STABILITY OF A LANDAU DAMPED SYSTEM*

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Introduction

Dispersion relation serves as a base for all the Landau damping theory (Ref. 1 and 2). In this paper we rederive the dispersion relation for general oscillator systems from a different approach which hopefully gives a little clearer view of the problem.

Stability criteria is obtained by mapping technique. Two examples will be discussed to illustrate the sufficient condition for stability.

Theory

Consider a set of oscillators have spectrum defined by $\rho(\omega)$. The motion equation of each oscillator at frequency ω is simply

$$X''(\omega, t) + \omega^2 X(\omega, t) = f_s + f_e \quad (1)$$

$X(\omega, t)$ coordinate of oscillator with frequency ω

$f_s(t)$ — Driving force from self collective motion

$f_e(t)$ — External excitation

The force term f_s depends on collective motion of the whole set of oscillators and the environment. In many cases the response of environment to particle motion can be expressed as a linear differential

equation. We assume $f_s(t)$ satisfies:

$$\mathcal{L}\left(\frac{d}{dt}\right) f_s(t) = \langle x(t) \rangle \quad (2)$$

where $\mathcal{L}\left(\frac{d}{dt}\right)$ is a linear differential operator, describing the response of the environment to beam excitation. $\langle x(t) \rangle$ is the center of mass motion of the oscillator system.

(2) is general enough to include many of the impedances practically used. For a circular machine $f_s(t)$ may also depend on the particle motion in earlier turns, then we have to consider multiturn impedances.

The problem is now to solve these two coupled equation. We first combine these two equations into one integral equation. Equations (1) and (2) can be solved formally as:

$$x(t, \omega) = \int_0^t G(\omega, t-t') \left(f_s + \int_s(t_1) dt_1 + \int_e(t_1) \right) + x(0, \omega) \cos \omega t + \frac{x'(0, \omega)}{\omega} \sin \omega t \quad (3)$$

$$f_s(t) = \mathcal{L}^{-1}\left(\frac{d}{dt}\right) \langle x(t) \rangle \quad (4)$$

where $G(\omega, t-t') = \frac{\sin \omega(t-t')}{\omega}$ is the Green's function for our simple problem. Integral over the spectrum and insert (4) into (3), we have

$$\langle x(t) \rangle = \int_0^t \int_{-\infty}^{\infty} \rho(\omega) G(\omega, t-t') d\omega \left(\mathcal{L}^{-1}\left(\frac{d}{dt}\right) \langle x(t') \rangle + f_e \right) dt' + x(0, \omega) \cos \omega t + \frac{x'(0, \omega)}{\omega} \sin \omega t \quad (5)$$

This is a second kind Volterra type integral equation (Ref. 5). Equation (5) can be solved by using Laplace transformation. If we write $X(p)$ as the transform of $\langle x(t) \rangle$, the transformed equation of (5) is:

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$$X(p) = \int_{-\infty}^{\infty} \rho(\omega) G(p, \omega) \mathcal{Y}^{-1}(p) X(p) d\omega + \phi(p) + F(p) \int G(p, \omega) d\omega \quad (6)$$

where $G(p, \omega)$ — Laplace transform of $G(\omega, t)$

$\phi(p)$ — Laplace transform of terms from initial

$F(p)$ — Laplace transform of external driving force terms

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If we write

$$\langle G(p) \rangle = \int_{-\infty}^{\infty} \rho(\omega) G(p, \omega) d\omega \quad (7)$$

we have:

$$X(p) = \frac{\phi(p) + F(p) \langle G(p) \rangle}{1 - \langle G(p) \rangle \mathcal{Y}^{-1}(p)} \quad (8)$$

The Laplacian inverse of $X(p)$ gives the solution:

$$\langle x(t) \rangle = \int_{\tau - i\infty}^{\tau + i\infty} \frac{\phi(p) + F(p) \langle G(p) \rangle}{1 - \langle G(p) \rangle \mathcal{Y}^{-1}(p)} e^{-pt} dp \quad (9)$$

where τ is positive and must be bigger than all the real part of the poles of the integrand (this guarantees $\langle x(t) \rangle = 0$ before any disturbance happens).

Criteria for Stability and Dispersion Relation

From (9) we see clearly that each pole of the integrand contribute a possible oscillating mode of collective motion. Terms from external driving may contribute forced oscillations which we are not interested in here. Initial condition term $\phi(p)$ can not cause instability. We need only consider poles from the zero points of the denominator in (9). Therefore the sufficient condition for stability is that the zero points of equation:

$$1 - \langle G(p) \rangle \mathcal{Y}^{-1}(p) = 0 \quad (10)$$

all lie on the left half complex p -plane. Equation (10) is exactly the well known dispersion relation (Ref. 1).

Mapping

The stability of the system is now reduced into a classical algebra problem, i.e. to find all the zero points of a function of complex variable. Unfortunately, this is not trivial for a general spectrum.

Traditional way of handling this problem is by conformal mapping. Particularly when g^{-1} is a constant or approximately a constant (usually written as $U+iV$), we can solve for $U+iV$ from equations (9).

$$U+iV = \frac{1}{\langle G(p) \rangle} \quad (11)$$

This gives a mapping from p -plane to $U+iV$ plane. The stable region and unstable region are separated by the image curve on $U+iV$ plane of line $\text{Re}(p) = \epsilon$ ($\epsilon \rightarrow 0+$). In the case this image curve encircles an simple connected domain when p goes from $(0, -i\infty) \rightarrow (0, +i\infty)$ the stability criteria is simply that stable when $U+iV$ falls on the L.H.S. of the image curve and unstable when on R.H.S. But sometime the curve is rather complicated, in that case we have to investigate the property of the function over the whole p -plane.

It is found to be advisable to let p go around a loop as Fig. 1 and let $R \rightarrow \infty$.

FIG. 1

In the following two examples. We will use this method to find stability conditions for a Lorentz spectrum and a Gaussian spectrum.

Example 1.

Lorentz Spectrum:

$$\rho(\omega) = \frac{n \alpha}{\sqrt{2\pi}} \left(\frac{1}{(\omega - \omega_0)^2 + \alpha^2} + \frac{1}{(\omega + \omega_0)^2 + \alpha^2} \right) \quad (12)$$

and constant impedance: $U + iV$

$$\langle G(p) \rangle = \frac{N}{2} \frac{(\alpha + p)}{[(\alpha + p)^2 + \omega_0^2] \cdot p} \quad (13)$$

$$U + iV = \frac{2}{N} \cdot \frac{[(\alpha + p)^2 + \omega_0^2] p}{\alpha + p} \quad (14)$$

Figures 2 and 3 show the mapping results for two different cases, the parameters are also listed. The frequency spread is chosen arbitrarily.

By using the principle of argument of analytic functions (Ref. 5), we can easily determine the stability region:

Region 1, Stable: When p goes a long the whole loop c , the argument of $\langle G(p) \rangle^{-1} - (U + iV)$ does not change and also we know $\langle G(p) \rangle^{-1} - (U + iV)$ has no pole in right half plane, therefore Equation (10) has no zero point on the right half plane.

Region 2, Unstable:
When $U + iV$ located in Region 2, the argument changed 4π radian, there must be two zero points on right half p -plane.

Region 3, Unstable:
When $U + iV$ located in Region 3, the argument changed 2π radian. This means there is one zero points on right half plane.

Compare the two cases in Fig. 2 and Fig. 3 we see the stability region becomes smaller when tune spread is reduced.

Example 2,

Gaussian Distribution:

$$\rho(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\Delta} \left[e^{-\frac{(\omega-\omega_0)^2}{2\Delta^2}} + e^{-\frac{(\omega+\omega_0)^2}{2\Delta^2}} \right]$$

$$G(p) = \frac{\pi}{2p} \rho(-ip) + \frac{1}{2p} \int_0^\infty d\omega \frac{\rho(\omega-ip) - \rho(\omega+ip)}{\omega}$$

Here we have taken principal value for integral (7). The mapping is done by numerical calculation. Figures 4 and 5 show the corresponding curve on $U+iv$ plane, the parameters are listed in the caption.

By the same argument as the previous example, we can conclude that for this case Region 1 is stable and Region 2 is unstable.

Multiturn Impedance

It is possible that the same analysis can be applied to find stability criteria for some problems with multiturn impedance.

For the sake of simplicity, we examine again Lorentz distribution with a self exciting force damping away on successive turns.

$$f(t) (U+iv) = \sum_n x(t-n\tau) \cdot e^{-\alpha n\tau}$$

The dispersion relation is:

$$(U+iv) = \frac{(1 - \beta e^{-p\tau})}{\langle G(p) \rangle} \quad (\beta = e^{-\alpha\tau})$$

$\langle G(p) \rangle$ is defined in Equation (13).

Figures 6 and 7 give the mapping diagrams for two different set of parameters. We see that because of multiturn effect the stable region is quite reduced.

We like to mention here that the same analysis can be used to treat feedback problem as long as the feedback can be smoothed. An example is given in Appendix I which gives same result as a first order analysis given by S. Khaifets (Ref. 4).

Conclusion

The use of Laplacian transform has been found to be useful to analyse Landau damping effects. The collective oscillating modes of a coupled oscillator system are related to solutions of the dispersion relation. The mapping technique for stability is discussed in detail.

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APPENDIX I

The smoothed solution for a feedback system satisfies:

$$x'' + \omega^2 x = \Sigma \int_m x(t - m\tau)$$

The Laplacian transformation of this equation gives:

$$(p^2 + \omega^2)x(p) = (\Sigma \int_m e^{-mtp}) x(p) + \phi(p)$$

where p — is the Laplacian variable

$\phi(p)$ — is the term from initial condition

$x(p)$ — is the transformation of $x(t)$

This gives:

$$x(p) = \frac{\phi(p)}{p^2 + \omega^2 - \Sigma \int_m e^{-mtp}}$$

$$x(t) = \int_{\tau-i\infty}^{\tau+i\infty} \frac{\phi(p) e^{-pt} dp}{(p^2 + \omega^2 - \Sigma \int_m e^{-mpt})}$$

The coherent frequency is determined by the zero points of the denominator in the integrand. For small $\int_m$ and first order approximation

$$\Delta\omega + \frac{1}{\tau} = -\frac{1}{2} \omega \Sigma \int_m e^{-im\omega\tau}$$

where $\Delta\omega$ is the frequency shift of collective motion, $\frac{1}{\tau}$ is the growth rate of collection motion.

Incidentally it is not hard to find higher order solution of equation:

$$p^2 + \omega^2 - \Sigma \int_m e^{-mpt} = 0$$

But, the smooth approximation we used makes higher order solution unnecessary.

FIG. 4

FIG. 5