_{jk} = 0 .$$

The summations extend over cyclic permutations of i, j, and k, and a symbol $D_i E_{jk}$ denotes an operator product such that $D_i E_{jk} u = D_i(E_{jk} u)$.

Suppose that u is a solution of $E_{ij} u = 0$ and of $E_{jk} u = 0$. Since (6.4) and (6.6) then show that $D_j E_{ik} u = 0$ and $b_j E_{ik} u = 0$, we have $E_{ik} u = 0$ provided that $b_j \neq 0$. The system (6.1) can therefore be deduced from n - 1 differential equations; for, if any one of the b's is nonzero, say b_j, all the equations (6.1) follow from $E_{ij}\, u = 0$ for $i = 1, \ldots, j-1, j+1, \ldots, n$.

There are two groups of transformations which carry any solution of (6.1) into another solution. One of these is obviously the group of permutations of the subscripts $1, \ldots, n$ or, in other words, simultaneous permutations of the b's and z's. The function obtained by applying any such permutation to any solution $u(b_1,\ldots,b_n; z_1,\ldots,z_n)$ not only is another solution but also is homogeneous in the z's if u is. Solutions which belong to an irreducible representation of the group may be of special interest; in particular, the R function belongs to the identical representation according to (2.2).

The second group consists of transformations induced by homographic transformations of the z's. We need to extend only slightly the results of Appell and Darboux [4, pp. 58-60] for a single Euler-Poisson equation. If α, β, γ, and δ are complex numbers restricted only by $\alpha\delta - \beta\gamma \neq 0$, the transformations

$$(6.7) \qquad v(z_1, \ldots, z_n) = \left[\prod_{i=1}^{n} (\gamma z_i + \delta)^{-b_i}\right] u\left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \ldots, \frac{\alpha z_n + \beta}{\gamma z_n + \delta}\right)$$

form a group of which the infinitesimal operators are

$$(6.8) \qquad H = \sum_{i=1}^{n} z_i D_i \quad ,$$

$$(6.9) \qquad D_+ = \sum_{i=1}^{n} D_i \quad ,$$

$$(6.10) \qquad D_- = \sum_{i=1}^{n} z_i (z_i D_i + b_i) \; .$$

It is easy to verify the operator equations

$$(6.11)\qquad E_{ij}\,H = (H+1)\,E_{ij} \quad ,$$

$$(6.12)\qquad E_{ij}\,D_+ = D_+\,E_{ij} \quad ,$$

$$(6.13)\qquad E_{ij}\,D_- = (D_- + z_i + z_j)\,E_{ij} \quad .$$

If $u(z_1,\ldots,z_n)$ is a solution of $E_{ij}\,u = 0$, it follows that $H\,u$, $D_+ u$, and $D_- u$ also are solutions. Examples are furnished by (5.7), (5.10), and (5.13). Moreover, since the finite transformations of the group are generated by the infinitesimal ones, it follows also that $v(z_1,\ldots,z_n)$ as given by (6.7) is a solution. If u is homogeneous in the z's , then $H\,u$, $D_+ u$, and $D_- u$ will be homogeneous but v will in general not be; an important exception is

$$(6.14)\qquad v(z_1,\ldots,z_n) = \left(\prod_{i=1}^{n} z_i^{-b_i}\right) u(z_1^{-1},\ldots,z_n^{-1}) \quad .$$

Equation (2.8) is an example of this exceptional case, which is thus seen to underlie an Euler transformation of F_D .

A solution of the system (6.1) is found by separation of variables to be

$$(6.15)\qquad u = G(t;\, z_1,\ldots,z_n) = f(t)\,(t+z_1)^{-b_1} \cdots (t+z_n)^{-b_n} \quad ,$$

where $-t$ is a separation constant and $f(t)$ is an arbitrary function. Integrating (6.15) with respect to t gives additional solutions of the form

$$(6.16)\qquad \int_C G(t;\, z_1,\ldots,z_n)\,dt \quad .$$

The contour C may be either a closed loop or an open path; if open, its endpoints may be either independent of the z's or chosen from among the points $-z_1, \dots, -z_n$. That the latter values are permissible is seen by considering

$$E_{ij} \int^{-z_i} G(\tau;\, z_1, \dots, z_n)\, d\tau = \Big[-(z_i - z_j) D_j G + b_j G \Big]_{\tau = -z_i}$$

$$= b_j \left[\frac{\tau + z_i}{\tau + z_j}\, G \right]_{\tau = -z_i} = 0 .$$

The last step assumes $b_i < 1$, a condition that is required in any event for the existence of the integral.

We now ask under what conditions the solution (6.16) will be, like R, homogeneous of degree $-a$ in the z's. With the notation of (2.9) we have

$$\sum_{i=1}^{n} z_i D_i G = \left(\frac{\tau}{f} \frac{df}{d\tau} + 1 - c \right) G - \frac{\partial}{\partial \tau}(\tau G)$$

$$= -a G - \frac{\partial}{\partial \tau}(\tau G)$$

if $f(t) = t^{a'-1}$. Hence, if it exists, the integral

$$u(z_1, \dots, z_n) = \int_C dt\; t^{a'-1} \prod_{i=1}^{n} (t + z_i)^{-b_i} \tag{6.17}$$

is a solution of (6.1) that is homogeneous of degree $-a$ if C is either a circuit closed on the Riemann surface of the integrand or else a simple path with endpoints chosen from among the values $0, \infty, -z_1, \dots, -z_n$, homogeneity being obvious in the latter case. The $(n+1)(n+2)/2$ integrals along simple paths are similar to solutions obtained for the differential equations of F_1 by Picard and extended to F_D by Appell and Kampé de Fériet

[2, pp. 55, 120]. Integrals around certain closed loops were shown by Erdélyi [5] to be of fundamental importance in the theory of F_1 and related functions.

The transformation theory of the various integrals (6.17) and their expressions in terms of the R function will not be considered in the present paper, with the exception of a single case to be discussed in the next section.

7. Integral representations. A particular case of (6.17) is

$$(7.1)\quad B(a,a')\, R(a;\, b_1,\dots,b_n;\, z_1,\dots,z_n) = \int_0^\infty dt\; t^{a'-1} \prod_{i=1}^{n} (t + z_i)^{-b_i} ,$$

where a and a' must have positive real parts. For $|\arg z_i| < \pi$ $(i = 1, \dots, n)$ the integral is a single-valued analytic function of the z's if the integrand is defined by $\arg t = 0$ and $|\arg(t + z_i)| < \pi$. Provided that $|1 - z_i| < 1$ $(i = 1, \dots, n)$, the integrand can be expanded in ascending powers of $(1 - z_1), \dots, (1 - z_n)$ and integrated term by term; the integral is thus found to coincide with the left side of (7.1) as defined by the series (2.1) and its analytic continuation.

Replacement of t by t^{-1} gives the representation

$$(7.2)\quad B(a,a')\, R(a;\, b_1,\dots,b_n;\, z_1,\dots,z_n) = \int_0^\infty dt\; t^{a-1} \prod_{i=1}^{n} (1 + tz_i)^{-b_i} ,$$

again valid when a and a' have positive real parts. Comparison of (7.1) and (7.2) shows that

$$(7.3) \quad R(a;\, b_1,\ldots,b_n;\, z_1,\ldots,z_n) = \left(\prod_{i=1}^{n} z_i^{-b_i}\right) R(a';\, b_1,\ldots,b_n;\, z_1^{-1},\ldots,z_n^{-1}) \,.$$

By using the relations (4.2) and (4.3) between contiguous functions, it is easy to prove that if (7.3) holds for R functions with parameters (a,a'), then it holds also for functions with parameters $(a-1,a')$ and $(a,a'-1)$. The Euler transformation (7.3) is therefore valid without restriction on a and a'. We have already commented on (7.3) in connection with (2.8) and (6.14), and have used it in deriving (3.9) and (3.11).

A new type of integral representation, not known previously for F_D, gives R as the integral of another R function with one less variable:

$$(7.4) \quad B(c-b_n,b_n)\, R(a;\, b_1,\ldots,b_n;\, z_1,\ldots,z_n) = \int_0^\infty dt\; t^{b_n-1}(t+1)^{-a'} R(a;\, b_1,\ldots,b_{n-1};\, z_1+tz_n,\ldots,z_{n-1}+tz_n) \,.$$

Convergence of the integral requires $\operatorname{Re} c > \operatorname{Re} b_n > 0$. By means of (2.4) and (7.1), the right side of (7.4) can be written as

$$\left[B(a,a'-b_n)\right]^{-1} z_n^{-a} \int_0^\infty\!\!\int_0^\infty dt\, dx\; t^{b_n-1}(t+1)^{-a'} x^{a'-b_n-1} \prod_{i=1}^{n-1}(x+z_i z_n^{-1}+t)^{-b_i} \,.$$

If $x+t$ and x/t are taken as new variables, the integration over the second of these can be carried out; the remaining integral immediately reduces to the left side of (7.4) by virtue of (7.1) and (2.4).

In the particular case of the Gauss hypergeometric function ($n = 2$), the right sides of (7.1) and (7.4) have the same form because of (3.5); comparison of the two integrals shows that

$$(7.5) \qquad z_2^{\,a}\, R(a;\, b_1,b_2;\, z_1,z_2) = z_2^{\,b_1}\, R(b_1;\, a,a';\, z_1,z_2)\ .$$

This equation, which appears to have no analogue for $n > 2$, is equivalent to the familiar symmetry of the Gauss function $F(a,b;c;z)$ in the parameters a and b .

Iteration of (7.4) yields the more general representation

$$(7.6) \quad B(b_1+\cdots+b_k, b_{k+1},\ldots,b_n)\, R(a;\, b_1,\ldots,b_n;\, z_1,\ldots,z_n)$$
$$= \int_0^\infty \cdots \int_0^\infty dt_{k+1}\cdots dt_n \left(\prod_{k+1}^{n} t_i^{\,b_i-1}\right)\left(1+\sum_{k+1}^{n} t_i\right)^{-a'}$$
$$\cdot\, R\left(a;\, b_1,\ldots,b_k;\, z_1+\sum_{k+1}^{n} t_i z_i,\ldots,z_k+\sum_{k+1}^{n} t_i z_i\right)\ .$$

The index k can have any one of the values 1, 2, ..., $n - 1$; convergence requires that the quantities $b_1+\cdots+b_k$, b_{k+1}, ..., b_n all have positive real parts. An important special case is $k = 1$:

$$(7.7) \quad B(b_1,\ldots,b_n)\, R(a;\, b_1,\ldots,b_n;\, z_1,\ldots,z_n)$$
$$= \int_0^\infty \cdots \int_0^\infty dt_2\cdots dt_n \left(\prod_{i=2}^{n} t_i^{\,b_i-1}\right)\left(1+\sum_{i=2}^{n} t_i\right)^{-a'}\left(z_1+\sum_{i=2}^{n} t_i z_i\right)^{-a},$$

where $\mathrm{Re}\, b_i > 0$ ($i = 1, \ldots, n$).

From (7.7) we can derive another representation of the same type as (7.4) but with a different R function in the integrand. We interchange the subscripts 1 and n in (7.7) and substitute $t_1 = ut$, $t_2 = (1-u)t$. Since the integral with respect to all variables but u is again of the form (7.7), we find

(7.8) $$B(b_1,b_2)\, R(a;\, b_1,\dots,b_n;\, z_1,\dots,z_n)$$
$$= \int_0^1 du\, u^{b_1-1}(1-u)^{b_2-1}\, R(a;\, b_1+b_2,b_3,\dots,b_n;\, uz_1+(1-u)z_2,z_3,\dots,z_n) ,$$

provided that b_1 and b_2 have positive real parts. Iteration of (7.8) gives

(7.9) $$B(b_1,\dots,b_k)\, R(a;\, b_1,\dots,b_n;\, z_1,\dots,z_n)$$
$$= \int_0^1 \cdots \int_0^1 du_1 \cdots du_k\ \delta(1-\sum_{i=1}^{k} u_i) \left(\prod_{i=1}^{k} u_i^{b_i-1} \right)$$
$$\cdot\, R(a;\, \sum_{i=1}^{k} b_i, b_{k+1},\dots,b_n;\ \sum_{i=1}^{k} u_i z_i, z_{k+1},\dots,z_n) ,$$

where δ denotes the Dirac delta function and where Re $b_i > 0$ ($i = 1, \dots, k$). The index k can take any one of the values 2, 3, ..., n ; in particular, for k = n, we have

(7.10) $$B(b_1,\dots,b_n)\, R(a;b_1,\dots,b_n;\, z_1,\dots,z_n)$$
$$= \int_0^1 \cdots \int_0^1 du_1 \cdots du_n\ \delta(1-\sum_{i=1}^{n} u_i) \left(\prod_{i=1}^{n} u_i^{b_i-1} \right) \left(\sum_{i=1}^{n} u_i z_i \right)^{-a} ,$$

valid for Re $b_i > 0$ ($i = 1, \dots, n$). If one substitutes $u_i = v_i^2$, the right sides of (7.9) and (7.10) become integrals over part of the surface of a hypersphere. Equation (7.10), which can alternatively be obtained from (7.7) by a change of variables, is equivalent to a known representation of F_D [2, p. 115].

Another new type of integral representation gives the product of two R functions as the integral of a single R function:

$$(7.11)\quad B(a,a')\,B(\alpha,\alpha')\,R(a;\,b_1,\ldots,b_n;\,z_1,\ldots,z_n)\,R(\alpha;\,\beta_1,\ldots,\beta_\nu;\,\zeta_1,\ldots,\zeta_\nu)$$

$$= B(a+\alpha,a'+\alpha')\int_0^\infty dt\; t^{\alpha-1}$$

$$\cdot\, R(a+\alpha;\,b_1,\ldots,b_n,\beta_1,\ldots,\beta_\nu;\,z_1,\ldots,z_n,t\zeta_1,\ldots,t\zeta_\nu)\,.$$

The parameter α' is defined by $\alpha+\alpha'=\beta_1+\cdots+\beta_\nu$; by using the limits given at the end of §3, one finds that convergence requires the real parts of a, a', α, and α' to be positive. If the representation (7.1) is substituted in the integrand, the two integrations can be carried out in reverse order by means of (7.2) and (7.1) to yield the left side of (7.11). Repeated application of (7.11) can be used to represent the product of $p+1$ functions as a p-fold integral of a single R function.

A procedure similar to that used in proving (7.11) permits evaluation of the integral

$$(7.12)\quad B(a+s,a')\int_0^\infty dt\; t^{s-1}\,R(a+s;\,b_1,\ldots,b_{n-1},b_n+s;\,z_1,\ldots,z_{n-1},z_n+t)$$

$$= B(s,b_n)\,B(a,a')\,R(a;\,b_1,\ldots,b_n;\,z_1,\ldots,z_n)$$

if the real parts of s, a, and b_n are all positive. After substituting (7.1) in the integrand, we can carry out the two integrations in reverse order to get the right side of (7.12).

In view of (2.1), the R function can also be represented as a multiple integral of the Mellin-Barnes type:

$$(7.13)\quad B(b_1,\dots,b_n)\, R(a;\, b_1,\dots,b_n;\, z_1,\dots,z_n) = (2\pi i)^{-n}\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty} ds_1\cdots ds_n$$

$$\cdot\, B(a+s_1+\cdots+s_n, -s_1,\dots,-s_n)\, B(b_1+s_1,\dots,b_n+s_n)\prod_{i=1}^{n}(z_i-1)^{s_i}.$$

It is assumed that none of a, $b_1,\dots,b_n$ is zero or a negative integer. The paths of integration are to be indented in such a way that the path in the s_i plane separates the poles at $s_i = 0, 1, 2, \dots$ from the other poles of the integrand.

8. Elliptic integrals. It has been pointed out recently [3] that the three standard kinds of elliptic integral are hypergeometric functions of the type F_D. They are included as special cases of the integral

$$(8.1)\quad \int_0^u (\mathrm{sn}\, v)^{2a-1}(\mathrm{sn}^2 u - \mathrm{sn}^2 v)^{a'-1}(\mathrm{cn}\, v)^{1-2b_1}(\mathrm{dn}\, v)^{1-2b_2}(1+\nu\, \mathrm{sn}^2 v)^{-b_3}\, dv$$

$$= \tfrac{1}{2} B(a,a')\, (\mathrm{sn}\, u)^{2c-2}\, R(a;\, b_1,b_2,b_3,b_4;\, \mathrm{cn}^2 u,\, \mathrm{dn}^2 u,\, 1+\nu\, \mathrm{sn}^2 u,\, 1)\ ,$$

where $a+a' = c = b_1+\cdots+b_4$. Convergence of the integral requires a and a' to have positive real parts. Substitution of $\mathrm{sn}\, v = (1+t)^{-\frac{1}{2}}\, \mathrm{sn}\, u$ puts the integral in the form (7.1)

In the case of complete elliptic integrals ($\mathrm{cn}\, u = 0$) or incomplete integrals of the first or second kinds ($b_3 = 0$), the number of variables in the R function can be reduced by using the relations given in § 3.

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