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**Tensor Pressure Tokamak
Equilibrium and Stability**

W. Anthony Cooper

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✓ TENSOR PRESSURE TOKAMAK
EQUILIBRIUM AND STABILITY

W. Anthony Cooper

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CONTENTS

LIST OF ILLUSTRATIONS	v
ACKNOWLEDGEMENTS	xiii
ABSTRACT	xv
1. INTRODUCTION	1
2. EQUILIBRIUM	11
2.1 THE TENSOR PRESSURE EQUILIBRIUM EQUATION	12
2.2 FORCE BALANCE RELATION AND POLOIDAL BETA	15
2.3 PRESSURE MOMENTS FROM A BEAM DISTRIBUTION FUNCTION	19
2.4 MIRROR AND FIREHOSE STABILITY	22
2.5 NUMERICAL METHOD	23
2.6 SAMPLE EQUILIBRIA	26
3. STABILITY	39
3.1 THE ENERGY PRINCIPLE	40
3.2 THE EULER EQUATION	44
3.3 THE TENSOR PRESSURE BALLOONING MODE EQUATION	46
3.4 NUMERICAL METHOD	49
3.5 STABILITY OF SAMPLE EQUILIBRIA	50
4. STABILITY OF ISX-B MODEL EQUILIBRIA	95
5. SUMMARY AND CONCLUSIONS	102
APPENDICES	105
A. THE FORCE BALANCE RELATION	105
B. THE CURRENT DENSITIES	107
C. THE DOUBLE ADIABATIC ENERGY PRINCIPLE	111
D. THE ENERGY PRINCIPLE IN ASYMMETRIC TORI	115
E. MINIMIZATION OF THE ENERGY PRINCIPLE	119
F. THE CONNOR-HASTIE-TAYLOR TRANSFORMATIONS	131
G. EQUILIBRIUM EXPRESSIONS IN (ψ, B) SPACE	135

H. EXPANSION OF THE SECULAR TERM AND THE CURVATURES FOR NUMERICAL TREATMENT	139
I. EVALUATION OF $(\vec{\nabla}\psi \cdot \vec{\nabla})B^2$ AND $(\vec{B} \cdot \vec{\nabla})B^2$	143
REFERENCES	147

LIST OF ILLUSTRATIONS

- Fig. 1.1. Beam distribution function level contours from a numerical solution of the Fokker-Planck equation at a location halfway between the magnetic axis and the edge of the plasma ($r/a = 0.5$), with average electron density $\bar{n}_e = 5 \times 10^{13} \text{ cm}^{-3}$ and $Z_{\text{eff}} = 1.0$ 2
- Fig. 1.2. Beam distribution function level contours from a numerical solution of the Fokker-Planck equation at a location halfway between the magnetic axis and the edge of the plasma ($r/a = 0.5$), with average electron density $\bar{n}_e = 5 \times 10^{13} \text{ cm}^{-3}$ and $Z_{\text{eff}} = 5.0$ 3
- Fig. 2.1. Plasma shaping with elongation $b/a = 1.6$, triangularity $d = 0.75$, and quadrangularity $q = 0.55$. . 25
- Fig. 2.2. Flux contours and toroidal current density contours with parallel beam injection ($\theta_b = 25^\circ$, where θ_b is the angle between the magnetic field and the direction of the beam at the outer midplane point of the plasma) for $\beta_j = 1.1$ and $\beta_j = 2.2$ 29
- Fig. 2.3. Flux contours and toroidal current density contours with perpendicular beam injection ($\theta_b = 43.57^\circ$, where θ_b is the angle between the magnetic field and the direction of the beam at the outer midplane point of the plasma) for $\beta_j = 1.1$ and $\beta_j = 2.2$ 30
- Fig. 2.4. Parallel pressure contours (solid curves) overlaid on flux surfaces (dashed curves) of equilibria with broad and peaked tensor pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection 32
- Fig. 2.5. Perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) of equilibria with broad and peaked tensor pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection 34

Fig. 2.6.	Broad and peaked profiles for background pressure p_0 , perpendicular beam pressure $p_{b\perp}$, and parallel beam pressure $p_{b\parallel}$ of tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection	35
Fig. 2.7.	Toroidal current density contours of tensor pressure equilibria with broad and peaked pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection	37
Fig. 2.8.	Toroidal current density profiles of tensor pressure equilibria with broad and peaked pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection	38
Fig. 3.1.	Variation of the eigenfunctions on different flux surfaces with arc length. The eigenmodes near the magnetic axis are flutelike (with a small ballooning modulation), whereas the modes near the edge of the plasma are ballooning in character (with a small interchange tail)	51
Fig. 3.2.	Convergence of the eigenvalues on an outer, a middle, and an inner flux surface with arc length. (N_e represents the number of poloidal circuits, starting from the outer edge, that an eigenmode makes before it vanishes)	52
Fig. 3.3.	Parallel and perpendicular pressure surfaces (solid curves) overlaid on flux surfaces (dashed curves) of scalar pressure equilibria and of tensor pressure equilibria with moderately broad pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection	55
Fig. 3.4.	Toroidal current density contours of scalar pressure equilibria and of tensor pressure equilibria with moderately broad pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection	56

- Fig. 3.5. Sufficient stability limits with finite arc length (7.5 poloidal transits) and $\beta \equiv (2/3) \langle p_{\parallel} + 2p_{\perp} \rangle$ as a function of β_j for ideal MHD and for tensor pressure equilibria with moderately broad pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection 58
- Fig. 3.6. Sufficient stability limits with finite arc length (7.5 poloidal transits) and $\beta \equiv \langle p_{\parallel} + p_{\perp} \rangle$ as a function of β_j for ideal MHD and for tensor pressure equilibria with moderately broad pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection 59
- Fig. 3.7. Convergence of critical β values with arc length for equilibria susceptible to interchange modes 60
- Fig. 3.8. Parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection with slightly peaked pressure profiles 62
- Fig. 3.9. Sufficient stability limits as a function of β_j for different arc lengths (N_e = the number of poloidal transits of the eigenmodes) of tensor pressure equilibria induced by parallel injection ($\theta_b = 25^\circ$) with slightly peaked pressure profiles 64
- Fig. 3.10. Sufficient stability limits as a function of β_j for different arc lengths (N_e = the number of poloidal transits of the eigenmodes) of tensor pressure equilibria induced by perpendicular injection ($\theta_b = 43.57^\circ$) with slightly peaked pressure profiles . . 65
- Fig. 3.11. Necessary and sufficient stability limits as a function of β_j for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection with slightly peaked pressure profiles . . 66
- Fig. 3.12. Marginally stable β (in %) as a function of β_j imposed by interchanges (dashed lines) and ballooning modes (solid lines) for tensor pressure equilibria with slightly peaked pressure profiles and injection angles (with respect to the magnetic field) of 25° and 43.57° . . 68

- Fig. 3.13a. Parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 0.5$ 71
- Fig. 3.13b. Parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 1.5$ 72
- Fig. 3.13c. Parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 2.5$ 73
- Fig. 3.14a. Beam and background pressure profiles and toroidal current density profiles for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 0.5$ 74
- Fig. 3.14b. Beam and background pressure profiles and toroidal current density profiles for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 1.5$ 75
- Fig. 3.14c. Beam and background pressure profiles and toroidal current density profiles for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 2.5$ 76
- Fig. 3.15a. Necessary and sufficient stability limits as a function of safety factor q on axis for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and $\beta_j = 0.5$ 78

- Fig. 3.15b. Necessary and sufficient stability limits as a function of safety factor q on axis for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and $\beta_j = 2.5$ 79
- Fig. 3.16a. Sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and $\beta_j = 0.5$ 80
- Fig. 3.16b. Sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and $\beta_j = 2.5$ 81
- Fig. 3.17a. Necessary and sufficient stability limits as a function of β_j for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and safety factor $q_a = 1.0$ on axis 83
- Fig. 3.17b. Necessary and sufficient stability limits as a function of β_j for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and safety factor $q_a = 1.4$ on axis 84
- Fig. 3.18. Necessary and sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles 85
- Fig. 3.19a. Necessary and sufficient stability limits as a function of safety factor q on axis for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and $\beta_j = 0.5$ 86

- Fig. 3.19b. Necessary and sufficient stability limits as a function of safety factor q on axis for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and $\beta_j = 2.5$ 87
- Fig. 3.20a. Sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and $\beta_j = 0.5$ 89
- Fig. 3.20b. Sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and $\beta_j = 2.5$ 90
- Fig. 3.21a. Necessary and sufficient stability limits as a function of β_j for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and safety factor $q_a = 1.0$ on axis 91
- Fig. 3.21b. Necessary and sufficient stability limits as a function of β_j for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and safety factor $q_a = 1.4$ on axis 92
- Fig. 3.22. Necessary and sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles . . 93
- Fig. 4.1. Model ISX-B toroidal current density profile, background and beam pressure profiles, and flux surfaces with $\beta = 1.44\%$. (The unstable region is bounded by the dashed contour.) 98

Fig. 4.2. Model ISX-B toroidal current density profile, scalar pressure profiles, and flux surfaces with $\beta = 1.44\%$. (The unstable region is bounded by the dashed contour.) 100

Fig. 4.3. Model ISX-B toroidal current density profile, background and beam pressure profiles, and flux surfaces with $\beta = 0.83\%$. (The unstable region is bounded by the dashed contour.) 101

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ABSTRACT

We investigate the equilibrium and magnetohydrodynamic (MHD) stability of tokamaks with tensor pressure and examine, in particular, the effects of anisotropies induced by neutral beam injection. Perpendicular and parallel beam pressure components are evaluated by taking moments of a distribution function obtained from the solution of a Fokker-Planck equation that models the injection of high-energy neutral beams into a tokamak. We numerically generate D-shaped beam-induced tensor pressure equilibria. From these equilibria, we find that the level perpendicular pressure contours can be quite distorted and shifted away from the outer boundary (relative to the flux surfaces), especially with perpendicular injection into broad pressure profile equilibria, and that the level parallel pressure contours undergo a smaller shift that is concentrated in the central core of the plasma. On the other hand, in equilibria with peaked pressure profiles, the pressure contours nearly coincide with the flux surfaces. In addition, a force balance relation, a representation for the poloidal beta (β_p), and the expressions for the current densities are derived from the MHD equilibrium relations for an axisymmetric tensor pressure tokamak.

A double adiabatic energy principle is derived from a modified version of the guiding center plasma energy principle. The asymptotic expansion (in large toroidal mode number n) and subsequent minimization of this energy principle yield an Euler equation from which we obtain a necessary stability criterion for the guiding center plasma to oscillations of large toroidal mode number. If we neglect the terms peculiar to the double adiabatic theory, we obtain a sufficient stability condition for the guiding center plasma. The stabilizing term due to the bending of the field lines is almost equivalent to the corresponding term in ideal MHD. The driving term is modified, however, because the pressures are no longer constant on a flux surface and the additional double adiabatic term is stabilizing. We use a shooting method to examine the stability of tensor pressure equilibria

to perturbations of large toroidal mode number. We find that because the double adiabatic term inhibits perturbations in high-pressure regions, the difference between the necessary and the sufficient criteria is significant for plasmas susceptible to flute modes that naturally concentrate near the magnetic axis. The difference between criteria is small for ballooning stability because the pressures are weak near the edge of the plasma, where this type of mode concentrates. The critical β values imposed by stability to ballooning modes are higher for perpendicular beam injection than for parallel beam injection or for corresponding scalar pressure equilibria with comparable pressure profiles. Perpendicular injection causes an inward displacement of the $p_{\perp}$ surfaces with respect to the flux contours, and this displacement improves the stability conditions. The difference between the stability limits of broad, as compared with peaked, pressure profile equilibria is large for interchange unstable plasmas and tends to coalesce for ballooning unstable plasmas.

Finally, we apply the tensor pressure ballooning mode equation to computed equilibria that model experimentally determined ISX-B discharge profiles with high-power neutral beam injection. We predict that the plasma is unstable to flutelike modes in the central core of the discharge as a result of the pressure profile peakedness induced by the beams.

1. INTRODUCTION

In addition to heating tokamak plasmas to reactor-like temperatures, neutral beam injection is expected to produce pressure anisotropy, plasma flows, and plasma currents. Here, we will concentrate on the first of the three effects.

Injection experiments carried out recently on ISX-B¹ and previously on ORMAK, TFR, DITE, T11, and PLT²⁻⁶ indicate that the behavior of the injected hot ions is classical. This implies that when the injection energy is sufficiently high, the beam particles tend to slow down faster than they scatter in pitch angle. The resulting distribution function can display a strong energy anisotropy, the appropriate moments of which yield pressures along the field lines that are different from the pressures across them. The experimental observation of an energy anisotropy is difficult. For example, the observation of energy anisotropies in a sequence of low-density, high-temperature discharges in PLT was complicated by the presence of the beams, which tended to swamp the signal in the parallel charge exchange analyzer.⁷ In that respect, the presence of beam-induced pressure anisotropies remains to be experimentally verified.

Transport simulations of a beam-heated ISX-B-like tokamak using a Fokker-Planck module to treat the slowing down and pitch angle scattering of the high-energy ions yield a fast-particle distribution function that displays a significant velocity space anisotropy when the impurity content of the plasma is low ($Z_{\text{eff}} \approx 1$), as illustrated in Fig. 1.1.⁸ By contrast, when $Z_{\text{eff}} \approx 5$, the beam distribution is rapidly isotropized, as seen in Fig. 1.2.

Neutral beam injection is not the only method by which a tensor pressure can be developed and maintained. The perpendicular charge exchange analyzer has shown an enhancement of the tail of the ion distribution function during lower hybrid heating experiments in such machines as PETULA and WEGA.^{9,10} Ion cyclotron heating is expected to produce a similar tail. Theoretical estimates of electron cyclotron heating predict an increase in the perpendicular electron

Fig. 1.1. Beam distribution function level contours from a numerical solution of the Fokker-Planck equation at a location halfway between the magnetic axis and the edge of the plasma ($r/a = 0.5$), with average electron density $\bar{n}_e = 5 \times 10^{13} \text{ cm}^{-3}$ and $Z_{\text{eff}} = 1.0$. (Courtesy of J. T. Hogan)

temperature,¹¹ although the processes that anomalously enhance the electron transport in present-day tokamaks may rapidly isotropize the pressures. In reactors, the anomalous loss of trapped alpha particles may produce $p_{\parallel} \neq p_{\perp}$ because the pressure they introduce is comparable to the background plasma pressure near the magnetic axis.¹²

Fig. 1.2. Beam distribution function level contours from a numerical solution of the Fokker-Planck equation at a location halfway between the magnetic axis and the edge of the plasma ($r/a = 0.5$), with average electron density $\bar{n}_e = 5 \times 10^{13} \text{ cm}^{-3}$ and $Z_{eff} = 5.0$. (Courtesy of J. T. Hogan)

The efficient utilization of the magnetic field demands the operation of a tokamak at the highest possible beta (β). It is generally believed that the limiting value of β is imposed by stability to ballooning modes. These perturbations arise after a threshold pressure gradient is reached and are driven by the interaction of that

gradient with the destabilizing curvature of the magnetic field. In those regions where the curvature is unfavorable (destabilizing), the perturbation causes the plasma to bulge out and the magnetic field lines to bend. The field line bending acts as a restoring force. The pressure anisotropy can influence the magnetohydrodynamic (MHD) stability of a tokamak in two ways. First, it can directly alter the driving and stabilizing mechanisms and thus affect the marginal stability points and the evolution of ballooning and interchange modes (both of which are pressure driven). Second, the tensor pressure elements can indirectly affect the MHD stability through their influence on the equilibrium.

The ballooning instability is analogous to the Rayleigh-Taylor instability, which is driven by a gravitational force.¹³ In tokamak geometry, the centrifugal force produced by particles moving along the field lines and the force on the magnetic moment owing to the gradient of the magnetic field correspond roughly to a gravitational force.¹⁴ These forces, which have approximately the same direction, are balanced by the bending and stretching of the magnetic field lines that they induce. The centrifugal force experienced by ions and electrons as they move along the field lines is proportional to $v_{\parallel}^2$ and the gradient $|\vec{B}|$ force is proportional to $(1/2)v_{\perp}^2$, with roughly the same constant of proportionality. Parallel injection would tend to enhance the centrifugal force as the β of the plasma is raised, whereas perpendicular injection would enhance the gradient $|\vec{B}|$ force on the magnetic moment. From a fluid point of view, the centrifugal force is related to the interaction of the parallel pressure gradient with the magnetic field line curvature, and the gradient $|\vec{B}|$ force is related to the interaction of the perpendicular pressure gradient with radial gradients in $|\vec{B}|$. Large spectral^{15,16} or initial value^{17,18} computer codes must be used to study the stability of tokamak plasmas to ballooning modes with large wavelengths in the toroidal direction. Ballooning instabilities that display a lot of structure in the toroidal direction seem to set a limit on β .

Dobrott et al. investigated ballooning modes with large toroidal mode number n for ideal MHD plasmas.¹⁹ Their study consisted of minimizing an energy principle that has been expanded in powers of $(1/n)$. The problem then reduces to solving one second-order ordinary differential equation on each flux surface (i.e., an Euler equation). To obtain a solution, they assumed that the perturbation vanishes at the point on each flux surface where the magnitude of the magnetic field is largest. Connor et al. improved on these results by introducing a transformation that extended the periodic domain of the poloidal angle to infinity.²⁰ In this manner they were able to construct a perturbation in an eikonal form¹³ that varies slowly along the field lines and rapidly across the field lines in a magnetic field structure with shear, without having to impose any constraints on the perturbation.^{21,22} The structure of the Euler equation and hence of the instability is modified very little by inclusion of nonideal effects,²³ kinetic effects,²⁴ or pressure anisotropy.²⁵

Fielding and Haas²⁵ have found that when particles are injected almost perpendicular to the magnetic field, but with enough parallel velocity that the beam is not trapped, a distribution function results that can sustain a pressure anisotropy in which the perpendicular pressure varies on a flux surface even in a large-aspect-ratio device. They obtained an Euler equation from a version of the guiding center plasma energy principle which neglects a usually small term known as the kinetic integral.²⁶ After expanding the equation in large aspect ratio, they found that the gradient of the perpendicular pressure along the poloidal angle on a flux surface can be stabilizing, if the pressure is largest at the inner edge of the torus. The resulting inward shift of the perpendicular pressure surfaces can be sustained because the injected fast particles move more slowly along the field lines on the inner edge of the torus than they do on the outer edge, where the curvature is destabilizing.

An instability related to the ballooning mode is the interchange or flute mode. The interchange stability criterion (also known as the Mercier criterion) for an anisotropic pressure, axisymmetric torus in generalized geometry was derived by Mercier and Cotsaftis²⁷ for a double adiabatic plasma and by Connor and Hastie²⁸ for a guiding center plasma.

In Sect. 2 we find that because distribution functions featuring an energy anisotropy in velocity space have a pressure moment $p_{\parallel}$ along the field lines that can be quite different from the component $p_{\perp}$ across them, the resulting pressure anisotropy can modify the MHD equilibrium significantly. In effect, the auxiliary heating methods can quite drastically distort the perpendicular pressure without seriously affecting the poloidal flux topology because the contribution of the $p_{\perp}$ gradients to the toroidal current density is small.

We also derive the tensor pressure equilibrium equation from the basic MHD equations, in which the pressure is a tensor with only diagonal elements. We obtain, as well, a force balance relation and expressions for the poloidal β and the current densities for an axisymmetric torus of arbitrary cross section and aspect ratio. The results obtained reduce to the comparable ideal MHD results presented by Callen and Dory when $p_{\parallel} = p_{\perp}$.²⁹

We concentrate here on the effects of a tensor pressure induced by neutral beams injected into an axisymmetric device of arbitrary cross section and aspect ratio. The distribution function is obtained from a solution of the Fokker-Planck equation with a source term of high-energy particles where the pitch angle scattering operator is ignored³⁰ and is valid in the limit that the beam particle energy is very large. The tensor pressure elements obtained represent limiting beam effects because any amount of pitch angle scattering would reduce the anisotropy. Fielding and Haas have used this particular distribution function to examine the ballooning mode stability of a device with large aspect ratio.²⁵ The mirror stability criterion precludes the treatment of trapped beam particles in the absence of pitch angle scattering.

We use a Buneman fast Poisson solver³¹ to solve the finite difference form of the equilibrium equation used to generate D-shaped tensor pressure equilibria. The flux surface topology does not change drastically with increased diamagnetism. We consider here broad and peaked pressure profiles with parallel and perpendicular injection. As is well known, in tensor pressure, the parallel and perpendicular pressure surfaces do not necessarily coincide with the flux surfaces. Nevertheless, with very peaked pressure profiles, the pressure contours nearly match the flux surfaces when $p_{\parallel} \neq p_{\perp}$. For broad pressure profile equilibria, however, the parallel pressure contours undergo a significant inward shift with respect to the flux surfaces, but only near the magnetic axis. This shift should disappear in the large-aspect-ratio limit.²⁵ For cases in which the pressure profiles are broad and the beams are injected at such angles that the perpendicular pressure is enhanced, the $p_{\perp}$ contours undergo a significant inward shift with respect to the flux surfaces. For cases in which the pressure profiles are broad and the beams are injected at angles more closely colinear with the magnetic field, the $p_{\perp}$ surfaces show virtually no shift with respect to the flux contours. One can anticipate an improvement in β limits set by ballooning modes when perpendicular injection is used, because the inward shift of the perpendicular pressure surfaces is equivalent to the shift of pressure from the destabilizing curvature region to the stabilizing curvature region.

Energy principles that have pressure anisotropies can be divided into two categories:

- 1) The guiding center plasma,³² Kruskal-Oberman,³³ collisionless,³⁴ or kinetic²⁸ energy principle and
- 2) The guiding center fluid,³² hydromagnetic,³³ Chew, Goldberger and Low,³⁵ or double adiabatic energy principle.

We shall mainly use guiding center plasma (GCP) and double adiabatic model (DA) to distinguish between the energy principles in these two categories.

In Sect. 3 we obtain a double adiabatic energy principle to examine plasma stability by replacing the kinetic integral (which reduces to the plasma compression term in ideal MHD) of the guiding center plasma energy principle with a double adiabatic contribution.³³ If the guiding center plasma energy principle predicts a plasma to be stable, this assures that the double adiabatic energy principle will also predict stability, provided the distribution function is monotonically decreasing in energy.

To obtain an Euler equation from the asymptotic expansion (in large toroidal mode number n) and subsequent minimization of the double adiabatic energy principle, we demand that the perturbation have no component along the field lines. We are fully justified in this procedure because we are interested in obtaining a necessary criterion for the stability of a guiding center plasma from this energy principle.³⁶ The imposition of a constraint on the eigenfunction in the minimization of the double adiabatic energy principle implies that the perturbation we construct is not the most unstable. For this reason, the derived stability criterion is necessary for the double adiabatic model and, consequently, for the guiding center plasma. The field bending term in the Euler equation is virtually unaltered from that of the ideal MHD. The driving term is modified by the variation of the pressures along the poloidal angle on a flux surface. In addition, we have a stabilizing contribution peculiar to the double adiabatic theory. If we neglect this contribution, the Euler equation is identical to that derived by Fielding and Haas,²⁵ which can be considered to be sufficient for the stability of a guiding center plasma. This is because the kinetic integral they neglected is positive definite whenever the distribution function is monotonically decreasing in energy. By solving the Euler equation with and without the double adiabatic contribution, we can bracket the marginal stability of the guiding center plasma without ever having to evaluate the complicated kinetic integral that was replaced by the double adiabatic term.

We use a shooting method to determine the stability of computed tensor pressure equilibria. We note at this point that the pressure-driven modes near the magnetic axis are referred to as interchanges and are uniform on a flux surface. The pressure-driven modes near the edge of the plasma are referred to as ballooning modes and display a strong variation on a flux surface, concentrating at the outer edge of a flux surface, where the curvature is most destabilizing. We first examine the stability of moderately broad pressure profile equilibria and find that the stability predictions are sensitive to the definition of β we use, because there is no unique definition when $p_{\parallel} \neq p_{\perp}$. We next apply the ballooning mode equation to a sequence of slightly peaked pressure profile equilibria. The difference between the necessary and the sufficient stability criteria is large for interchange unstable plasmas, but coalesces for ballooning unstable plasmas. Perpendicular beam injection is preferable to parallel beam injection for ballooning mode stability, but competing effects preclude a similar conclusion for interchange stability. Finally, we present a survey of necessary and sufficient stability results for very broad tensor and scalar pressure profile equilibria. We find that these equilibria are less susceptible to flute modes than are the more peaked pressure profile equilibria, but the stability limits tend to coalesce when ballooning modes dominate. The stability predictions of tensor pressure equilibria induced by perpendicular beam injection are enhanced beyond those of scalar pressure equilibria with comparable pressure profiles and those of tensor pressure equilibria induced by parallel beam injection. Perpendicular injection induces an inward shift of the $p_{\perp}$ contours away from the destabilizing curvature region, which improves the stability conditions.

In Sect. 4 we generate tensor and scalar pressure equilibria to model the ISX-B tokamak and find that the theoretical predictions obtained from the ballooning mode equation indicate that the device should be unstable to pressure-driven modes in the central core of the plasma because of the pressure profile peakedness induced by the neutral beams.

In Sect. 5 we present the summary and the conclusions.

All physical expressions in this work are normalized to dimensionless form. The magnetic fields are thus normalized to the vacuum field $B_{\phi 0}$ at the geometric center of the discharge chamber; the pressures are normalized to twice the vacuum magnetic energy density at the discharge chamber midpoint; the currents are normalized to $j_0 = B_{\phi 0} / (\mu_0 a)$, where $\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of free space; the distances are normalized to a , the half-width of the discharge chamber; and the densities are normalized to the central density ρ_0 .

2. EQUILIBRIUM

When the auxiliary heating power, of neutral beams for example, exceeds the ohmic heating power by an order of magnitude or more, the ion distribution function can display a significant velocity-space anisotropy that causes the pressure moment $p_{\parallel}$ along the field lines to be different from $p_{\perp}$, the component across the field lines.

In this section we derive a force balance relation and expressions for the poloidal β in an axisymmetric torus with tensor pressure, arbitrary aspect ratio, and arbitrary cross section, following the ideal MHD calculations of Callen and Dory.²⁹ We concentrate here on the effect of a tensor pressure induced by neutral beams injected into a noncircular tokamak of low aspect ratio. The parallel and perpendicular pressure components vary on a flux surface and thus can be expressed as functional forms of the poloidal flux function ψ and the magnitude of the magnetic field B in an axisymmetric toroidal device. We obtain $p_{\parallel}$ and $p_{\perp}$ by evaluating the moments of a distribution function that is the solution of the Fokker-Planck equation with a source term of energetic particles in which the pitch angle scattering operator is ignored.³⁰ Consequently, we need not define how the pressures depend on B . Fielding and Haas have used this particular distribution function to examine the ballooning mode stability of tokamaks of large aspect ratio.^{25,37} The mirror stability criterion precludes the consideration of trapped beam particles if they do not scatter in pitch angle. Although numerically computed tensor pressure equilibria have been previously reported,³⁸ the dependences of the pressure on B were chosen rather arbitrarily.

We compute beam-induced, low-aspect-ratio, D-shaped tensor pressure equilibria on a rectangular grid. We find that the pressure contours and the flux surfaces nearly coincide if the pressure profiles are peaked. The perpendicular pressure surfaces are noticeably distorted and displaced inward with respect to every flux surface for

broad pressure profile equilibria with injection at such angles that $p_{\perp}$ is significantly enhanced by the beam. For injection at angles more closely colinear to the magnetic field, the beam contribution to $p_{\perp}$ is negligible, and the perpendicular pressure contours match the flux surfaces. The $p_{\parallel}$ contours of beam-induced tensor pressure equilibria with broad pressure profiles are shifted quite noticeably inward with respect to the flux surfaces, but only in the neighborhood of the magnetic axis.

2.1. THE TENSOR PRESSURE EQUILIBRIUM EQUATION

The macroscopic equilibrium equations are given by

$$\vec{\nabla} \cdot \vec{P} = \vec{j} \times \vec{B}, \quad (2.1)$$

$$\vec{\nabla} \times \vec{B} = \vec{j}, \quad (2.2)$$

$$\vec{\nabla} \cdot \vec{B} = 0. \quad (2.3)$$

The pressure tensor $\vec{P} = p_{\perp} \vec{I} + (p_{\parallel} - p_{\perp}) \hat{b} \hat{b}$, where $\hat{b}$ is the unit vector along a magnetic field line. With this expression we can write Eq. (2.1) as

$$\begin{aligned} \vec{\nabla} \cdot \vec{P} &= \vec{\nabla} p_{\perp} + \hat{b} (\hat{b} \cdot \vec{\nabla}) (p_{\parallel} - p_{\perp}) + (p_{\parallel} - p_{\perp}) \left[\vec{\kappa} - \frac{\hat{b} (\hat{b} \cdot \vec{\nabla}) B^2}{2B^2} \right] \\ &= \vec{j} \times \vec{B}, \end{aligned} \quad (2.4)$$

where $\vec{\kappa}$ is the curvature of the field lines, given by

$$\vec{\kappa} \equiv (\hat{b} \cdot \vec{\nabla})\hat{b} = \frac{\vec{\nabla}B^2 - \hat{b}(\hat{b} \cdot \vec{\nabla})B^2}{2B^2} + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{B^2} . \quad (2.5)$$

The component of Eq. (2.4) along the field line yields

$$(\hat{b} \cdot \vec{\nabla})p_{\parallel} = \frac{p_{\parallel} - p_{\perp}}{2B^2} (\hat{b} \cdot \vec{\nabla})B^2 . \quad (2.6)$$

Combining the last three equations we can get

$$\sigma(\vec{\nabla} \times \vec{B}) \times \vec{B} = \vec{\nabla}_{\perp} p_{\perp} + \frac{p_{\parallel} - p_{\perp}}{2B^2} \vec{\nabla}_{\perp} B^2 , \quad (2.7)$$

where the operator $\vec{\nabla}_{\perp} \equiv \vec{\nabla} - \hat{b}(\hat{b} \cdot \vec{\nabla})$ and $\sigma \equiv 1 - \frac{p_{\parallel} - p_{\perp}}{B^2}$. A rearrangement of terms results in

$$[\vec{\nabla} \times (\sigma \vec{B})] \times \vec{B} = \vec{\nabla}_{\perp} p_{\parallel} - \frac{p_{\parallel} - p_{\perp}}{2B^2} \vec{\nabla}_{\perp} B^2 . \quad (2.8)$$

In a tokamak the magnetic field is

$$\vec{B} = B_{\phi} \hat{e}_{\phi} + B_{\chi} \hat{e}_{\chi} = I \vec{\nabla} \phi + \vec{\nabla} \phi \times \vec{\nabla} \psi , \quad (2.9)$$

where $I \equiv RB_{\phi}$, ϕ is the toroidal angle, and $2\pi\psi$ is the poloidal flux function. In any orthogonal (ϕ, ψ, χ) coordinate system, the gradient operator is

$$\vec{\nabla} = \left(\frac{\hat{e}_\phi}{R}\right) \frac{\partial}{\partial \phi} + \hat{e}_\psi \frac{RB}{\chi} \frac{\partial}{\partial \psi} + \left(\frac{\hat{e}_\chi}{JB_\chi}\right) \frac{\partial}{\partial \chi}, \quad (2.10)$$

where J is the Jacobian.

If we expand Eq. (2.8) using this flux coordinate system and noting the requirement of axisymmetry, we get two equations,²⁷

$$\frac{\partial}{\partial \chi}(\sigma I) = -\frac{I}{B^2} \left[\frac{\partial p_\parallel}{\partial \chi} - \frac{(p_\parallel - p_\perp)}{2B^2} \frac{\partial B^2}{\partial \chi} \right] \quad (2.11)$$

and

$$\frac{1}{J} \frac{\partial}{\partial \psi}(\sigma JB_\chi^2) = -\frac{I}{R^2} \frac{\partial}{\partial \psi}(\sigma I) - \frac{\partial p_\parallel}{\partial \psi} + \frac{(p_\parallel - p_\perp)}{2B^2} \frac{\partial B^2}{\partial \psi} \quad (2.12)$$

If we expand Eq. (2.6), we obtain

$$\frac{\partial p_\parallel}{\partial \chi} = \frac{(p_\parallel - p_\perp)}{2B^2} \frac{\partial B^2}{\partial \chi}, \quad (2.13)$$

which implies that σI is a function only of ψ . After rearranging some terms and writing the left-hand side of Eq. (2.12), the basic equilibrium equation can be written as

$$\Delta^* \psi + \frac{1}{\sigma} \vec{\nabla} \psi \cdot \vec{\nabla} \sigma = -\frac{I}{\sigma} \frac{\partial}{\partial \psi}(\sigma I) - \frac{R^2}{\sigma} \left[\frac{\partial p_\parallel}{\partial \psi} - \frac{(p_\parallel - p_\perp)}{2B^2} \frac{\partial B^2}{\partial \psi} \right], \quad (2.14)$$

where the operator $\Delta^* \equiv R^2 \vec{\nabla} \cdot (R^{-2} \vec{\nabla})$. If we take the pressure to be functional forms of ψ and $B \equiv |\vec{B}|$, then

$$\frac{\partial p_{\parallel}}{\partial \psi} \Big|_{\chi} = \frac{\partial p_{\parallel}}{\partial \psi} \Big|_B + \frac{\partial p_{\parallel}}{\partial B} \Big|_{\psi} \frac{\partial B}{\partial \psi} \Big|_{\chi} . \quad (2.15)$$

With $p_{\parallel} = p_{\parallel}(\psi, B)$, we get from Eq. (2.6) that $\frac{\partial p_{\parallel}}{\partial B} = \frac{p_{\parallel} - p_{\perp}}{B}$; therefore,

$$\frac{\partial p_{\parallel}}{\partial \psi} \Big|_B = \frac{\partial p_{\parallel}}{\partial \psi} \Big|_{\chi} - \frac{(p_{\parallel} - p_{\perp})}{B} \frac{\partial B}{\partial \psi} \Big|_{\chi} . \quad (2.16)$$

The analog of the Grad-Shafranov equation with tensor pressure is

$$\Delta^* \psi + \frac{1}{\sigma} \vec{\nabla} \psi \cdot \vec{\nabla} \sigma = - \frac{I}{\sigma} \frac{\partial}{\partial \psi} (\sigma I) - \frac{R^2}{\sigma} \frac{\partial p_{\parallel}}{\partial \psi} \Big|_B . \quad (2.17)$$

This form of the equilibrium equation was shown by Grad.^{39,40} The parallel pressure must be treated as a function of ψ and B in Eq. (2.17). The perpendicular pressure surfaces can be quite distorted without seriously affecting the equilibrium because $p_{\perp}$ contributes indirectly to Eq. (2.17) through the small term $\vec{\nabla} \psi \cdot \vec{\nabla} \sigma$.

2.2 FORCE BALANCE RELATION AND POLOIDAL BETA

In a tensor pressure plasma, it is convenient to define β as

$$\beta \equiv \frac{2}{3} \langle p_{\parallel} + 2p_{\perp} \rangle , \quad (2.18)$$

where the angle brackets indicate a volume average. To obtain an expression for the poloidal β , we need to rearrange the terms in Eq. (2.12) to get

$$\frac{1}{J} \frac{\partial}{\partial \psi} (\sigma J B_{\chi}^2) = - \frac{I}{R^2} \frac{\partial}{\partial \psi} (\sigma I) - \frac{1}{4} \frac{\partial}{\partial \psi} (p_{\parallel} + 2p_{\perp}) + B^2 \frac{\partial \sigma}{\partial \psi} - \frac{1}{4} \frac{\partial p_{\parallel}}{\partial \psi} .$$

Then using Eq. (2.12) again to replace the last term on the right-hand side, we obtain

$$\frac{1}{J} \frac{\partial}{\partial \psi} (\sigma J B_{\chi}^2) = - \frac{I}{R^2} \frac{\partial}{\partial \psi} (\sigma I) - \frac{1}{3} \frac{\partial}{\partial \psi} (p_{\parallel} + 2p_{\perp}) + \frac{4}{3} B^2 \frac{\partial \sigma}{\partial \psi} - \frac{(p_{\parallel} - p_{\perp})}{6B^2} \frac{\partial B^2}{\partial \psi} .$$

We now integrate this equation over ϕ and χ on a flux surface to get

$$\begin{aligned} \frac{1}{V'(\psi)} \frac{\partial}{\partial \psi} [V'(\psi) \langle \sigma B_{\chi}^2 \rangle_{\psi}] &= - \frac{1}{3} \langle \frac{\partial}{\partial \psi} (p_{\parallel} + 2p_{\perp}) \rangle_{\psi} - \langle \frac{I}{R^2} \frac{\partial}{\partial \psi} (\sigma I) \rangle_{\psi} + \frac{4}{3} \langle R^2 \frac{\partial \sigma}{\partial \psi} \rangle_{\psi} \\ &\quad - \langle \frac{p_{\parallel} - p_{\perp}}{6B^2} \frac{\partial B^2}{\partial \psi} \rangle_{\psi} , \end{aligned} \quad (2.19)$$

where

$$\langle X \rangle_{\psi} \equiv \frac{2\pi}{V'(\psi)} \oint \frac{d\ell}{B_{\chi}} X ,$$

(2.20)

$$V'(\psi) \equiv \frac{dV}{d\psi} \equiv 2\pi \oint \frac{d\ell}{B_{\chi}} ,$$

and $d\ell = JB_X d\chi$. To obtain a force balance relation we follow very closely the derivation of Callen and Dory²⁹ for a scalar pressure plasma. The next step is to integrate Eq. (2.19) from ψ to ψ_b , where ψ_b is the value of the flux at the boundary (or at any other arbitrarily chosen flux surface). We then multiply the equation with $V'(\psi)d\psi$ and integrate it from ψ_a , the flux at the magnetic axis, to ψ_b to get the force balance equation. The detailed derivation of the force balance relation is shown in Appendix A. It can be written as

$$\frac{1}{3} [P_{\parallel}(\psi) - P_{\parallel}(\psi_b) + 2P_{\perp}(\psi) - 2P_{\perp}(\psi_b)] = \frac{1}{2} \langle \sigma B_X^2 \rangle_{\psi_b} + \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V(\psi_b)} \left(\langle \frac{1}{\sigma R^2} \rangle_{\psi} \frac{\partial}{\partial \psi} (\sigma I)^2 + \right. \\ \left. \langle \sigma B_X^2 \rangle_{\psi} \frac{\partial}{\partial \psi} \left\{ \ln \left[\frac{V'^2(\psi) \langle \sigma B_X^2 \rangle_{\psi}}{V(\psi)} \right] \right\} - \frac{8}{3} \langle B^2 \frac{\partial \sigma}{\partial \psi} \rangle_{\psi} + \frac{1}{3} \langle \frac{P_{\parallel} - P_{\perp}}{B^2} \frac{\partial B^2}{\partial \psi} \rangle_{\psi} \right), \quad (2.21)$$

where we have defined

$$[P_{\parallel}(\psi) - P_{\parallel}(\psi_b) + 2P_{\perp}(\psi) - 2P_{\perp}(\psi_b)] = - \frac{\int_{\psi_a}^{\psi_b} d\psi V'(\psi) \int_{\psi}^{\psi_b} d\psi' \langle \frac{\partial}{\partial u} (p_{\parallel} + 2p_{\perp}) \rangle_u}{V(\psi_b)}. \quad (2.22)$$

An expression for the poloidal beta, β_p , can be obtained from the force balance relation if we define

$$\beta_p \equiv \frac{2}{3 \langle B_X^2 \rangle_{\psi_b}} [P_{\parallel}(\psi) - P_{\parallel}(\psi_b) + 2P_{\perp}(\psi) - 2P_{\perp}(\psi_b)]. \quad (2.23)$$

Then

$$\beta_p \equiv 1 - \frac{\frac{\langle p_{\parallel} - p_{\perp} B^2 \rangle_{\chi} \psi_b}{B^2}}{\langle B_{\chi}^2 \rangle_{\psi_b}} + \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V(\psi_b) \langle B_{\chi}^2 \rangle_{\psi_b}} \left(\frac{1}{\sigma R^2} \psi \frac{\partial}{\partial \psi} (\sigma I)^2 + \langle \sigma B_{\chi}^2 \rangle_{\psi} \frac{\partial}{\partial \psi} \left\{ \ln \left[\frac{V'^2(\psi) \langle \sigma B_{\chi}^2 \rangle_{\psi}}{V(\psi)} \right] \right\} \right. \\ \left. - \frac{8}{3} \langle B^2 \frac{\partial \sigma}{\partial \psi} \rangle_{\psi} + \frac{1}{3} \frac{\langle p_{\parallel} - p_{\perp} \frac{\partial B^2}{\partial \psi} \rangle_{\psi}}{B^2} \right) ; \quad (2.24)$$

when evaluated at the plasma boundary, the second term on the right-hand side must vanish. The last two terms under the integral sign are absent in the scalar pressure case. In the limit $p_{\parallel} = p_{\perp}$, the force balance relation and consequently β_p are equivalent to those derived by Callen and Dory. In a tensor pressure plasma, the pressures are functions of the flux and the magnitude of the magnetic field, so a strict separation of pressure and magnetic contributions to the force balance relation is no longer possible. The pressures can be considered to be balanced not only by the fields, but also by the degree of anisotropy. The diamagnetic or paramagnetic character of the plasma is still determined mainly by the sign of $\frac{\partial}{\partial \psi} (\sigma I)^2$, although for very anisotropic pressure plasmas, the contribution of $\langle B^2 \frac{\partial \sigma}{\partial \psi} \rangle_{\psi}$ could conceivably be significant. A very diamagnetic plasma must have $\frac{\partial}{\partial \psi} (\sigma I)^2$ large and positive.

The expression for β_p [Eq. (2.24)] is cumbersome to evaluate. A much simpler and more useful expression is given by

$$\beta_j \equiv \frac{2 \langle p_{\parallel} + 2p_{\perp} \rangle}{3 (I_{\phi} / \phi d\ell)^2} , \quad (2.25)$$

where the angle brackets represent a volume average, I_ϕ is the toroidal current, and $\oint dl$ is an approximation of the poloidal circumference given by

$$\oint dl \equiv \left(\iint dS_\phi \right) \frac{(a+b)}{ab},$$

where $\iint dS_\phi$ is the area of a cross-sectional cut of the largest flux surface contained within the box of half-width a and half-height b .

We derive expressions for the current densities in Appendix B when $p_\parallel \neq p_\perp$ in the MHD equilibrium.

2.3. PRESSURE MOMENTS FROM A BEAM DISTRIBUTION FUNCTION

The total pressures are evaluated from the moments of the distribution functions of a background Maxwellian and a beam component. The beam distribution function is the solution of a Fokker-Planck equation with a fast-particle source term in which the pitch angle scattering operator has been ignored.³⁰ This solution is given by

$$f_b = \frac{1}{2\pi B_o(\psi)} \frac{I_b \tau_s S(\psi)}{E^{3/2} + E_c^{3/2}} \left(\frac{m_b}{2E_o} \right)^{3/2} \delta \left[\frac{\mu}{E} - \mu_e(\psi) \right], \quad (2.26)$$

where I_b is the beam current per unit charge, τ_s is the slowing down time, $S(\psi)$ is the number of injected particles per unit volume within the flux surface ψ , and $B_o(\psi)$ is the normalized magnetic field at the outer midplane point of the flux contour ψ . The magnetic moment is μ , and its value at the outer midplane point of the flux surface ψ is $\mu_e(\psi)$. The magnetic moments are normalized to the ratio $E_o/B_{\phi o}$, the energies are normalized to the injected beam particle energy E_o , and $B_{\phi o}$ is the vacuum field at the geometric center of the discharge chamber. The critical energy $E_c = \frac{1}{2} \frac{m_b}{m_i} \left(\frac{3\sqrt{\pi}}{4} \right)^{2/3} \left(\frac{m_i}{m_e} \right)^{1/3} \left(\frac{T_e}{E_o} \right)$, where m_b ,

m_i , and m_e are the beam particle, ion, and electron masses, respectively, and T_e is the electron temperature. In this model, all the beam particles born with a given pitch angle on a flux surface conserve pitch angle as they slow down because of the neglect of the pitch angle scattering operator when determining f_b . This distribution function is valid in the limit $E_c \ll E_0$, because then the beam particles slow down on the electrons, and only negligibly pitch angle scatter off the thermal ions. If we include pitch angle scattering in the Fokker-Planck equation, the δ function in f_b is replaced by a sum of Legendre polynomials in the midplane pitch angle,⁴¹ and as E_c approaches E_0 (i.e., unity), the beam distribution function becomes more and more isotropic in velocity space, thus allowing force balance to be well approximated by the scalar pressure equilibrium equation.

The pressure moments of f_b given by Eq. (2.26) can be evaluated analytically. The perpendicular beam pressure component is

$$p_{\perp b} = 2\pi s \int_b \left(\frac{E_0}{m_b}\right)^{3/2} \int_0^1 dE \int_0^{E/B} d\mu \frac{B}{\sqrt{2(E-\mu B)}} \mu B f_b, \quad (2.27)$$

and the beam parallel pressure is given by

$$p_{\parallel b} = 2\pi s \int_b \left(\frac{E_0}{m_b}\right)^{3/2} \int_0^1 dE \int_0^{E/B} d\mu \frac{B}{\sqrt{2(E-\mu B)}} 2(E-\mu B) f_b, \quad (2.28)$$

where the factor $s = 2\mu_0 E_0 / B_{\phi 0}^2$. Using Eq. (2.27), we obtain the total perpendicular pressure

$$p_{\perp}(\psi, B) = p_0(\psi) + \frac{0.5 P(\psi) \mu_e(\psi) B^2}{\sqrt{1 - \mu_e(\psi) B}} H[1 - \mu_e(\psi) B], \quad (2.29)$$

and the beam parallel pressure is given by

$$p_{\parallel}(\psi, B) = p_0(\psi) + P(\psi) B \sqrt{1 - \mu_e(\psi)B} H[1 - \mu_e(\psi)B] \quad , \quad (2.30)$$

where $p_0(\psi)$ is the isotropic background pressure profile, the beam deposition profile $P(\psi)$ is given by

$$P(\psi) = \frac{s I_b \tau_s S(\psi)}{2 B_0(\psi)} \int_0^1 dE \frac{E^{3/2}}{E^{3/2} + E_c^{3/2}} \quad ,$$

and $H[1 - \mu_e(\psi)B]$ is a step function. We have assumed that the beam contains only one species of particles.

We note that once $p_{\parallel}(\psi, B)$ is specified, $p_{\perp}(\psi, B)$ is automatically determined by the general relation

$$p_{\perp}(\psi, B) = p_{\parallel}(\psi, B) - B \frac{\partial p_{\parallel}}{\partial B} \quad . \quad (2.31)$$

In a high-beta, large-aspect-ratio device, $1 - (\mu_e B) \sim 1$ for preferentially parallel injection. Then, to get $p_{\parallel b}$ on the order of the inverse aspect ratio, we need $P(\psi)$, which is a measure of the source strength, to be $O(\epsilon)$. With injection that is mainly perpendicular but has enough parallel velocity to avoid trapping the beam, we could have $1 - (\mu_e B)$ on the order of the inverse aspect ratio ϵ . In this case, for $p_{\perp b}$ to be $O(\epsilon)$, i.e., the same as the background plasma pressure, we only need $P(\psi)$ on the order of $\epsilon^{3/2}$. Therefore, a significant anisotropy can be most effectively sustained with almost perpendicular injection. We also notice that for perpendicular injection in particular, the beam pressures can be made larger at the inner edge of the torus where the magnetic field is highest, and, consequently, away from the destabilizing curvature region where

ballooning instabilities are driven. From a particle point of view, this is because the hot ions move more slowly along the field lines on the inner edge of the torus and more rapidly on the outer edge. In addition, the slower a particle moves along a field line, the larger the fraction of time it spends in the region where the curvature is stabilizing.³⁷ The limiting case corresponds to a particle that is barely trapped; it spends virtually all its time at the inner midplane point of the flux surface on which it resides.

Enhancement of the pressures in the high magnetic field region results from the discrete nature of the beam particle pitch angles. In the limit that the distribution of injected ions is isotropic, the enhanced pressure produced by the particles at the inner edge of the torus is balanced by a population of trapped beam particles that never contributes to the pressure in the high field region.

The tensor pressure components calculated from f_b [Eq. (2.26)] represent limiting beam-induced anisotropies, because any amount of pitch angle scattering would act to reduce them by making the beam distribution function more isotropic in velocity space.

2.4. MIRROR AND FIREHOSE STABILITY

The condition for mirror stability is given by³⁹

$$\tau \equiv 1 + \frac{1}{B} \frac{\partial p_{\perp}}{\partial B} > 0 \quad . \quad (2.32)$$

Using Eq. (2.29) for the perpendicular pressure, we get

$$1 + \frac{P(\psi)\mu_e(\psi)}{\sqrt{1-\mu_e(\psi)B}} \left\{ \left[\frac{1-0.75\mu_e(\psi)B}{1-\mu_e(\psi)B} \right] H[1-\mu_e(\psi)B] - 0.5\mu_e(\psi)B \delta[1-\mu_e(\psi)B] \right\} > 0.$$

(2.33)

We can see from Eq. (2.33) that the equilibrium is mirror unstable wherever $\mu_e(\psi)B = 1$ anywhere in the plasma (which corresponds to the banana-tip locations of the beam particle orbits on the flux surface ψ), whereupon the equilibrium equation ceases to be elliptic.^{39,40} The discrete nature of the beam particle pitch angles implies that all those particles with the same pitch angle, and thus with common turning points where $\mu_e(\psi)B = 1$, cause the perpendicular beam pressure to become singular [as is seen in Eq. (2.29)] and discontinuous across those points. Consequently, when the pitch angle scattering operator is neglected in determining the beam distribution function f_b , the mirror stability criterion is satisfied only if all beam particles are circulating. Therefore, to generate tensor pressure equilibria, the $\mu_e(\psi)$ profile must be chosen to exclude trapped beam particles.

The condition for firehose stability is given by^{27,39,40}

$$\sigma \equiv 1 - \frac{1}{B} \frac{\partial p_{\parallel}}{\partial B} > 0 \quad . \quad (2.34)$$

Using Eq. (2.30), this gives the condition

$$1 - \frac{P(\psi)}{B} \left[\frac{1 - 1.5\mu_e(\psi)B}{\sqrt{1 - \mu_e(\psi)B}} \right] H[1 - \mu_e(\psi)B] + P(\psi)\mu_e(\psi) \sqrt{1 - \mu_e(\psi)B} \delta[1 - \mu_e(\psi)B] > 0, \quad (2.35)$$

which is easily satisfied unless the beam power is exceedingly high.

2.5. NUMERICAL METHOD

We use a Buneman fast Poisson solver³¹ to invert the Δ^* operator of the tensor pressure equilibrium equation [Eq. (2.17)] on a rectangular grid to generate equilibria with $p_{\parallel} \neq p_{\perp}$. A careful

selection of the flux on the rectangular box as a boundary condition can be used to shape the plasma when solving Eq. (2.17). This is given by

$$\psi = \psi_w \left[\underbrace{\left(\frac{x}{a}\right)^2 + \left(\frac{z}{b}\right)^2}_{\text{ellipticity}} + \underbrace{t_g \left(\frac{x}{a}\right) \left(\frac{z}{b}\right)}_{\text{triangularity}} - \underbrace{q_g \left(\frac{x}{a}\right)^2 \left(\frac{z}{b}\right)^2}_{\text{quadrangularity}} \right], \quad (2.36)$$

where a is the half-width of the box, b is the half-height of the box, t_g produces triangular distortions of the plasma, and q_g is used to make quadrangular deformations of the plasma.⁴² We assume that there is some conceivable distribution of poloidal field coils and currents in those coils that, when combined with the plasma current, will make ψ of the form described by Eq. (2.36) on the rectangular boundary. Vertical fields implicit in the triangularity factor t_g prevent large shifts of the magnetic axis and shaping of the inner flux surfaces. We produce D-shaped plasmas by choosing $t_g > 0$ and $q_g \geq 0$ and inverse D-shaped plasmas when $t_g < 0$ and $q_g \geq 0$. The flux at the inner wall is a constant when $t_g + q_g = 1$. The plasma shaping is also very sensitive to the coefficient ψ_w , as is illustrated in Fig. 2.1. As ψ_w is increased, separatrices move into the rectangular chamber at the D-tips and erode the size of the plasma.

To compute tensor pressure equilibria we must provide the profiles for the pressures and the function (σI) . With initial guesses for ψ and σ , we calculate $B^2 = [(\sigma I)^2/\sigma^2 + (\nabla\psi)^2]/R^2$, evaluate the pressures and (σI) , and then obtain ψ from the Buneman solver. This procedure is repeated until we generate a converged solution. Two convergence criteria must be satisfied simultaneously. The maximum difference between (1) $\Delta\psi$ and the source term used to obtain that value of ψ and (2) the σ arrays of two consecutive iterations must not exceed small chosen tolerance levels.

We have found that we can compute tensor pressure equilibria on a CDC-7600 in about 9 s using a 33×33 grid and 30 s using a 65×65 grid, with absolute and relative error limits of one part in 10^5 .

Fig. 2.1. Plasma shaping with elongation $b/a = 1.6$, triangularity $d = 0.75$, and quadrangularity $q = 0.55$.

2.6 SAMPLE EQUILIBRIA

To generate beam-induced tensor pressure equilibria we need to specify the isotropic background plasma pressure $p_0(\psi)$, the beam deposition profile $P(\psi)$, the function (σI) [defined by Eq. (2.11)], and the magnetic moment profile $\mu_e(\psi)$. The magnetic moment profile $\mu_e(\psi)$ must be determined consistent with the geometry of a neutral-beam, low-aspect-ratio system. We will consider only neutral beams injected at the midplane of the device. The $\mu_e(\psi)$ profile is then chosen as

$$\mu_e(\psi) = \mu_e(\psi_b) \left[1 - \frac{\epsilon}{2} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right) \frac{1}{\sin^2 \theta_b} \right], \quad (2.37)$$

where ψ_a and ψ_b are the fluxes at the magnetic axis and the plasma boundary, respectively, ϵ is the inverse aspect ratio, and θ_b is the angle between the beam and the magnetic field at the outer midplane point of the boundary flux surface. This choice for Eq. (2.37) is compatible with the following properties of the beam geometry. As the trajectory of the beam penetrates the flux surfaces, the angle between $\vec{B}$ and the beam decreases because of the toroidicity, and this should act to enhance $p_{\parallel}$ as the high-energy neutrals approach the magnetic axis. The last term in Eq. (2.37) models this geometric effect. The neutral beam is more nearly tangent to the inner flux surfaces. This should be less significant for larger-aspect-ratio devices and for more perpendicular angles of injection (since the high-energy neutrals must travel a much shorter toroidal distance before reaching the magnetic axis). For simplicity, we have assumed that the fast particles are ionized only at the outer (and not at all at the inner) midplane point of each flux surface. We note that $\mu_e(\psi_b)B_b = \sin^2 \theta_b$, where B_b is the normalized midplane magnetic field at the outer boundary; since $(1 + \epsilon)B_b = 1$, then $\mu_e(\psi_b) = (1 + \epsilon)\sin^2 \theta_b$. This choice for $\mu_e(\psi)$ tends to peak the $p_{\parallel}$ profile and broaden the $p_{\perp}$ profile. An upper bound for $\mu_e(\psi)$ is obtained from the condition for mirror stability, i.e., $\mu_e(\psi)B_{\max} < 1$ (no trapped beam particles).

Rather than present a survey of tensor pressure equilibria with different values of aspect ratio and noncircularity, in this section we will concentrate on D-shaped equilibria with elongation 1.6:1 in $\epsilon = 1/3$ devices.

Equilibria obtained here from more nearly perpendicular injection have $\theta_b = 43.57^\circ$, even though most of the beam pressure is deposited in the parallel component. A further increase in θ_b would produce trapped beam particles that would cause $p_\perp$ to become singular and the equilibrium to become mirror unstable owing to the neglect of pitch angle scattering. Equilibria obtained here from nearly parallel injection have $\theta_b = 25^\circ$.

2.6.1 Equilibria with Different β_j

In this subsection we present D-shaped tensor pressure equilibria computed on a 33×33 grid. The parallel pressure profile is given by

$$p_\parallel(\psi, B) = S_c \left[\left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^2 - \frac{1}{2} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^3 \right] [1 + 2 B \sqrt{1 - \mu_e(\psi)B}]. \quad (2.38)$$

The perpendicular pressure profile is obtained using Eq. (2.31), and the function (σI) is given by

$$(\sigma I)^2 = 1 + 2 S_c \left[d_p \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^2 + q_x \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^3 + \frac{2}{5} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^5 \right], \quad (2.39)$$

where the coefficient S_c is used to raise or lower β , the coefficient d_p is used to alter β_j , and q_x is adjusted to give the desired safety factor q value at the magnetic axis. The last term in Eq. (2.39) prevents the toroidal current from becoming hollow because of the choice made for the pressure profiles and causes the plasma to become locally paramagnetic near the magnetic axis. We find that it is numerically difficult to obtain well-converged equilibria that satisfy force balance with hollow current profiles.

We show the flux surfaces and contours and toroidal current density contours of tensor pressure equilibria with high and low values of β_j induced by parallel injection ($\theta_b = 25^\circ$) in Fig. 2.2 and by perpendicular injection ($\theta_b = 43.57^\circ$) in Fig. 2.3. We notice that the effects on the flux and on the toroidal current density of raising β_j from 1.1 to 2.2 are very similar if we compare the parallel injection to the perpendicular injection equilibria. The nesting of the flux surfaces closer to the outer edge of the plasma is only marginally enhanced with increased β_j , because the implicit vertical fields imposed by the triangular deformation of the plasma prevent a large shift of the magnetic axis. As the boundary conditions are rescaled in an effort to maintain the D-shapedness of the plasma with increased β_j , we notice that the flux contours lose some elongation as separatrices move into the chamber at the D-tips. The surfaces of the toroidal current density for $\beta_j = 1.1$ are indicative of broad j_ϕ profiles. For the more diamagnetic plasmas ($\beta_j = 2.2$), the j_ϕ profiles become much more peaked at the outer edge of the plasma, and small regions of reversed current appear at the inner edge. Nevertheless, for both angles of injection, the plasma remains locally paramagnetic at the magnetic axis at high β_j because of the chosen behavior of the pressure and (σI) profiles in that area, which are incompatible with the more typically crescent-shaped j_ϕ contours of diamagnetic plasmas.

2.6.2. Profile Effects

In this subsection, we compare the effects of parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection on tensor pressure plasma equilibria when the pressures are very peaked and very broad. The peaked pressure profile equilibria are given by

$$p_{\parallel}(\psi, B) = S_c \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^3 [1 + \alpha B \sqrt{1 - \mu_e(\psi)B}] \quad (2.40)$$

and

$\beta = 1.01\%$
 $\beta_b/\beta = 0.40$
 $\beta_j = 1.12$
 $\theta_b = 25^\circ$

$\beta = 1.04\%$
 $\beta_b/\beta = 0.39$
 $\beta_j = 2.21$
 $\theta_b = 25^\circ$

FLUX CONTOURS

TOROIDAL CURRENT
 DENSITY CONTOURS

Fig. 2.2. Flux contours and toroidal current density contours with parallel beam injection ($\theta_b = 25^\circ$, where θ_b is the angle between the magnetic field and the direction of the beam at the outer midplane point of the plasma) for $\beta_j = 1.1$ and $\beta_j = 2.2$.

$\beta = 1.04\%$
 $\beta_b/\beta = 0.48$
 $\beta_j = 1.11$
 $\theta_b = 43.57^\circ$

$\beta = 1.26\%$
 $\beta_b/\beta = 0.46$
 $\beta_j = 2.24$
 $\theta_b = 43.57^\circ$

FLUX CONTOURS

TOROIDAL CURRENT
DENSITY CONTOURS

Fig. 2.3. Flux contours and toroidal current density contours with perpendicular beam injection ($\theta_b = 43.57^\circ$, where θ_b is the angle between the magnetic field and the direction of the beam at the outer midplane point of the plasma) for $\beta_j = 1.1$ and $\beta_j = 2.2$.

$$(\sigma I)^2 = 1 + 2 S_c \left[d_p \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^3 + q_x \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^5 \right] , \quad (2.41)$$

while the broad pressure profile equilibria are determined by

$$p_{\parallel}(\psi, B) = S_c \left[\left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^2 - \frac{2}{2.1} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^{2.1} \right] [1 + \alpha B \sqrt{1 - \mu_e(\psi)B}] \quad (2.42)$$

and

$$(\sigma I)^2 = 1 + 2S_c \left\{ d_p \left[\left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^2 - \frac{2}{2.1} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^{2.1} \right] + q_x \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^5 \right\}. \quad (2.43)$$

The $p_{\perp}(\psi, B)$ profiles are determined from Eq. (2.31). The coefficients S_c , d_p , q_x , and α are simultaneously adjusted to make $\beta = 4.70\%$, $\beta_j = 1.50$, the safety factor $q = 1$ on axis, and $\beta_b/\beta = 0.474$, where β_b is the beam contribution to β , for each equilibrium we generate.

In Fig. 2.4 we display the parallel pressure contours overlaid on flux surfaces for equilibria with broad and peaked tensor pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection. We notice that the peaked $p_{\parallel}$ contours are virtually flux surfaces, whereas the broad $p_{\parallel}$ contours are noticeably displaced inward with respect to the flux surfaces near the magnetic axis. The perpendicular pressure contours are overlaid on flux surfaces for equilibria with broad and peaked tensor pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection in Fig. 2.5. Here, the peaked $p_{\perp}$ contours for both angles of injection are virtually flux surfaces, and the broad $p_{\perp}$ contours obtained with parallel injection match the flux surfaces because the beam contributes negligibly to the perpendicular pressure. The $p_{\perp}$ contours obtained with perpendicular injection, however, are significantly distorted near the magnetic axis and significantly shifted away from the destabilizing curvature region.

ORNL-DWG 79-3064 FED

Fig. 2.4. Parallel pressure contours (solid curves) overlaid on flux surfaces (dashed curves) of equilibria with broad and peaked tensor pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

Fig. 2.5. Perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) of equilibria with broad and peaked tensor pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

The displacement of the pressure contours of the broad pressure profile equilibria with respect to the flux surfaces is due to the weak dependence of the pressures on ψ and to the ensuing dominant dependence on B in the central region of the plasma. The dependence of $p_{\parallel}$ and $p_{\perp}$ on B for equilibria with peaked pressure profiles is very weak everywhere, so the pressure contours are only insignificantly distorted away from the flux surfaces. It is interesting to note that the peaked pressure and flux contours are somewhat more elliptical, yet less D-shaped, than the broad pressure and flux surfaces, respectively.

Figure 2.6 illustrates three classes of broad and peaked pressure profiles for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection: isotropic background profiles p_0 , perpendicular beam profiles $p_{b\perp}$, and parallel beam profiles $p_{b\parallel}$. These clearly indicate no distortions of the two peaked beam pressure profiles as compared with the peaked isotropic background pressure profile. The broad parallel pressure profiles are distorted, especially in the central part of the plasma, as compared with the broad p_0 . With parallel beam injection, the beam contributes insignificantly to $p_{\perp}$, but with perpendicular beam injection, the contribution of the beam to $p_{\perp}$ is much larger, and $p_{b\perp}$ is significantly distorted and shifted away from the destabilizing curvature region because of its strong dependence on B . The weak dependence of the pressures on ψ for equilibria with broad pressure profiles makes $p_{\parallel}$ and $p_{\perp}$ vary slowly from flux surface to flux surface in the central part of the plasma, and the change of the pressures within a flux surface tends to amplify the inward shift of the pressure surfaces. With more peaked pressure profiles, the change in pressures from one flux surface to the next is so rapid that it obscures the variation of $p_{\parallel}$ and $p_{\perp}$ within a flux contour and thus prevents a significant shift of the pressure surfaces away from that contour. That the $p_{\perp}$ contours are more dramatically displaced than the $p_{\parallel}$ surfaces can be most easily understood from a single-particle point of view. The fast ions that spend a relatively larger fraction of their time in the stabilizing curvature region move more slowly along the field lines and therefore contribute preferentially to the perpendicular, rather than to the

Fig. 2.6. Broad and peaked profiles for background pressure p_0 , perpendicular beam pressure $p_{b\perp}$, and parallel beam pressure $p_{b\parallel}$ of tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

parallel, beam pressure component. We must note that even with $\theta_b = 43.57^\circ$, most of the beam pressure is deposited in the parallel component.

Figure 2.7 shows the toroidal current density contours for peaked and broad pressure profile equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) injection. We notice that the differences of the j_ϕ contours between parallel and perpendicular injection are insignificant. However, there are noticeable differences between the j_ϕ contours of equilibria having broad pressure profiles and those having peaked profiles. With peaked pressure profiles, the inner j_ϕ surfaces are significantly crescent-shaped, whereas with broad pressure profiles, the j_ϕ contours are only incipiently crescent-shaped and have small regions of reversed current at the inner edge of the torus.

The toroidal current density profiles for tensor pressure equilibria induced by parallel and perpendicular injection (Fig. 2.8) are remarkably similar despite disparate pressure profiles, especially when those profiles are broad. The j_ϕ profiles are peaked with peaked pressure profiles and broad with broad pressure profiles. There are small regions of reversed current at the inner edge of the torus with broad toroidal current profiles.

In summary, the changes in flux surface topology are small with increasing β_j , but the variation of j_ϕ contours is much more significant. We find no shifts or distortions of the pressure surfaces as compared with the flux contours of equilibria with peaked pressure profiles. The $p_\parallel$ contours are shifted somewhat inward with respect to the flux surfaces for broad pressure profile equilibria induced by parallel and perpendicular injection. The $p_\perp$ contours of broad pressure profile equilibria with parallel injection match the flux surfaces. On the other hand, with perpendicular injection, the beam contributes more heavily to $p_\perp$, and the perpendicular pressure contours are significantly distorted and displaced away from the outer edge of the torus.

Fig. 2.7. Toroidal current density contours of tensor pressure equilibria with broad and peaked pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

Fig. 2.8. Toroidal current density profiles of tensor pressure equilibria with broad and peaked pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

3. STABILITY

The study of stability of tokamak plasmas to oscillations with long toroidal wavelengths requires the use of large spectral^{15,16} or initial value codes.^{17,18} Yet, pressure-driven instabilities having very short wavelengths in the toroidal direction are predicted to occur at high β .

In this section, we perform the asymptotic expansion (in large toroidal mode number n) and subsequent minimization¹⁹ of a modified version of the Kruskal-Oberman energy principle³³ and apply the Fourier transform technique²⁰ to deal with the periodicity conditions of the perturbation in a sheared magnetic field. We thus derive Euler equations that give the necessary and the sufficient criteria for the stability of a guiding center plasma^{39,40} to oscillations of high toroidal mode number.^{19,20} In this particular version of the energy principle, we replace the kinetic integral (which reduces to the plasma compression term in ideal MHD) with a double adiabatic contribution³³ to obtain the necessary stability equation. When we disregard altogether the kinetic integral (which is positive definite for monotonic distribution functions), we obtain an equation that predicts sufficient stability criteria for a guiding center plasma. This equation is equivalent to that derived by Fielding and Haas.²⁵ When we apply these ballooning mode equations to beam-induced tensor pressure equilibria, we find that the difference between the necessary and the sufficient criteria is significant for plasmas susceptible to flutelike modes, but small for ballooning stability. The critical β value imposed by stability to ballooning modes is higher for perpendicular than for parallel beam injection. The difference between the stability limits of broad peaked pressure profile equilibria is large for interchange unstable plasmas and tends to coalesce for ballooning unstable plasmas.

3.1 THE ENERGY PRINCIPLE

Energy principles that allow pressure anisotropy have a multiplicity of names that are used interchangeably and can cause confusion. Effectively, they can be divided into two categories:

- 1) the guiding center plasma,³² Kruskal-Oberman,³³ collisionless,³⁴ or kinetic²⁸ energy principle and
- 2) the guiding center fluid,³² hydromagnetic,³³ Chew, Goldberger, and Low,³⁵ or double adiabatic energy principle.

The difference between these two categories is that in the first category, the pressures are determined by the appropriately weighted velocity space integral of the distribution function, whereas in the second, the pressures are obtained from approximations to the pressure moment equations that eliminate the parallel heat flow.

The double adiabatic energy principle δW_{DA} can be expressed as³³

$$\delta W_{DA} = \delta W_U + \delta W_{DAC} \quad , \quad (3.1)$$

where δW_U is commonly referred to as the universal term, and δW_{DAC} contains terms unique to the double adiabatic theory. The double adiabatic energy principle can be written as

$$\begin{aligned} \delta W_{DA} = & \frac{1}{2} \int d^3x \left(Q^2 + \vec{j} \cdot (\vec{\xi} \times \vec{Q}) + (\vec{\xi} \cdot \vec{\nabla}) p_{\perp} (\vec{\nabla} \cdot \vec{\xi}) \right. \\ & + p_{\perp} [2(\vec{\nabla} \cdot \vec{\xi})^2 - 4(\vec{\nabla} \cdot \vec{\xi})q + 2q^2] - C(q - \vec{\nabla} \cdot \vec{\xi})^2 \\ & + (p_{\parallel} - p_{\perp}) \{ -(\vec{\xi} \cdot \vec{\nabla})q + q^2 - \hat{b} \cdot [(\hat{b} \cdot \vec{\nabla})\vec{\xi} \cdot \vec{\nabla}] \vec{\xi} \\ & + \hat{b} \cdot [(\vec{\xi} \cdot \vec{\nabla})\hat{b} \cdot \vec{\nabla}] \vec{\xi} - [(\hat{b} \cdot \vec{\nabla})\vec{\xi}]^2 + (\hat{b} \cdot \vec{\nabla})\vec{\xi} \cdot (\vec{\xi} \cdot \vec{\nabla})\hat{b} \} \Big) \\ & + \frac{1}{2} \int d^3x \{ p_{\perp} [2(\vec{\nabla} \cdot \vec{\xi})q + q^2] + 3(p_{\parallel} - p_{\perp}) q^2 + C(q - \vec{\nabla} \cdot \vec{\xi})^2 \} \quad , \end{aligned} \quad (3.2)$$

where the first integral is δW_U and the second is δW_{DAC} . The unit vector along the field lines is $\hat{b}$, ξ is the perturbation, $\vec{Q} = \vec{\nabla} \times (\vec{\xi} \times \vec{B})$ is the perturbed magnetic field, and $q = \hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}$.

The Kruskal-Oberman energy principle δW_{KO} can be written as³³

$$\delta W_{KO} = \delta W_U + \delta W_K, \quad (3.3)$$

where the universal term contains all the purely fluid elements and δW_K is called the kinetic integral. In this paper we will use the double adiabatic energy principle.

By means of a Schwartz inequality, it can be shown that the kinetic integral δW_K can be reduced to a fluid form that is less than or equal to δW_{DAC} , provided the distribution function is monotonically decreasing in energy.³³ Consequently, $\delta W_{KO} \leq \delta W_{DA}$. This implies that any criteria derived from the guiding center fluid can be viewed as necessary for the stability of the guiding center plasma. This is a reasonable expectation, even though δW_{DA} does not include the stabilizing effects of trapped particles in a tokamak,⁴³ because in deriving the adiabatic laws that govern the evolution of the pressures in the double adiabatic theory, heat flow along the magnetic field lines (which is by no means small) was ignored.

The parallel and perpendicular pressures are given by the appropriate moments of the distribution function

$$[p_{\parallel}, p_{\perp}] = 2\pi s \sum_i \left(\frac{E_0}{m_i}\right)^{3/2} \int_0^1 dE \int_0^{E/B} d\mu \frac{B}{\sqrt{2(E-\mu B)}} [2(E-\mu B), \mu B] f_i, \quad (3.4)$$

and the pressure-like moment C is given by

$$C = 2\pi s \sum_i \left(\frac{E_0}{m_i}\right)^{3/2} \int_0^1 dE \int_0^{E/B} d\mu \frac{B(\mu B)^2}{\sqrt{2(E-\mu B)}} \left(\frac{\partial f_i}{\partial E}\right), \quad (3.5)$$

where $f_i = f_i(E, \mu, \psi)$ and the factor $s = 2\mu_0 E_0 / B_{\phi 0}^2$.

The motivation for writing the double adiabatic energy principle as the universal term of the guiding center plasma energy principle plus double adiabatic corrections is that δW_U is independent of the component of the perturbation along the field lines. In that case, we need only be concerned with $\xi_{\parallel}$ when treating δW_{DAC} .

The next procedure is to try to obtain δW_{DA} in a form that is more concise and transparent. A more detailed derivation is left to Appendix C. The double adiabatic energy principle can be written as

$$\begin{aligned} \delta W_{DA} = & \frac{1}{2} \int d^3x \left(\sigma Q_{\perp}^2 + \sigma j_{\parallel} \hat{b} \cdot (\xi_{\perp} \times \hat{Q}_{\perp}) + \sigma B^2 \left(1 - \frac{\sigma}{\tau} \right) (\xi_{\perp} \cdot \hat{k})^2 \right. \\ & + \tau B^2 [\hat{\nabla} \cdot \xi_{\perp} + (1 + \frac{\sigma}{\tau}) (\xi_{\perp} \cdot \hat{k})]^2 - (\xi_{\perp} \cdot \hat{k}) (\xi_{\perp} \cdot \hat{\nabla}) (p_{\parallel} + p_{\perp}) \\ & + \frac{p_{\perp}^2}{3p_{\parallel}} [\hat{\nabla} \cdot \xi_{\perp} + (1 - \frac{3p_{\parallel}}{p_{\perp}}) (\xi_{\perp} \cdot \hat{k}) + \xi_{\parallel} (\hat{\nabla} \cdot \hat{b}) + \frac{3p_{\parallel}}{p_{\perp}} (\hat{b} \cdot \hat{\nabla}) \xi_{\parallel}]^2 \\ & \left. + (C - \frac{p_{\perp}^2}{3p_{\parallel}}) \{ [\hat{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \hat{k} + \xi_{\parallel} (\hat{\nabla} \cdot \hat{b})]^2 - (\hat{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \hat{k})^2 \} \right), \end{aligned} \quad (3.6)$$

where $\tau \equiv 1 + \frac{2p_{\perp} - C_1}{B^2}$, $\sigma \equiv 1 - \frac{p_{\parallel} - p_{\perp}}{B^2}$, $\hat{k} \equiv (\hat{b} \cdot \hat{\nabla}) \hat{b}$ is the curvature of the magnetic field, and we identify $C \equiv C_{\eta} \equiv \eta p_{\perp}^2 / 3p_{\parallel}$, so that $C_1 \equiv p_{\perp}^2 / 3p_{\parallel}$.

For a Maxwellian or bi-Maxwellian plasma, the evaluation of C from Eq. (3.5) gives $C = 2p_{\perp}^2 / p_{\parallel}$, which implies $\eta = 6$ for such distributions. On the other hand, if the distribution function is a delta function in magnetic moment and a step function in energy, then $C = p_{\perp}^2 / 3p_{\parallel}$, and as a consequence, $\eta = 1$. For a guiding center plasma $1 \leq \eta \leq 6$. The energy principle obtained from the universal term δW_U of the guiding center plasma energy principle²⁵ can be easily obtained from δW_{DA} by replacing C_1 with C_{η} in τ and by neglecting the last two terms, which are unique to the double adiabatic theory.

The minimization of this double adiabatic energy principle [Eq. (3.6)] with respect to the component of the perturbation along the magnetic field lines is exceedingly difficult. Our interest in deriving a double adiabatic energy principle and examining its stability properties is that any criterion derived from it is necessary for the stability of a guiding center plasma. From this vantage, we are fully justified in assuming that the perturbation has only perpendicular components to the magnetic field. This is equivalent to imposing the constraint on the perturbation that $\xi_{\parallel} = 0$. Therefore, any criterion obtained from the double adiabatic energy principle with this constraint imposed will predict a more optimistic stability condition than when the constraint is absent. The result is an energy principle that gives criteria that are even more necessary for the stability of a guiding center plasma than if $\xi_{\parallel}$ were arbitrary. Taking $\xi_{\parallel} = 0$ considerably simplifies δW_{DA} and the task of minimizing it. The last term in δW_{DA} disappears, and the other term peculiar to the double adiabatic theory can be combined with the second term of δW_{DA} , so that the energy principle reduces to

$$\begin{aligned} \delta W_{DA} = \frac{1}{2} \int d^3x \{ & \underbrace{\sigma Q_{\perp}^2}_{\text{field bending}} + \underbrace{\tau_0 B^2 [\vec{\nabla} \cdot \vec{\xi}_{\perp} + (1 + \frac{\gamma}{\tau_0}) (\vec{\xi}_{\perp} \cdot \vec{k})]}_{\text{field compression}}^2 \\ & + \underbrace{\sigma B^2 (1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_0}) (\vec{\xi}_{\perp} \cdot \vec{k})^2}_{\text{anisotropic-double adiabatic}} \\ & + \underbrace{\sigma j_{\parallel} \hat{b} \cdot (\vec{\xi}_{\perp} \times \vec{Q}_{\perp})}_{\text{kink}} - \underbrace{(\vec{\xi}_{\perp} \cdot \vec{k}) (\vec{\xi}_{\perp} \cdot \vec{\nabla}) (p_{\parallel} + p_{\perp})}_{\text{ballooning}} \} , \end{aligned} \quad (3.7)$$

where $\gamma \equiv 1 - \frac{p_{\parallel}}{B^2}$. The variable τ_0 is defined in Appendix C. The first term, $\sigma Q_{\perp}^2$, is commonly referred to as the field bending term, and it acts to stabilize ballooning modes. We note that it is slightly modified by anisotropy through σ , the modification being of order β . It is of interest to recognize that because the pressures are functions not only of the poloidal flux but also of the magnitude of the magnetic field, the pressures and the fields are no longer independent of one another. So, when a magnetic field line bends in response to a

perturbation, it also stretches, causing the magnitude of the field to decrease, and this in turn affects the pressures.

The next term, $\tau_0 B^2 [\vec{\nabla} \cdot \vec{\xi}_\perp + (1 + \gamma/\tau_0)(\vec{\xi}_\perp \cdot \vec{k})]^2$, is a field compression element. As with the field bending term, this term is also modified by the anisotropy. Compression of the magnetic field alters its magnitude and, consequently, the pressures. In the limit that $p_\parallel = p_\perp$ we do not recover the ideal magnetohydrodynamic field compression term, which is $B^2 [\vec{\nabla} \cdot \vec{\xi}_\perp + 2(\vec{\xi}_\perp \cdot \vec{k})]^2$. This is expected, because we cannot obtain the ideal MHD energy principle from δW_{DA} . Consequently, the δW_{DA} field compression term is also slightly different from the field compression term of the guiding center plasma energy principle (from which the ideal MHD is recovered). We notice that the field compression is positive definite and, hence, stabilizing.

The term $\sigma B^2 (1 + 3p_\parallel/\sigma B^2 - \gamma^2/\sigma\tau_0)(\vec{\xi}_\perp \cdot \vec{k})^2$ combines effects of anisotropy and double adiabaticity and has no equivalent in ideal MHD. The anisotropic part is $\sigma B^2 (1 - \sigma/\tau)$, which is zero when $p_\parallel = p_\perp$. A very similar term appears in the guiding center plasma energy principle that is due to the anisotropy.²⁵ The double adiabatic part can be written as $\sigma B^2 (\sigma/\tau + 3p_\parallel/\sigma B^2 - \gamma^2/\sigma\tau_0)$ and is finite even if we take $p_\parallel = p_\perp$. The double adiabatic part can be shown to be positive definite and hence stabilizing, whereas the anisotropic part is not necessarily so.

In δW_{DA} , $\sigma j_\parallel \hat{b} \cdot (\vec{\xi}_\perp \times \vec{Q}_\perp)$ is an anisotropy-modified term that drives kink instabilities, and $(\vec{\xi}_\perp \cdot \vec{k})(\vec{\xi}_\perp \cdot \vec{\nabla})(p_\parallel + p_\perp)$ drives ballooning modes and interchanges. These two terms reduce to the comparable expressions that appear in ideal MHD if we take $p_\parallel = p_\perp$. Despite all the modifications, the structure of the guiding center fluid energy principle, δW_{DA} , is remarkably similar to that of the ideal MHD energy principle.

3.2 THE EULER EQUATION

The double adiabatic energy principle is expanded in an orthogonal (ϕ, ψ, χ) flux coordinate system, with the metric $(d\ell)^2 = (Rd\phi)^2 + (d\psi/RB_\chi)^2 + (JB_\chi d\chi)^2$, where J is the Jacobian as shown in Appendix D. Because the equilibrium is axisymmetric, the different toroidal Fourier components of the perturbation are uncoupled. Therefore, we need only

consider a single Fourier mode $\exp(-in\phi)$ of the displacement $\vec{\xi}_\perp$, where ϕ is the toroidal angle and n is the mode number. The component of the perturbation normal to the flux surface is given by $X = \vec{\xi}_\perp \cdot \nabla\psi$, and the geodesic component is given by $U = (\vec{\xi}_\perp \cdot \nabla\psi \times \vec{B}) / (R^2 B_\chi^2)$.

In the large toroidal mode number limit, we expand the perturbation $\vec{\xi}_\perp$ in inverse powers of n , following Dobrott et al.¹⁹ We find that the energy principle is large and positive (and hence would always predict stability) unless the gradients of the normal and geodesic displacements along the field lines are of higher order and $\vec{\xi}_\perp$ is incompressible to lowest order. The detailed minimization of the double adiabatic energy principle to obtain the Euler equation is presented in Appendix E. We find that the term that drives kink instabilities vanishes to lowest order. The field compression term can be algebraically eliminated by minimizing δW_{DA} with respect to the first-order component of the perturbation in the geodesic direction. The minimization of the energy principle to the zeroth-order normal component of the perturbation yields, after extensive algebraic manipulations, a second-order ordinary differential equation called the Euler equation [Eq. (E.18) of Appendix E], given by

$$\begin{aligned}
 & B k_\parallel \left\{ \frac{\sigma B}{R^2 B_\chi^2} \left[1 - \left(\frac{R^2 B_\chi^2}{Bn} \right)^2 \frac{\partial^2}{\partial \psi^2} \right] k_\parallel X_o \right\} \\
 & - \left[\frac{\partial}{\partial \psi} (p_\parallel + p_\perp) - \frac{B \phi (\kappa_\psi / \kappa_s)}{R B J B_\chi^2} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) \right] \left(\frac{\kappa_\psi}{R B_\chi} X_o - \frac{i R B_\chi \kappa_s}{Bn} \frac{\partial X_o}{\partial \psi} \right) \\
 & + \frac{B^2 (p_\perp \sigma + 3 p_\parallel \tau)^2}{\tau (p_\perp^2 + 3 p_\parallel B^2 \tau)} \left[\left(\frac{\kappa_\psi}{R B_\chi} \right)^2 X_o - \frac{2 i \kappa_\psi \kappa_s}{n B} \frac{\partial X_o}{\partial \psi} + \left(\frac{R B_\chi \kappa_s}{n B} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right] = 0,
 \end{aligned} \tag{3.8}$$

where the first term corresponds to the field bending, the second drives ballooning and flute modes, and the last term is the double adiabatic addition. Here, X_o is the lowest-order component of X ,

$\kappa_\psi \equiv (\vec{k} \cdot \vec{\nabla}\psi)/(RB_\chi)$ is the normal curvature, $\kappa_s \equiv \vec{k} \cdot (\vec{B} \times \vec{\nabla}\psi)/(RB_\chi B)$ is the geodesic curvature, and the operator $k_\parallel \equiv \frac{1}{iJB} \left(\frac{\partial}{\partial \chi} + i\nu \right)$, where $\nu \equiv JB_\phi/R$. This is the analog of the ideal MHD equation first derived by Dobrott et al.,¹⁹ which they solved numerically assuming the perturbation vanished at the inner edge of the torus on every flux surface.

3.3 THE TENSOR PRESSURE BALLOONING MODE EQUATION

To deal with the periodicity conditions of the mode in a sheared magnetic field, we apply the Fourier transform technique to Eq. (3.8), as outlined in Appendix F, to extend the ballooning mode equation to an infinite domain.²⁰ The perturbation takes the form

$$X_0(\psi, \chi) = \frac{1}{2\pi} \sum_m \exp\left(-\frac{2\pi i m \chi}{\oint d\chi}\right) \int_{-\infty}^{\infty} dy \exp\left(\frac{2\pi i m y}{\oint d\chi}\right) \hat{X}_0(\psi, y) \quad . \quad (3.9)$$

This transformation can be obtained by expanding the displacement $X_0(\psi, \chi)$ in a Fourier series. The Fourier coefficients a_m represent a sequence of points in the complex plane that by analytic continuation can generate a function $a(m)$. The inverse Fourier transform of this function results in Eq. (3.9).

Because the transformed perturbation need not be periodic in the infinite domain, it can be expressed in a quasi-mode or eikonal form:¹³

$$\hat{X}_0(\psi, y) = F(\psi, y) \exp(-in \int^y v dy) \quad , \quad (3.10)$$

where F is a slowly varying function of ψ and y , and all the rapid cross-field variations are contained in the exponent.

The Euler equation then is given by

$$\begin{aligned}
 & \frac{1}{J} \frac{\partial}{\partial y} \left\{ \frac{\sigma}{JR^2 B_\chi^2} \left[1 + \left(\frac{R^2 B_\chi^2}{B} \int^y dy \frac{\partial v}{\partial \psi} \right)^2 \right] \frac{\partial F}{\partial y} \right\} \\
 & \quad \text{field bending} \\
 & + \left[\frac{\partial}{\partial \psi} (p_\parallel + p_\perp) - \frac{B_\phi (\kappa_\psi / \kappa_s)}{RB_\chi BJB_\chi} \frac{\partial}{\partial y} (p_\parallel + p_\perp) \right] \left(\frac{\kappa_\psi}{RB_\chi} - \frac{RB_\chi \kappa_s}{B} \int^y dy \frac{\partial v}{\partial \psi} \right) F \\
 & \quad \begin{array}{cc} \text{normal} & \text{poloidal} \\ \text{pressure} & \text{gradients} \end{array} \quad \begin{array}{cc} \text{normal} & \text{geodesic} \\ \text{curvatures} & \end{array} \\
 & + \frac{B^2 (p_\perp \sigma + 3p_\parallel \tau)^2}{\tau (p_\perp^2 + 3p_\parallel B^2 \tau)} \left(\frac{\kappa_\psi}{RB_\chi} - \frac{RB_\chi \kappa_s}{B} \int^y dy \frac{\partial v}{\partial \psi} \right)^2 F(\psi, y) = 0. \quad (3.11) \\
 & \quad \text{double} \quad \text{adiabatic}
 \end{aligned}$$

As indicated, the first term is the field bending and is only slightly modified by the factor σ from the scalar pressure case. The second term shows the coupling between the pressure gradients with the curvatures, which drives ballooning instabilities. The main difference between Eq. (3.11) and that obtained from ideal MHD is that because the pressures are no longer constant on a flux surface, an additional term appears owing to the gradient of the pressures in the poloidal direction. The last term is a double adiabatic contribution that we see is stabilizing whenever $\tau > 0$, a requirement that must be met for the equilibrium to be mirror stable and thus well posed. If we neglect the double adiabatic contribution, the Euler equation is virtually identical to that derived by Fielding and Haas²⁵ from the universal term of the Kruskal-Oberman energy principle. By excluding the stabilizing double adiabatic contribution, the solutions of the Euler equation will give criteria for marginal stability that are sufficient for the stability of a guiding center plasma; by including it, the criteria are necessary for the stability of the guiding center plasma. In that sense, we can bracket the actual marginal stability point.

The boundary conditions are obtained by requiring that the perturbation be square integrable, that is, that $\int_{-\infty}^{\infty} dy |F|^2$ be finite. If $|F|^2$ is proportional to $1/y$, the integral diverges logarithmically. If it is proportional to some power of the inverse of y that is slightly larger than unity, then the integral converges. That gives the natural boundary condition

$$|y|^{1/2} F \rightarrow 0 \quad \text{as} \quad |y| \rightarrow \infty .$$

Although $X_0(\psi, y)$ is not the actual perturbation, a real and periodic $X_0(\psi, \chi)$ can be constructed from it by a superposition of quasi-modes.

The pressures in Eq. (3.11) are expressed as a function of ψ and the poloidal variable y . A form of the Euler equation that is more useful for numerical treatment and more elegant can be obtained if the pressures instead are expressed as functions of ψ and B . (The variable B is the magnitude of the magnetic field.) The transformation of equilibrium expressions from orthogonal to nonorthogonal (ψ, B) space appears in Appendix G. We also add a term obtained by minimizing the perpendicular kinetic energy to Eq. (3.11), which was obtained by minimizing a potential energy, to make it an eigenmode equation.

The ensuing tensor pressure balloon equation is

$$(\vec{B} \cdot \vec{\nabla}) [S_B (\vec{B} \cdot \vec{\nabla}) F] + D(1 - \lambda)F = 0 . \quad (3.12)$$

Here, λ is the eigenvalue ($\lambda > 0$ for instability), the field bending term is

$$S_B = \frac{\sigma}{R^2 B_X^2} \left[1 + \left(\frac{R^2 B_X^2}{B} \int_{\chi_0}^y dy \frac{\partial v}{\partial \psi} \right)^2 \right] , \quad (3.13)$$

and the driving term is

$$D = \left(\frac{\partial p_{\parallel}}{\partial \psi} + \frac{\sigma}{\tau} \frac{\partial p_{\perp}}{\partial \psi} \right) \left(\frac{\kappa_{\psi}}{RB_{\chi}} - \frac{RB_{\chi} \kappa_s}{B} \int_{\chi_0}^y dy \frac{\partial v}{\partial \psi} \right)$$

pressure gradients curvatures

$$- \frac{B^2 (p_{\perp} \sigma + 3p_{\parallel} \tau)^2}{\tau (p_{\perp}^2 + 3p_{\parallel} B^2 \tau)} \left(\frac{\kappa_{\psi}}{RB_{\chi}} - \frac{RB_{\chi} \kappa_s}{B} \int_{\chi_0}^y dy \frac{\partial v}{\partial \psi} \right)^2 \quad (3.14)$$

double adiabatic term

We can alternatively define $\sigma \equiv 1 - \frac{1}{B} \frac{\partial p_{\parallel}}{\partial B}$ and $\tau \equiv 1 + \frac{1}{B} \frac{\partial p_{\perp}}{\partial B}$. The destabilizing effect of the perpendicular pressure gradient can be reduced in magnitude whenever $\sigma < 1$ and $\tau > 1$ (when $p_{\parallel}$ and $p_{\perp}$ are monotonic in B). The double adiabatic contribution is small near the plasma edge, but it can become significant in the neighborhood of the magnetic axis, where the pressures are large.

3.4 NUMERICAL METHOD

To investigate the large n stability of computed tensor pressure equilibria induced by neutral beam injection, we numerically solve the Euler equation [Eq. (3.12)] using a shooting method to obtain the eigenvalue λ on each flux surface. The equilibrium quantities, which are provided by the equilibrium code on a rectangular grid, are linearly interpolated from that grid onto the flux contours, where the ballooning equation is actually solved. We try to minimize the number of interpolations to reduce the numerical error. To do this, we use equilibrium relations to transform the ballooning mode equation [Eq. (3.12)] into a form more suitable for the numerics, as shown in Appendix H. In this equation, the eigenvalue is a measure of how much the driving term must be either increased (if $\lambda < 0$) or decreased (if $\lambda > 0$) to make it cancel the field bending stabilization on each contour.

The eigenvalue is adjusted on each flux surface until the mode just vanishes at the inner edge of the torus after some arbitrarily selected number of poloidal transits starting from the outer edge. The actual eigenvalue on each contour is obtained only in the limit of infinite arc length (i.e., the number of poloidal circuits), but can be approximated if we carefully examine the convergence of the eigenfunction as we vary the arc length. The eigenfunctions on flux surfaces near the magnetic axis are flutelike (they vary slowly in poloidal angle variable), with a small ballooning modulation; consequently, the eigenvalues are very sensitive to the poloidal arc length of integration. On surfaces near the plasma edge, the modes are ballooning in nature, with a small interchange tail, and the eigenvalues are insensitive to the arc length. Figure 3.1 illustrates the eigenmode behavior; Fig. 3.2 shows the convergence of the eigenvalues on different flux surfaces.

To obtain solutions of Eq. (3.12), we integrate this second-order ordinary differential equation using an implicit trapezoidal rule. We have also implemented a much more accurate, complicated, and time-consuming scheme based on the routine ODE by Shampine and Gordon⁴⁴ and have found very good agreement.

3.5 STABILITY OF SAMPLE EQUILIBRIA

In this section, we undertake a stability study of beam-driven, D-shaped, tensor and scalar pressure tokamak equilibria with pressure profiles of varying peakedness. The nearly parallel injection equilibria we present have $\theta_b = 25^\circ$, where θ_b is the angle between the magnetic field and the direction of the neutral beam at the outer midplane point of the tokamak device. The nearly perpendicular injection equilibria obtained here have $\theta_b = 43.57^\circ$. A further increase in θ_b would correspondingly increase the contribution of the beam to the perpendicular pressure (and diminish its contribution to $p_{\parallel}$). This would produce trapped beam particles, however, which would cause $p_{\perp}$ to become singular and the equilibrium to become mirror unstable.^{39,40}

Fig. 3.1. Variation of the eigenfunctions on different flux surfaces with arc length. The eigenmodes near the magnetic axis are flutelike (with a small ballooning modulation), whereas the modes near the edge of the plasma are ballooning in character (with a small interchange tail).

The magnetic moment profile $\mu_e(\psi)$ used to compute equilibria is given by Eq. (2.37).

The high- β second ballooning stability region⁴⁵⁻⁵⁰ can be obtained for equilibria that display a large shift of the magnetic axis. This shift, with the ensuing steep gradients of the poloidal flux, increases the poloidal magnetic field in the destabilizing curvature region and shortens the connection length for ballooning modes. We have not numerically investigated this second region of

Fig. 3.2. Convergence of the eigenvalues on an outer, a middle, and an inner flux surface with arc length. (N_e represents the number of poloidal circuits, starting from the outer edge, that an eigenmode makes before it vanishes.)

stability because the implicit vertical fields of the particular plasma shaping function that we use [Eq. (2.36)] prevent shifts of the magnetic axis large enough to make the poloidal and toroidal fields comparable in magnitude at the outer edge of the torus.

3.5.1 Moderately Broad Pressure Profiles

We first examine the stability of equilibria with moderately broad pressure profiles on a 33×33 grid. The main purpose of this set of examples is to show that there is not a unique manner to define β in tensor pressure equilibria, and the stability results obtained are very sensitive to the definition chosen. The β limits presented are somewhat optimistic because they were obtained by solving the ballooning mode equation within a poloidal domain that included only 7.5 poloidal transits the short way around. Nevertheless, they also serve to show that a difference exists between the stability limits with parallel and perpendicular injection and between tensor and scalar pressure equilibria.

We compute D-shaped equilibria ($t_g = 0.8$, $q_g = 0.2$) for an $\epsilon = 1/3$ device with elongation of 1.6:1. In this set of examples, we compare the stability of tensor pressure equilibria with parallel and perpendicular injection and scalar pressure equilibria with safety factor $q = 1$ on axis and different β_j . The scalar pressure equilibria are obtained by specifying

$$p_{\parallel}(\psi) = p_{\perp}(\psi) = 3 S_c \left[\left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^2 - \frac{1}{2} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^3 \right] \quad (3.15)$$

The tensor pressure equilibrium profiles, on the other hand, are given by

$$p_{\parallel}(\psi, B) = S_c \left[\left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^2 - \frac{1}{2} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^4 \right] [1 + 2 B \sqrt{1 - \mu_e(\psi)B}], \quad (3.16)$$

where the magnetic moment profile is given by Eq. (2.37). The function (σI) defined by Eq. (2.11) is specified by

$$(\sigma I)^2 = 1 + 2 S_c \left[d_p \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^2 + q_x \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^3 + \frac{3}{2} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^4 \right]. \quad (3.17)$$

The coefficient S_c is a scale factor used to raise or lower β , the coefficient d_p is used to vary β_j , and q_x is adjusted to make the safety factor $q > 1$ on axis. The last term in the polynomial for $(\sigma I)^2$ prevents the toroidal current density from becoming hollow because of the choice made for the pressure profiles.

Figure 3.3 presents parallel and perpendicular pressure surfaces (solid curves) overlaid on flux contours (dashed curves) for three sample equilibria with $\beta_j \approx 0.64$. We notice that the scalar pressure surfaces are indeed flux surfaces. The parallel pressure contours for both angles of injection are slightly shifted inward with respect to the flux surfaces near the magnetic axis for tensor pressure equilibria. For $\theta_b = 25^\circ$, the perpendicular pressure contours virtually coincide with the flux surfaces because the beam pressure component contributes negligibly to $p_\perp$. For $\theta_b = 43.57^\circ$, the perpendicular pressure contours display a marked inward shift with respect to each flux surface. We expect this to have a stabilizing effect on ballooning modes, because the inward displacements represent a relaxation of the perpendicular pressure gradient normal to the flux surfaces in the destabilizing curvature region. In addition, the gradients $\frac{\partial p_\parallel}{\partial B}$ and $\frac{\partial p_\perp}{\partial B}$ are both positive, which makes $\sigma < 1$ and $\tau > 1$, thus reducing the contribution of $\frac{\partial p_\perp}{\partial \psi}$ to the driving term of the ballooning mode equation [Eq. (3.12)]. The equilibria we generate for each value of β_j have very similar toroidal current densities, as illustrated by the j_ϕ contours in Fig. 3.4. The shape of the j_ϕ surfaces shown is characteristic of fairly paramagnetic equilibria.

Fig. 3.3. Parallel and perpendicular pressure surfaces (solid curves) overlaid on flux surfaces (dashed curves) of scalar pressure equilibria and of tensor pressure equilibria with moderately broad pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

Fig. 3.4. Toroidal current density contours of scalar pressure equilibria and of tensor pressure equilibria with moderately broad pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

With tensor pressure equilibria, there is no unique way of defining β . The most widely used definition is given by Eq. (2.18). The definition

$$\beta_+ = \langle p_{\parallel} + p_{\perp} \rangle, \quad (3.18)$$

where the angle brackets represent a volume average, could also be considered useful and convenient. The stability limits we obtain are quite sensitive to the definition of β we use, as illustrated in Figs. 3.5 and 3.6. The marginal stability curves are obtained here by solving the ballooning equation over 7.5 poloidal transits. We find that perpendicular injection enhances stability conditions even beyond ideal MHD predictions because of the inward displacement of the perpendicular pressure surfaces. The stability predictions with parallel injection are much more pessimistic when we use $\beta \equiv (2/3) \langle p_{\parallel} + 2p_{\perp} \rangle$ because this definition is more heavily weighted in favor of the $p_{\perp}$ component. When we use $\beta_+ \equiv \langle p_{\parallel} + p_{\perp} \rangle$, we find that parallel injection predicts higher stability than does ideal MHD for low values of β_j . This improvement in stability conditions is a result of the stabilizing effect of the inward shift of the $p_{\parallel}$ surfaces near the magnetic axis, where the most unstable contours are concentrated. We have neglected the double adiabatic contribution to the ballooning mode equation in the results presented in this subsection.

Despite the pressure profiles chosen, the most unstable surfaces in these equilibria were located near the magnetic axis. Therefore, the critical β values are very sensitive to the arc length of integration owing to the interchange nature of the instability, as is illustrated in Fig. 3.7. Consequently, the results presented in this subsection should be considered only as indicative of trends.

Fig. 3.5. Sufficient stability limits with finite arc length (7.5 poloidal transits) and $\beta \equiv (2/3) \langle p_{\parallel} + 2p_{\perp} \rangle$ as a function of β_J for ideal MHD and for tensor pressure equilibria with moderately broad pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

Fig. 3.6. Sufficient stability limits with finite arc length (7.5 poloidal transits) and $\beta \equiv \langle p_{\parallel} + p_{\perp} \rangle$ as a function of β_j for ideal MHD and for tensor pressure equilibria with moderately broad pressure profiles induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection.

Fig. 3.7. Convergence of critical β values with arc length for equilibria susceptible to interchange modes.

Henceforth, we will use only the definition $\beta \equiv (2/3) \langle p_{\parallel} + 2p_{\perp} \rangle$ unless explicitly stated otherwise.

3.5.2 Slightly Peaked Pressure Profiles

We again compute D-shaped equilibria with triangularity $t_g = 0.8$ and quadrangularity $q_g = 0.2$ in an $\epsilon = 1/3$ device with elongation of 1.6:1 on a 33×33 grid. In this set of examples, we vary the input parameters until we generate an equilibrium (with a chosen value of β_j and $q = 1$ on axis) that is marginally stable ($\lambda = 0$ on the most unstable surface) for a given arc length of integration. The β limits are then determined by extrapolating the results obtained from the solutions of the ballooning mode equation over three finite arc lengths to infinity. This procedure was possible only because of the speed with which a computer like the CDC-7600 can compute equilibria on a 33×33 grid.

Here we will compare necessary and sufficient stability criteria of parallel and perpendicular beam-induced tensor pressure equilibria having safety factor $q = 1$ on axis and different values of β_j . The parallel pressure profile is given by Eq. (2.38), and the (σI) profile is given by Eq. (2.39).

The effect of neutral beam injection on plasma equilibria for mildly peaked pressure profiles is shown in Fig. 3.8. The pressure contours (solid curves) are overlaid on flux surfaces (dashed curves), and we see that for both angles of injection, the parallel pressure and flux contours virtually coincide. The beam contribution to the $p_{\perp}$ component is negligible for $\theta_b = 25^\circ$, so the perpendicular pressure surfaces match the flux contours resulting from parallel injection. For $\theta_b = 43.57^\circ$, the beam contribution to $p_{\perp}$ is larger and the perpendicular pressure contours are shifted noticeably inward with respect to each flux surface. When we compare Fig. 3.3 and Fig. 3.8, however, we notice that the inward displacements of the tensor pressure surfaces are much less significant with the more peaked pressure profiles than with the broader ones.

ORNL DWG 79-2657R FED

$$\begin{aligned}\theta_b &= 25^\circ \\ \beta &= 1.04\% \\ \beta_b / \beta &= 0.39 \\ \beta_J &= 2.21\end{aligned}$$

$$\begin{aligned}\theta_b &= 43.57^\circ \\ \beta &= 1.27\% \\ \beta_b / \beta &= 0.46 \\ \beta_J &= 2.24\end{aligned}$$

$P_{||}$

$P_{\perp}$

Fig. 3.8. Parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection with slightly peaked pressure profiles.

Stability results are shown in Figs. 3.9-3.12. At low β_j , the sufficient stability limits are set by interchange modes that appear near the magnetic axis; at high β_j , the limits are set by ballooning modes on flux surfaces near the edge of the plasma. We must mention that β changes independently of β_j because q_{edge} is being varied even though q_{axis} is fixed. We take $q_{\text{axis}} = 1$ and do not consider cases with $q_{\text{axis}} < 1$, since sawtooth oscillations prevent the safety factor from dropping far below unity.⁵¹

To generate each point in Figs. 3.9 and 3.10, we vary the input parameters until we obtain a marginally stable equilibrium ($\lambda = 0$ on the most unstable flux surface) with the desired β_j and $q = 1$ on axis for a given arc length. The asymptotic ($N_e \rightarrow \infty$, N_e is the number of poloidal transits of the mode) critical β is then determined by extrapolating the marginal β values obtained with finite arc lengths, as shown in Fig. 3.7. The marginal β values vary significantly with different arc lengths for interchange unstable plasmas but are insensitive to the arc length when ballooning modes near the edge of the plasma dominate. It is interesting to note that the transition from interchange unstable to ballooning mode unstable occurs at a somewhat lower β_j with parallel, rather than perpendicular, injection.

Figure 3.11 compares the necessary and the sufficient stability limits for parallel and for perpendicular injection. The necessary stability conditions are imposed by ballooning modes that appear near the edge of the plasma. We wish to draw attention to the large difference between the necessary and the sufficient criteria at low β_j . This difference is due to the double adiabatic term of the ballooning mode equation [Eq. (3.12)], which strongly inhibits any perturbations that appear where the pressures are largest (i.e., near the magnetic axis). At high β_j , both criteria are dominated by ballooning modes near the edge of the plasma, where low pressure diminishes the magnitude of the double adiabatic term and therefore makes the necessary and the sufficient stability limits coalesce.

Fig. 3.9. Sufficient stability limits as a function of β_j for different arc lengths (N_e = the number of poloidal transits of the eigenmodes) of tensor pressure equilibria induced by parallel injection ($\theta_b = 25^\circ$) with slightly peaked pressure profiles.

Fig. 3.10. Sufficient stability limits as a function of β_J for different arc lengths (N_e = the number of poloidal transits of the eigenmodes) of tensor pressure equilibria induced by perpendicular injection ($\theta_b = 43.57^\circ$) with slightly peaked pressure profiles.

Fig. 3.11. Necessary and sufficient stability limits as a function of β_J for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection with slightly peaked pressure profiles.

The salient stability results for slightly peaked pressure profiles are compiled in Fig. 3.12. We find that perpendicular injection is favorable for stability for β limits set by ballooning modes (solid curves). This improvement in stability is caused in part by the inward shift of the $p_{\perp}$ contours with respect to the flux surfaces and the ensuing profile changes that remove pressure from the destabilizing curvature region. This agrees with results previously found in the large-aspect-ratio limit.²⁵ The inward $p_{\perp}$ shift also accounts for the transition from interchange to ballooning unstable to occur at higher β_j with perpendicular, rather than with parallel, injection, as observed in Figs. 3.9 and 3.10. (Note that the definition of β we use is weighted in favor of the $p_{\perp}$ contribution.) Nevertheless, we observe that the steeper $p_{\parallel}$ gradients associated with parallel injection, which are due to the geometric effect of the beam particles being more nearly tangent to the inner flux surfaces, enhance the shear near the magnetic axis, stabilizing interchanges. This stabilization can be strong enough to overcome the unfavorable weighting of the $p_{\parallel}$ contribution to β when the shear at the axis is sufficiently large. Thus, for equilibria that are marginally stable to flute modes (dashed curve), we find that parallel injection can predict higher stability limits than can perpendicular injection.

The marginal β for ballooning mode stability decreases with β_j because q_{edge} increases with β_j (roughly $\beta \propto \epsilon \beta_j / q^2$, where q is the safety factor at the most unstable flux surface). The interchange stability, on the other hand, increases with β_j because the resulting increase in q_{edge} enhances the shear near the magnetic axis (q_{axis} is fixed). In summary, the necessary and the sufficient stability limits are markedly different at low β_j and coalesce at high β_j . Perpendicular injection is favorable for ballooning mode stability. Competing effects preclude a similar conclusion for interchange stability.

Fig. 3.12. Marginally stable β (in %) as a function of β_j imposed by interchanges (dashed lines) and ballooning modes (solid lines) for tensor pressure equilibria with slightly peaked pressure profiles and injection angles (with respect to the magnetic field) of 25° and 43.57° .

3.5.3 Broad Pressure Profiles

We finally examine the stability of a set of tensor and scalar pressure equilibria computed on a 65×65 grid. These equilibria have very broad pressure profiles and weak pressure gradients near the magnetic axis. The particularly broad profiles were chosen to assure that the equilibria in the β_j range of interest are all very stable in the neighborhood of the magnetic axis, where the lack of resolution because of the few grid points and the interchange nature of the instability yield uncertain extrapolations of the critical β limits. Time consumption considerations preclude the search of marginally stable equilibria for given arc lengths of integration on a 65×65 grid. Consequently, we solve the ballooning mode equation with different arc lengths of integration and extrapolate the eigenvalues obtained to infinite arc length for a sequence of three equilibria with chosen values of β_j and q on axis that bracket the marginal stability points on at least one or more flux surfaces. We then pass a quadratic interpolator to obtain the critical β value of each surface that goes unstable.

We compute D-shaped equilibria with triangularity $t_g = 1.0$ and quadrangularity $q_g = 0$ in an $\epsilon = 1/3$ device with elongation of 1.6:1. In this subsection, we will present a survey of necessary and sufficient stability conditions for different values of β_j and safety factor q on axis. In the process, we will compare the stability limits of tensor pressure equilibria induced by parallel and perpendicular injection and of tensor pressure equilibria induced by perpendicular injection only with scalar pressure equilibria having similar pressure profiles.

The tensor pressure profiles are specified by Eq. (2.42), and the (σI) profile is given by Eq. (2.43). To obtain scalar pressure equilibria that we can compare with the tensor pressure equilibria, we choose an isotropic pressure profile roughly comparable to one-third the trace of the anisotropic pressure tensor:

$$p_{\parallel}(\psi) = p_{\perp}(\psi) = S_c \left[\left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^2 - \frac{2}{2.1} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^{2.1} \right] \left[1 + \frac{\alpha}{3\sqrt{1-\mu_e(\psi)B}} \right], \quad (3.19)$$

where we use the coefficient S_c to raise or lower β and where the coefficient α is adjusted so that the ratio of the beam component of β to the total β is roughly the same for all equilibria investigated. The tensor and scalar pressure equilibria we compare will have $\theta_b = 43.57^\circ$ in the $\mu_e(\psi)$ expression given by Eq. (2.37).

We present the parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for beam-induced tensor pressure sample equilibria with parallel and perpendicular injection and scalar pressure sample equilibria with $\beta_j = 0.5$ (Fig. 3.13a), $\beta_j = 1.5$ (Fig. 3.13b), and $\beta_j = 2.5$ (Fig. 3.13c). If we compare the tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) injection, we notice that the $p_{\parallel}$ contours are displaced somewhat inward with respect to the flux surfaces near the magnetic axis. The $p_{\perp}$ contours virtually match the flux surfaces for $\theta_b = 25^\circ$ because the beam component contributes negligibly to the perpendicular pressure. The perpendicular pressure contours, on the other hand, display a significant inward shift with respect to each flux surface for $\theta_b = 43.57^\circ$, since the beam contribution to $p_{\perp}$ is much larger for this angle of injection. As we would expect, the scalar pressure contours are flux surfaces. The effect of inward shifts of pressure contours is largest for these broad pressure profiles, as seen by comparing Figs. 3.13a-3.13c with Fig. 3.8. The beam and isotropic background pressure profiles and the toroidal current density profiles associated with the contour plots of Figs. 3.13a-3.13c are shown for $\beta_j = 0.5$ (Fig. 3.14a), $\beta_j = 1.5$ (Fig. 3.14b) and $\beta_j = 2.5$ (Fig. 3.14c). We notice that the beam parallel pressure components $p_{b\parallel}$ are shifted inward near the magnetic axis for both angles of injection with respect to the isotropic background pressures p_0 . We also see that the beam perpendicular pressure components $p_{b\perp}$ are negligible for $\theta_b = 25^\circ$, whereas they are significant, quite distorted, and noticeably shifted inward with respect to the isotropic background pressures p_0 and the scalar pressure beam components p_b for $\theta_b = 43.57^\circ$. Even for $\theta_b = 43.57^\circ$, we find that $p_{b\parallel}$ is greater than $p_{b\perp}$. The toroidal current density profiles of the tensor pressure equilibria with $\theta_b = 25^\circ$ and

Fig. 3.13a. Parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 0.5$.

ORNL-DWG 79-3051 FED

Fig. 3.13b. Parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 1.5$.

Fig. 3.13c. Parallel and perpendicular pressure contours (solid curves) overlaid on flux surfaces (dashed curves) for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 2.5$.

Fig. 3.14a. Beam and background pressure profiles and toroidal current density profiles for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 0.5$.

Fig. 3.14b. Beam and background pressure profiles and toroidal current density profiles for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 1.5$.

Fig. 3.14c. Beam and background pressure profiles and toroidal current density profiles for tensor pressure equilibria induced by parallel ($\theta_b = 25^\circ$) and perpendicular ($\theta_b = 43.57^\circ$) beam injection and for scalar pressure equilibria, all having broad pressure profiles and $\beta_j = 2.5$.

$\theta_b = 43.57^\circ$ are very similar despite their dissimilar pressure profiles. If we compare the perpendicular beam-driven tensor pressure and scalar pressure j_ϕ profiles, we find them also quite similar.

We compare the necessary and the sufficient stability limits of tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) neutral beam injection in Figs. 3.15-3.18. The stability limits as a function of q_a (the safety factor at the magnetic axis) are displayed for $\beta_j = 0.5$ (Fig. 3.15a) and $\beta_j = 2.5$ (Fig. 3.15b). We do not examine stability conditions for $q_a < 1$, since sawtooth oscillations prevent the safety factor from dropping too far below unity.⁵¹ Perpendicular injection is favorable for stability because the inward shift of the $p_\perp$ surfaces and the ensuing profile changes remove pressure from the destabilizing curvature region. The stability limits decrease with q_a roughly consistent with the scaling $\beta \propto \epsilon \beta_j q^{-2}$, where q is the safety factor of the most unstable surface. [Although the safety factor at the edge of the plasma (q_b) increases with q_a , the increase is not proportional to q_a .] The sufficient marginal stability points on different flux surfaces for $q_a = 1.0$ and $q_a = 1.4$ are shown for $\beta_j = 0.5$ (Fig. 3.16a) and $\beta_j = 2.5$ (Fig. 3.16b). These figures show that regardless of the value of β_j , the flux surfaces near the magnetic axis are very stable. This is due to the very broad pressure profiles with weak gradients near the magnetic axis we have chosen. Also, this behavior distinguishes these equilibria from previously examined equilibria that had more peaked pressure profiles and were consequently unstable to interchanges near the magnetic axis at low β_j . Nevertheless, the most unstable surfaces at low β_j concentrate halfway between the magnetic axis and the edge of the plasma, thus giving these instabilities incipient flute character. At high β_j , the instabilities are much nearer the edge of the plasma and therefore are very ballooning in nature. For a given value of β , the unstable region is much broader with $q_a = 1.4$ than with $q_a = 1.0$, consistent with the q^{-2} scaling.

Fig. 3.15a. Necessary and sufficient stability limits as a function of safety factor q on axis for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and $\beta_j = 0.5$.

Fig. 3.15b. Necessary and sufficient stability limits as a function of safety factor q on axis for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and $\beta_j = 2.5$.

ORNL-DWG 79-3099

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Fig. 3.16a. Sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and $\beta_J = 0.5$.

Fig. 3.16b. Sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and $\beta_J = 2.5$.

The stability limits as a function of β_j for tensor pressure equilibria induced by parallel (dashed curves) and perpendicular (solid curves) beam injection are displayed for $q_a = 1.0$ (Fig. 3.17a) and $q_a = 1.4$ (Fig. 3.17b). The stability limits decrease with β_j because q_b increases with β_j . The necessary and sufficient stability conditions are markedly different at low β_j , since the most unstable surfaces are localized to regions of fairly high pressure. High pressures make the double adiabatic term of the ballooning mode equation large and, hence, very stabilizing. At high β_j , the stability criteria coalesce because the most unstable surfaces are located near the edge of the plasma, where low pressure makes the double adiabatic stabilization small. The necessary and sufficient marginal stability points on different flux surfaces are shown in Fig. 3.18. The unstable region bounded by the necessary marginal stability curve is localized to surfaces near the edge of the plasma and is much narrower than the unstable region bounded by the sufficient marginal stability points for a given value of β . The unstable region bounded by the necessary marginal stability curve is localized to the outer flux surfaces. This is due to the double adiabatic term, which inhibits any perturbations that appear where the pressures are high. The necessary and sufficient marginal stability curves thus coalesce near the edge of the plasma.

The necessary and sufficient stability limits of tensor and scalar pressure equilibria are shown in Figs. 3.19-3.22. This comparison closely parallels that between the stability of tensor pressure equilibria induced by perpendicular beam injection and the stability of those induced by parallel beam injection. The necessary and sufficient stability limits for $\beta_j = 0.5$ (Fig. 3.19a) and $\beta_j = 2.5$ (Fig. 3.19b) for different q_a show that the tensor pressure equilibria induced by perpendicular injection (solid curves) predict higher stability limits than do comparable scalar pressure equilibria (dashed curves). This improvement is caused by the stabilizing effect of the inward shift of the $p_\perp$ surfaces with respect to the flux contours, despite a somewhat peaked $p_\parallel$ profile. The stability limits decrease with increasing q_a , consistent with the rough q^{-2} scaling of critical β values.

Fig. 3.17a. Necessary and sufficient stability limits as a function of β_j for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and safety factor $q_a = 1.0$ on axis.

Fig. 3.17b. Necessary and sufficient stability limits as a function of β_j for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles and safety factor $q_a = 1.4$ on axis.

Fig. 3.18. Necessary and sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular (solid curves) and parallel (dashed curves) beam injection with broad pressure profiles.

Fig. 3.19a. Necessary and sufficient stability limits as a function of safety factor q on axis for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and $\beta_j = 0.5$.

Fig. 3.19b. Necessary and sufficient stability limits as a function of safety factor q on axis for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and $\beta_j = 2.5$.

We display the sufficient marginal stability points of tensor (solid curves) and scalar (dashed curves) pressure equilibria with $q_a = 1.0$ and $q_a = 1.4$ for $\beta_j = 0.5$ (Fig. 3.20a) and $\beta_j = 2.5$ (Fig. 3.20b). It is interesting to note that the marginal stability points for the scalar pressure equilibria are localized on surfaces closer to the edge of the plasma than in the tensor pressure equilibria. The enhanced ballooning character of the instability is due to the steeper pressure gradients near the plasma boundary (in the destabilizing curvature region) of the scalar pressure equilibria. The inward shift of the $p_\perp$ contours of the tensor pressure equilibria relaxes the pressure gradients near the outer edge and thus makes the plasma less susceptible to ballooning modes. The instability region is broader for any given value of β with $q_a = 1.4$ than it is with $q_a = 1.0$, consistent with the inverse q^2 scaling.

The necessary and sufficient stability limits as a function of β_j for $q_a = 1.0$ (Fig. 3.21a) and $q_a = 1.4$ (Fig. 3.21b) show that the tensor pressure equilibria induced by perpendicular injection (solid curves) are more stable than the scalar pressure equilibria (dashed curves). The stability limits decrease with β_j because q_b increases with β_j . The necessary critical β values are much larger than the sufficient critical β values at low β_j because the most unstable surfaces are located on flux contours halfway between the magnetic axis and the edge of the plasma, where the high pressure causes the double adiabatic term to cancel the driving mechanism of the instability in the ballooning mode equation. The necessary and sufficient stability limits coalesce at high β_j because the instability is dominated by ballooning modes closer to the edge of the plasma, where the low pressure reduces the magnitude of the stabilizing double adiabatic term. In effect, the necessary and the sufficient marginal stability curves are markedly different on the inner flux surfaces, as shown in Fig. 3.22, but coalesce at the edge of the plasma, where the pressures are low. Because of the stabilizing effect of the double adiabatic term in high-pressure regions, the unstable region bounded by the necessary marginal stability points is much narrower and more confined to the outer flux surfaces than is the unstable region bounded by the sufficient marginal stability curve.

ORNL-DWG 79-3101 FED

Fig. 3.20a. Sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and $\beta_j = 0.5$.

Fig. 3.20b. Sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and $\beta_j = 2.5$.

Fig. 3.21a. Necessary and sufficient stability limits as a function of β_j for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and safety factor $q_a = 1.0$ on axis.

Fig. 3.21b. Necessary and sufficient stability limits as a function of β_J for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles and safety factor $q_a = 1.4$ on axis.

Fig. 3.22. Necessary and sufficient stability limits as a function of flux surface for tensor pressure equilibria induced by perpendicular beam injection (solid curves) and for scalar pressure equilibria (dashed curves), both having broad pressure profiles.

Broad pressure profile equilibria are ballooning mode unstable for $0.5 < \beta_j < 2.5$ (although there is incipient flute character at low β_j). Peaked pressure profile equilibria are susceptible to ballooning modes at high β_j and are interchange unstable at low β_j . The predicted stability limits for broad and peaked pressure profile equilibria tend to coalesce at high β_j , where the steep pressure gradients of the broad profile equilibria at the edge of the plasma enhance their susceptibility to ballooning modes.

4. STABILITY OF ISX-B MODEL EQUILIBRIA

The principal goals of the ISX-B tokamak comprise high β studies, cross-section shaping, plasma fueling, and impurity studies. This device is a medium-sized tokamak (major radius $R_0 = 0.92$ m, minor radius $a = 0.27$ m) with a poloidal field system capable of generating noncircular plasmas (b/a up to 1.8) using a low toroidal field (B_T up to 1.8 T). To heat the plasma to high β , the system relies on two neutral beam injectors that can deliver up to 0.8-0.9 MW of power each at 40 keV, which exceeds the ohmic power input by a factor of 10 to 20.

The experimental results obtained so far have achieved $\beta > 1.8\%$ with 1.2 MW of beam power.¹ Some very interesting experimental observations can be interpreted as manifestations of pressure-driven mode activity.⁵² The pin diode detectors show a transition with increasing beam power from typical sawtooth oscillations, which develop near the magnetic axis, to high-frequency MHD activity (which masks any evidence of sawtoothing). The magnetic pick-up loops, on the other hand, show that the MHD activity with high-power injection can be interpreted as oscillations with large poloidal mode number m ($m \approx 14-15$), provided the modes rotate in the usual electron diamagnetic drift direction. The high poloidal m numbers imply the fluctuations have high toroidal n mode numbers if $q \approx 1$ where the instability is localized. In addition, a density clamping phenomenon was observed, which was correlated with the onset of the frequency MHD activity with high-power beam injection. These observations and the accompanying interpretations are consistent with the (theoretically predicted) most deleterious type of pressure-driven internal MHD activity at high β .⁵³ Alternatively, the oscillations from the pick-up loops could be explained as $m = 2$ activity if the neutral beams somehow reversed the normal rotation of the modes.

The typical ISX-B MHD equilibria with high-power neutral beam injection have extremely peaked pressure profiles owing to the beams. However, the overall beam contribution to the volume-averaged β is small, although it is comparable to the background plasma β component near the magnetic axis.

To generate model ISX-B equilibria we have to carefully tailor the background and beam pressure profiles to simultaneously match the experimentally determined contribution of the beam to β (β_b) and its contribution to β at the magnetic axis (β_{bo}); the contribution of the plasma background to β (β_p) and its contribution at the magnetic axis (β_{po}); and the ratios of β_b/β and β_{bo}/β_o , where β_o is the total β evaluated at the magnetic axis. We should not confuse β_p in this section with the poloidal beta defined in Sect. 2. In addition, we have to tailor the toroidal current density profile to match that of the total β_j , the beam contribution to β_j (β_{jb}) and the toroidal current I_T for a given toroidal magnetic field.

We choose the isotropic background pressure profile as

$$p_o(\psi) = S_c \left[\left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^{\ell_1} - \frac{\ell_1}{\ell_2} \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^{\ell_2} \right], \quad (4.1)$$

where the coefficient S_c is used to vary β and the powers ℓ_1 and ℓ_2 serve to adjust the peakedness of the background pressure profile to obtain the desired ratio of β_p to β_{po} with $\ell_2 > \ell_1$. The second term in Eq. (4.1) flattens $p_o(\psi)$ at the magnetic axis. The parallel component of the beam pressure profile is chosen as

$$p_{||}(\psi, B) = \alpha S_c \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^{\ell_3} B \sqrt{1 - u_e(\psi)B}, \quad (4.2)$$

where the magnetic moment profile $u_e(\psi)$ is given by Eq. (2.37) with injection angle $\theta_b = 45.573^\circ$; B is the magnitude of the magnetic field; and ψ , ψ_a , and ψ_b are the flux function, the flux function at the magnetic axis, and the flux function at the plasma boundary, respectively. We adjust the power ℓ_3 to obtain the desired ratio of β_b/β_{bo} and the coefficient α to get the desired ratio of β_b to β .

We also need to specify the function (σI) to solve the tensor pressure equilibrium equation. We choose

$$(\sigma I)^2 = 1 + 2 S_c \left[d_p \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^{\ell_1} + q_x \left(\frac{\psi - \psi_b}{\psi_a - \psi_b} \right)^{\ell_3} \right], \quad (4.3)$$

where d_p is used to vary β_j and q_x is adjusted to make $q \geq 1$ at the magnetic axis. The most accurate fits to the background and beam pressure profiles had $\ell_1 = 3$, $\ell_2 = 6$, and $\ell_3 = 5$.

Figure 4.1 shows a typical model ISX-B tensor pressure equilibrium of elongation 1.2:1, with $\beta = 1.44\%$, magnetic field $B_T = 1.12$ T, and current $I_T = 146$ kA. The toroidal current density j_ϕ is very peaked because the parallel beam pressure component and the isotropic background plasma pressure component are themselves very peaked. The boundary conditions for the equilibrium equation are given by Eq. (2.36), with $t_g = 0.8$ and $q_g = 0.2$. The implicit vertical fields contained within triangular deformation of the plasma prevent extreme shifts of the magnetic axis. This results in the D-shapedness of the most exterior flux surfaces, which are somewhat more pronounced than experimentally observed. For the purpose of determining stability properties, however, this is unimportant because of the very low pressure near the edge of the plasma. When we ignore the double adiabatic term, we find that the solution of the ballooning mode equation [Eq. (3.12)] for these equilibria with very peaked pressure profiles predicts the plasma to be unstable to interchange modes on the inner flux surfaces covering the region within the dashed curve in Fig. 4.1.

The beam-induced tensor pressure components given by Eq. (4.2) represent limiting anisotropies valid only when the slowing down of the fast ions on electrons dominates over the pitch angle scattering off the thermal ions. This occurs when the ratio $T_e/E_0 \ll 1/15$ (where T_e is the electron temperature and E_0 is the beam particle injection energy). In the limit $T_e/E_0 \gg 1/15$, on the other hand, the pitch angle scattering of the beam particles dominates over the slowing down, and the distribution function is approximately isotropic in velocity space. The ensuing beam contribution to the pressures therefore is isotropic

Fig. 4.1. Model ISX-B toroidal current density profile, background and beam pressure profiles, and flux surfaces with $\beta = 1.44\%$. (The unstable region is bounded by the dashed contour.)

in configurational space. Typically, $T_e/E_0 \approx 1/40$ for ISX-B, and so we treat both tensor and scalar pressure models to obtain predictions that bracket the true stability properties of the plasma.

We generate a scalar pressure equilibrium in Fig. 4.2 by adding an isotropic beam component to the polynomial in Eq. (4.1) to obtain a pressure profile comparable to the tensor pressure profiles shown in Fig. 4.1. The tensor and scalar pressure equilibria have almost identical β , β_j , q_{axis} , q_{edge} , I_T , and toroidal current density profiles. The solution of the ballooning mode equation shows that for very peaked pressure profiles, the region of instability (within the dashed curve) is virtually unchanged as compared with that obtained with tensor pressure.

Finally, in Fig. 4.3 we show how the stability of the ISX-B model tensor pressure equilibrium shown in Fig. 4.1, is affected by raising the toroidal field while maintaining the pressure and toroidal current density profiles fixed. The safety factor q at the axis increases with B_T , but the shear is comparable to the equilibrium in Fig. 4.1. The linear theory predicts that the discharge is extremely unstable in the central core of the plasma. Thus, when we raise B_T to 1.47 T to decrease β by a factor of 2, the region of instability is only imperceptibly diminished in size. This results because the neutral beams cause the pressures to peak very strongly at the magnetic axis.

In summary, the typical high- β ISX-B plasmas are theoretically predicted to be unstable to pressure-driven modes in the central region of the plasma as a result of the pressure profile peakedness induced by the neutral beams, whether the pressure is treated as a scalar or a tensor.

Fig. 4.2. Model ISX-B toroidal current density profile, scalar pressure profiles, and flux surfaces with $\beta = 1.44\%$. (The unstable region is bounded by the dashed contour.)

$\beta = 0.83\%$	$\beta_o = 5.21\%$	$\beta_J = 3.04$	$q_{axis} = 1.25$
$\beta_b = 0.23\%$	$\beta_{bo} = 2.61\%$	$\beta_{Jb} = 0.85$	$q_{edge} = 6.41$
$\beta_p = 0.60\%$	$\beta_{po} = 2.60\%$	$B_T = 1.47 \text{ T}$	$I_T = 145 \text{ kA}$

Fig. 4.3. Model ISX-B toroidal current density profile, background and beam pressure profiles, and flux surfaces with $\beta = 0.83\%$. (The unstable region is bounded by the dashed contour.)

5. SUMMARY AND CONCLUSIONS

In Sect. 2 we have concentrated on the effects of neutral beam injection into an axisymmetric torus of arbitrary cross section and aspect ratio and obtained the limiting tensor pressure moments from a beam distribution function that is a solution of the Fokker-Planck equation in the absence of pitch angle scattering. The mirror stability criterion precludes the consideration of trapped beam particles, because the discrete nature of the fast-ion pitch angles (when pitch angle scattering is ignored) implies that the beam particles have common turning points on their respective flux surfaces across which the perpendicular beam pressure component is discontinuous. In addition, we have shown general derivations for the tensor pressure equilibrium equation, the force balance relation, and the expressions for the poloidal beta and the current densities from the basic MHD equations, where the pressure is taken as a tensor with only diagonal elements.

We have generated D-shaped tensor pressure equilibria and found that the pressure contours and the flux surfaces nearly coincide if the pressure profiles are peaked. The perpendicular pressure surfaces are noticeably distorted and displaced inward with respect to every flux surface for broad pressure profile equilibria with injection at such angles that $p_{\perp}$ is significantly enhanced by the beam. For injection at angles more closely colinear to the magnetic field, the beam contribution to $p_{\perp}$ is negligible and the perpendicular pressure contours match the flux surfaces. The $p_{\parallel}$ contours of beam-induced tensor pressure equilibria with broad pressure profiles are shifted quite noticeably inward with respect to the flux surfaces, but only near the magnetic axis. We also found only minor modifications to the flux surface topology, with varying diamagnetism as a result of the boundary shaping function that was used.

In Sect. 3, we performed an asymptotic expansion (in large toroidal mode number n) and subsequent minimization of a modified version of the Kruskal-Oberman energy principle and applied the Fourier

transform technique to deal with periodicity conditions of the perturbation in a sheared magnetic field. We thus derived Euler equations that give the necessary as well as the sufficient criteria for the stability of a guiding center plasma to oscillations of high toroidal mode number. In this version of the energy principle, we replaced the kinetic integral (which reduces to the plasma compression term in ideal MHD) with a double adiabatic contribution to obtain the necessary stability equation. When we disregarded altogether the kinetic integral (which is positive definite for monotonic distribution functions), we obtained an equation that predicts sufficient stability criteria for a guiding center plasma.

We then applied these ballooning mode equations to beam-induced tensor pressure equilibria and found that because of the double adiabatic term, which inhibits any perturbations where the pressures are large, the difference between the necessary and the sufficient criteria is significant for plasmas susceptible to flutelike modes that appear near the magnetic axis. The stability limits coalesce when ballooning modes near the edge of the plasma (where the pressure is small) dominate. The enhanced stability of tensor pressure equilibria induced by perpendicular beam injection over those induced by parallel beam injection and over scalar pressure equilibria results from the inward shift of the $p_{\perp}$ surfaces away from the destabilizing curvature region. The weak pressure gradients of broad pressure profile equilibria in the central core of the plasma improve their stability to flutelike modes, and the critical β limits are thus markedly higher than those predicted for peaked pressure profile equilibria at low values of β_j . At high β_j , the stability criteria of peaked and broad pressure profile equilibria tend to coalesce, because they are dominated by ballooning modes driven by the steep pressure gradients near the edge of the plasma.

Finally, in Sect. 4, we found that the theoretical predictions obtained for ISX-B model equilibria indicate that the device is unstable to pressure-driven modes in the central core of the plasma as a result of the pressure profile peakedness induced by the high-power neutral beam injection.

In conclusion, we have obtained four new results in this work. First, we derived an equation from a double adiabatic energy principle by which we obtain a necessary condition for the stability of the guiding center plasma to perturbations of large toroidal mode number. This necessary condition is then used in conjunction with the sufficient condition derived by Fielding and Haas²⁵ to bracket the actual stability properties of an anisotropic pressure plasma. Second, we found that the perpendicular injection of beams into broad pressure profile equilibria produces the largest inward shift of the $p_{\perp}$ contours (with respect to the flux surfaces) and ensuing enhanced stability. Third, we have presented detailed, self-consistent, numerically determined equilibrium and stability results for a neutral-beam-injected model with $p_{\parallel} \neq p_{\perp}$ in D-shaped tokamaks with realistic aspect ratios. Fourth, we have obtained stability predictions for equilibria that model ISX-B measured profiles with high-power beam injection.

Appendix A

THE FORCE BALANCE RELATION

The force balance relation is obtained by first integrating the left-hand side of Eq. (2.19) by parts to get

$$\int_{\psi_a}^{\psi_b} d\psi \frac{dV}{d\psi} \int_{\psi}^{\psi_b} \frac{du}{V'(\psi)} \frac{\partial}{\partial u} [V'(u) \langle \sigma B_X^2 \rangle_u] = \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V'(\psi)} \left[V''(\psi) \langle \sigma B_X^2 \rangle_{\psi} + V'(\psi) \frac{\partial}{\partial \psi} \langle \sigma B_X^2 \rangle_{\psi} \right],$$

where we have made use of the fact that $V(\psi_a) = 0$. The right-hand side of this equation can be written as

$$\frac{1}{2} \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V'^2(\psi)} \left[2V'(\psi)V''(\psi) + V'^2(\psi) \frac{\partial}{\partial \psi} \langle \sigma B_X^2 \rangle_{\psi} \right] + \frac{1}{2} \int_{\psi_a}^{\psi_b} d\psi V(\psi) \frac{\partial}{\partial \psi} \langle \sigma B_X^2 \rangle_{\psi}.$$

An integration by parts on the last term transforms the expression to

$$\frac{1}{2} \int_{\psi_a}^{\psi_b} d\psi \frac{V^2(\psi)}{V'^2(\psi)} \left\{ \frac{1}{V(\psi)} \frac{\partial}{\partial \psi} [V'^2(\psi) \langle \sigma B_X^2 \rangle_{\psi}] - \frac{V'^2(\psi)}{V^2(\psi)} V'(\psi) \langle \sigma B_X^2 \rangle_{\psi} \right\} + \frac{V(\psi_b)}{2} \langle \sigma B_X^2 \rangle_{\psi_b},$$

so that after some algebraic manipulation, we obtain

$$\begin{aligned} \int_{\psi_a}^{\psi_b} d\psi \frac{dV}{d\psi} \int_{\psi}^{\psi_b} \frac{du}{V'(\psi)} \frac{\partial}{\partial u} [V'(u) \langle \sigma B_X^2 \rangle_u] &= \frac{1}{2} V(\psi_b) \langle \sigma B_X^2 \rangle_{\psi_b} \\ &+ \frac{1}{2} \int_{\psi_a}^{\psi_b} d\psi V(\psi) \langle \sigma B_X^2 \rangle_{\psi} \frac{\partial}{\partial \psi} \left\{ \ln \left[\frac{V'^2(\psi) \langle \sigma B_X^2 \rangle_{\psi}}{V(\psi)} \right] \right\}. \end{aligned} \quad (\text{A.1})$$

The integration of the right-hand side of Eq. (2.19) is considerably more trivial and the force balance relation can be written as:

$$\begin{aligned} \frac{1}{3} [\overline{P_{\parallel}(\psi) - P_{\parallel}(\psi_b) + 2P_{\perp}(\psi) - 2P_{\perp}(\psi_b)}] &= \frac{1}{2} \langle \sigma B_{\chi}^2 \rangle_{\psi_b} + \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V(\psi_b)} \left(\langle \frac{1}{\sigma R^2} \rangle_{\psi} \frac{\partial}{\partial \psi} (\sigma I)^2 \right. \\ &\quad \left. + \langle \sigma B_{\chi}^2 \rangle_{\psi} \frac{\partial}{\partial \psi} \left\{ \ln \left[\frac{V'^2(\psi) \langle \sigma B_{\chi}^2 \rangle_{\psi}}{V(\psi)} \right] \right\} - \frac{8}{3} \langle B^2 \frac{\partial \sigma}{\partial \psi} \rangle_{\psi} + \frac{1}{3} \langle \frac{P_{\parallel} - P_{\perp}}{B^2} \frac{\partial B^2}{\partial \psi} \rangle_{\psi} \right), \quad (A.2) \end{aligned}$$

where we have defined

$$[\overline{P_{\parallel}(\psi) - P_{\parallel}(\psi_b) + 2P_{\perp}(\psi) - 2P_{\perp}(\psi_b)}] = - \frac{\int_{\psi_a}^{\psi_b} d\psi V'(\psi) \int_{\psi}^{\psi_b} du \langle \frac{\partial}{\partial u} (p_{\parallel} + 2p_{\perp}) \rangle_u}{V(\psi_b)}. \quad (A.3)$$

Appendix B

THE CURRENT DENSITIES

When we express Ampere's law in flux coordinates, it shows the toroidal current density to be

$$j_{\phi} = \frac{R}{J} \frac{\partial}{\partial \psi} (JB^2_{\chi}) .$$

Using the equilibrium equation, Eq. (2.12), we can easily write the toroidal current density as

$$j_{\phi} = -\frac{R}{\sigma} \left(\frac{\partial p_{\perp}}{\partial \psi} \right)_{\chi} + \frac{p_{\parallel} - p_{\perp}}{2B^2} \frac{\partial B^2}{\partial \psi} - \frac{I}{R} \frac{\partial I}{\partial \psi} . \quad (B.1a)$$

If we express the pressures as functions of ψ and B , using the transformations described in Appendix G we find

$$j_{\phi} = -\frac{I^2}{\sigma R B^2} \frac{\partial p_{\parallel}}{\partial \psi} \Big|_B - \frac{R B^2_{\chi}}{\sigma B^2} \Big|_B - \frac{(\nabla \psi \cdot \nabla) B^2}{2\sigma R B^3} \frac{\partial}{\partial B} (p_{\parallel} + p_{\perp}) - \frac{1}{2\sigma^2 R} \frac{\partial}{\partial \psi} (\sigma I)^2 . \quad (B.1b)$$

From Ampere's law we also get for the poloidal current density

$$j_{\chi} = -B_{\chi} \frac{\partial I}{\partial \psi} , \quad (B.2a)$$

which in (ψ, B) space can be written as

$$j_{\chi} = -\frac{B_{\chi}}{\sigma} \frac{\partial}{\partial \psi}(\sigma I) - \frac{B_{\chi} I}{\sigma B^2} \frac{\partial}{\partial \psi}(p_{\parallel} - p_{\perp}) \Big|_B + \frac{I(\vec{\nabla} \psi \cdot \vec{\nabla}) B^2}{2\sigma R^2 B_{\chi} B^3} \frac{\partial}{\partial B}(p_{\parallel} + p_{\perp}) \quad , \quad (B.2b)$$

and for the radial current density

$$j_{\psi} = -\frac{B_{\phi}}{\sigma B^2 J B_{\chi}} \frac{\partial}{\partial \chi}(p_{\parallel} + p_{\perp}) \quad , \quad (B.3a)$$

and

$$j_{\psi} = -\frac{I(\vec{\nabla} \psi \cdot \vec{\nabla}) B^2}{2\sigma R B_{\chi} B^3} \frac{\partial}{\partial B}(p_{\parallel} + p_{\perp}) \quad . \quad (B.3b)$$

It is immediately noted that for scalar pressure plasmas, $j_{\psi} = 0$. The radial or normal current density in tensor pressure, axisymmetric toroidal devices is driven by the variation of the pressures along the poloidal angle on a flux surface; when the pressure is scalar, it is independent of the poloidal angle, which implies $j_{\psi} = 0$.

For the parallel current $\vec{j}_{\parallel} = (\vec{j} \cdot \vec{B}/B^2)\vec{B}$, we use Eqs. (B.1a) and (B.2a) to get

$$\vec{j}_{\parallel} = - \left[\left(\frac{\partial p_{\perp}}{\partial \psi} + \frac{p_{\parallel} - p_{\perp}}{2B^2} \frac{\partial B^2}{\partial \psi} \right) \frac{I}{\sigma B^2} + \frac{\partial I}{\partial \psi} \right] \vec{B} \quad , \quad (B.4a)$$

and we use Eqs. (B.1b) and (B.2b) to get

$$\vec{j}_{\parallel} = - \left[\frac{I}{\sigma B^2} \frac{\partial p_{\parallel}}{\partial \psi} \Big|_B + \frac{1}{\sigma} \frac{\partial}{\partial \psi}(\sigma I) \right] \vec{B} \quad . \quad (B.4b)$$

The perpendicular current density, obtained from $\vec{j} - \vec{j}_{\parallel}$, is

$$\vec{j}_{\perp} = - \left(\frac{\partial p_{\perp}}{\partial \psi} + \frac{p_{\parallel} - p_{\perp}}{2B^2} \frac{\partial B^2}{\partial \psi} \right) \frac{B_{\chi} (RB_{\chi} \hat{e}_{\phi} - I \hat{e}_{\chi})}{\sigma B^2}, \quad (B.5a)$$

and in (ψ, B) space it is

$$\vec{j}_{\perp} = - \frac{1}{\sigma} \left[\frac{\partial p_{\perp}}{\partial \psi} \right]_B + \frac{(\vec{\nabla} \psi \cdot \vec{\nabla}) B^2}{2\sigma R^2 B_{\chi}^2} \frac{\partial}{\partial B} (p_{\parallel} + p_{\perp}) \left] \frac{B_{\chi} (RB_{\chi} \hat{e}_{\phi} - I \hat{e}_{\chi})}{\sigma B^2}. \quad (B.5b)$$

The poloidal current density is unaltered from the case of scalar pressure. The anisotropy, however, modifies the toroidal current density as well as the parallel and perpendicular current densities. The extent of this modification consists of replacing $p'(\psi)$ of the scalar pressure case by

$$\frac{1}{\sigma} \left[\frac{\partial p_{\perp}}{\partial \psi} + \frac{(p_{\parallel} - p_{\perp})}{2B^2} \frac{\partial B^2}{\partial \psi} \right]$$

wherever it appears, which, as can be seen, reduces to the appropriate term when $p_{\parallel} = p_{\perp}$. As mentioned above, the anisotropy introduces a normal current density as well.

-110-

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Appendix C

THE DOUBLE ADIABATIC ENERGY PRINCIPLE

The double adiabatic energy principle is³³

$$\begin{aligned}
 \delta W_{DA} = & \frac{1}{2} \int d^3x \left(Q^2 + \vec{j} \cdot (\vec{\xi} \times \vec{Q}) + (\vec{\xi} \cdot \vec{\nabla}) p_{\perp} (\vec{\nabla} \cdot \vec{\xi}) \right. \\
 & + p_{\perp} [2(\vec{\nabla} \cdot \vec{\xi})^2 - 4(\vec{\nabla} \cdot \vec{\xi})q + 2q^2] - C (q - \vec{\nabla} \cdot \vec{\xi})^2 \\
 & + (p_{\parallel} - p_{\perp}) \{ -(\vec{\xi} \cdot \vec{\nabla})q + q^2 - \hat{b} \cdot [(\hat{b} \cdot \vec{\nabla})\vec{\xi} \cdot \vec{\nabla}]\vec{\xi} \\
 & + \hat{b} \cdot [(\vec{\xi} \cdot \vec{\nabla})\hat{b} \cdot \vec{\nabla}]\vec{\xi} - [(\hat{b} \cdot \vec{\nabla})\vec{\xi}]^2 + (\hat{b} \cdot \vec{\nabla})\vec{\xi} \cdot (\vec{\xi} \cdot \vec{\nabla})\hat{b} \} \quad (C.1) \\
 & \left. + \frac{1}{2} \int d^3x \{ p_{\perp} [2(\vec{\nabla} \cdot \vec{\xi})q + q^2] + 3(p_{\parallel} - p_{\perp}) q^2 + C (q - \vec{\nabla} \cdot \vec{\xi})^2 \} \right),
 \end{aligned}$$

where the first integral is δW_U and the second is δW_{DAC} and where $\hat{b}$ is the unit vector along the field lines, ξ is the perturbation, $\vec{Q} = \vec{\nabla} \times (\vec{\xi} \times \vec{B})$ is the perturbed magnetic field, and $q = \hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}$.

The next procedure is to try to obtain δW_{DA} in a form that is more concise and transparent. We closely follow the derivation in previous publications.^{54,55} To this effect we use such relations as

$$\vec{Q}_{\parallel} = \vec{B} \hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}_{\perp} - \hat{b} (\vec{\xi}_{\perp} \cdot \vec{\nabla})B - \vec{B} (\vec{\nabla} \cdot \vec{\xi}_{\perp}) ,$$

$$\frac{\vec{Q}_{\perp}}{B} = (\hat{b} \cdot \vec{\nabla})\vec{\xi}_{\perp} - (\vec{\xi}_{\perp} \cdot \vec{\nabla})\hat{b} - \hat{b} [\hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}_{\perp}] ,$$

and we expand $\vec{j} = j_{\parallel} \hat{b} + \vec{j}_{\perp}$ where $\vec{j}_{\perp} = [\vec{B} \times (\vec{\nabla} \cdot \vec{P})/B^2]$ ($\vec{P}$ is the pressure tensor), and note that $\hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}_{\perp} = -\xi_{\perp} \cdot \vec{\kappa}$. With this taken into account, the double adiabatic energy principle can be rewritten in the intermediate form

$$\begin{aligned}
 \delta W_{DA} = & \frac{1}{2} \int d^3x \left[\sigma Q_{\perp}^2 + Q_{\parallel}^2 + \sigma j_{\parallel} \hat{b} \cdot (\xi_{\perp} \times \vec{Q}_{\perp}) - C (\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k})^2 \right. \\
 & + (\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k} - \frac{Q_{\parallel}}{B}) (\xi_{\perp} \cdot \vec{\nabla})_{P_{\perp}} - (\xi_{\perp} \cdot \vec{k}) (\xi_{\perp} \cdot \vec{\nabla})_{P_{\parallel}} \\
 & \left. + 2p_{\perp} (\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k})^2 - (p_{\parallel} - p_{\perp}) (\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k} + \frac{Q_{\parallel}}{B}) (\xi_{\perp} \cdot \vec{k}) \right] \\
 & + \delta W_{DAC}
 \end{aligned} \tag{C.2}$$

Expanding $Q_{\parallel}$ in the form shown above and then making use of the equilibrium relation

$$\vec{\nabla} \left(p_{\perp} + \frac{B^2}{2} \right) = \hat{b} (\hat{b} \cdot \vec{\nabla}) \left(p_{\perp} + \frac{B^2}{2} \right) + \sigma B^2 \vec{k}, \tag{C.3}$$

where $\sigma = 1 - \frac{p_{\parallel} - p_{\perp}}{B^2}$ and $\vec{k} = (\hat{b} \cdot \vec{\nabla}) \hat{b}$ is the magnetic field curvature, we can get an expression for $(\xi_{\perp} \cdot \vec{\nabla}) B^2$. Then the energy principle reduces to

$$\begin{aligned}
 \delta W_{DA} = & \frac{1}{2} \int d^3x \left\{ \sigma Q_{\perp}^2 + \sigma j_{\parallel} \hat{b} \cdot (\xi_{\perp} \times \vec{Q}_{\perp}) + \sigma B^2 \left(1 - \frac{\sigma}{\tau_0} \right) (\xi_{\perp} \cdot \vec{k})^2 - C (\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k})^2 \right. \\
 & \left. + \tau_0 B^2 \left[(\vec{\nabla} \cdot \xi_{\perp} + (1 + \frac{\sigma}{\tau_0}) (\xi_{\perp} \cdot \vec{k}))^2 - (\xi_{\perp} \cdot \vec{k}) (\xi_{\perp} \cdot \vec{\nabla}) (p_{\parallel} + p_{\perp}) \right] \right\} \\
 & + \delta W_{DAC},
 \end{aligned} \tag{C.4}$$

$$\text{where } \tau_0 \equiv 1 + \frac{2p_{\perp}}{B^2}.$$

We now examine the double adiabatic correction δW_{DAC} and break up the perturbation into parallel and perpendicular components to the magnetic field,

$$\begin{aligned}
 \delta W_{\text{DAC}} = \frac{1}{2} \int d^3x \{ & -2p_{\perp}(\xi_{\perp} \cdot \vec{k})(\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k}) + 3p_{\parallel}(\xi_{\perp} \cdot \vec{k})^2 + C(\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k})^2 \\
 & + 2(\hat{b} \cdot \vec{\nabla})\xi_{\parallel} [p_{\perp}(\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k}) - 3p_{\parallel}(\xi_{\perp} \cdot \vec{k}) + p_{\perp}\xi_{\parallel}(\vec{\nabla} \cdot \hat{b})] \\
 & + C(\vec{\nabla} \cdot \hat{b})^2 \xi_{\parallel}^2 + 2\xi_{\parallel}(\vec{\nabla} \cdot \hat{b}) [C(\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k}) - p_{\perp}(\xi_{\perp} \cdot \vec{k})] \\
 & + 3p_{\parallel}[(\hat{b} \cdot \vec{\nabla})\xi_{\parallel}]^2 \} \quad . \quad (C.5)
 \end{aligned}$$

We note that the term $C(\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k})^2$ in δW_{DAC} and Eq. (C.4) will cancel each other out. We now complete the square on the term $(\hat{b} \cdot \vec{\nabla})\xi_{\parallel}$ and then complete the square on $\xi_{\parallel}(\vec{\nabla} \cdot \hat{b})$ in δW_{DAC} so that

$$\begin{aligned}
 \delta W_{\text{DAC}} = \frac{1}{2} \int d^3x \{ & -2p_{\perp}(\xi_{\perp} \cdot \vec{k})(\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k}) + 3p_{\parallel}(\xi_{\perp} \cdot \vec{k})^2 + C(\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k})^2 \\
 & - \frac{p_{\perp}^2}{3p_{\parallel}} [\vec{\nabla} \cdot \xi_{\perp} + (1 - \frac{3p_{\parallel}}{p_{\perp}})(\xi_{\perp} \cdot \vec{k})]^2 \quad (C.6) \\
 & + \frac{p_{\perp}^2}{3p_{\parallel}} [\vec{\nabla} \cdot \xi_{\perp} + (1 - \frac{3p_{\parallel}}{p_{\perp}})(\xi_{\perp} \cdot \vec{k}) + \xi_{\parallel}(\vec{\nabla} \cdot \hat{b}) + \frac{3p_{\parallel}}{p_{\perp}}(\hat{b} \cdot \vec{\nabla})\xi_{\parallel}]^2 \\
 & + (C - \frac{p_{\perp}^2}{3p_{\parallel}}) \{ [\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k} + \xi_{\parallel}(\vec{\nabla} \cdot \hat{b})]^2 - (\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{k})^2 \} \} .
 \end{aligned}$$

The first four terms in δW_{DAC} only contain components of the perturbation perpendicular to the magnetic field. After combining these with the second, third, and last terms in Eq. (C.4), the double adiabatic energy principle can be expressed as

$$\delta W_{DA} = \frac{1}{2} \int d^3x \left(\sigma Q_{\perp}^2 + \sigma j_{\parallel} \hat{b} \cdot (\xi_{\perp} \times \vec{Q}_{\perp}) + \sigma B^2 \left(1 - \frac{\sigma}{\tau_d}\right) (\xi_{\perp} \cdot \vec{r})^2 \right) \quad (C.7)$$

$$+ \tau_d B^2 \left[\vec{\nabla} \cdot \xi_{\perp} + \left(1 + \frac{\sigma}{\tau_d}\right) (\xi_{\perp} \cdot \vec{r}) \right]^2 - (\xi_{\perp} \cdot \vec{r}) (\xi_{\perp} \cdot \vec{\nabla}) (p_{\parallel} + p_{\perp})$$

$$+ \frac{p_{\perp}^2}{3p_{\parallel}} \left[\vec{\nabla} \cdot \xi_{\perp} + \left(1 - \frac{3p_{\parallel}}{p_{\perp}}\right) (\xi_{\perp} \cdot \vec{r}) + \xi_{\parallel} (\vec{\nabla} \cdot \hat{b}) + \frac{3p_{\parallel}}{p_{\perp}} (\hat{b} \cdot \vec{\nabla}) \xi_{\parallel} \right]^2$$

$$+ \left(C - \frac{p_{\perp}^2}{3p_{\parallel}}\right) \left\{ \left[\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{r} + \xi_{\parallel} (\vec{\nabla} \cdot \hat{b}) \right]^2 - (\vec{\nabla} \cdot \xi_{\perp} + \xi_{\perp} \cdot \vec{r})^2 \right\},$$

where $\tau_d \equiv 1 + \frac{2p_{\perp} - C_1}{B^2}$ and we identify $C = C_{\eta} = \frac{\eta p_{\perp}^2}{3p_{\parallel}}$ so that $C_1 = p_{\perp}^2/3p_{\parallel}$.

Appendix D

THE ENERGY PRINCIPLE IN AXISYMMETRIC TORI

We now expand the different terms appearing in δW_{DA} in an orthogonal (ϕ, ψ, χ) flux coordinate system with the metric $(dl)^2 = (Rd\phi)^2 + (d\psi/RB_\chi)^2 + (JB_\chi d\chi)^2$, where J is the Jacobian. Then

$$Q_1^2 = \frac{1}{J^2 R^2 B_\chi^2} \left(\frac{\partial X}{\partial \phi} + v \frac{\partial X}{\partial \phi} \right)^2 + \frac{R^2 B_\chi^2}{J^2 B^2} \left(\frac{\partial U}{\partial \chi} + v \frac{\partial U}{\partial \phi} - X \frac{\partial v}{\partial \psi} \right)^2 .$$

Here $X \equiv \vec{\xi}_\perp \cdot \vec{\nabla} \psi$ is related to the component of the perturbation normal to the flux surface; $U \equiv [\vec{\xi}_\perp \cdot (\vec{\nabla} \psi \times \vec{B})] / |\vec{\nabla} \psi|^2$ is the normalized component of the perturbation that lies on a flux surface in the geodesic direction; and $v \equiv JB_\phi/R$, where the safety factor $q(\psi) = (1/2\pi) \oint v d\chi$.

The divergence of the perturbation in this geometry is

$$\vec{\nabla} \cdot \vec{\xi}_\perp = \frac{R_\chi^2}{B^2} \frac{\partial U}{\partial \phi} + \frac{1}{J} \frac{\partial}{\partial \psi} (JX) - \frac{1}{J} \frac{\partial}{\partial \chi} \left(\frac{RB_\phi}{B^2} U \right) ,$$

and

$$\vec{\xi}_\perp \cdot \vec{\kappa} = \frac{\kappa_\psi}{RB_\chi} X - \frac{RB_\chi \kappa_s}{B} U ,$$

where $\kappa_\psi = (\vec{\nabla} \psi \cdot \vec{\kappa}) / |\vec{\nabla} \psi|$ gives the normal curvature and $\kappa_s = [(\vec{B} \times \vec{\nabla} \psi) \cdot \vec{\kappa}] / (B |\vec{\nabla} \psi|)$ gives the geodesic curvature.

For the kink term we get

$$\hat{b} \cdot (\vec{\xi}_1 \times \vec{Q}_1) = \frac{U}{JB} \left(\frac{\partial X}{\partial \chi} + v \frac{\partial X}{\partial \phi} \right) - \frac{X}{JB} \left(\frac{\partial U}{\partial \chi} + v \frac{\partial U}{\partial \phi} \right) + \frac{X^2}{JB} \frac{\partial v}{\partial \psi} \quad ,$$

and for the driving term,

$$(\vec{\xi}_1 \cdot \vec{\nabla})(p_{\parallel} + p_{\perp}) = X \frac{\partial}{\partial \psi}(p_{\parallel} + p_{\perp}) - \frac{RB\phi}{JB^2} U \frac{\partial}{\partial \chi}(p_{\parallel} + p_{\perp}) \quad .$$

As a consequence, the guiding center fluid energy principle is

$$\begin{aligned} \delta W_{DA} = \frac{1}{2} \int d^3x \{ & \frac{\sigma}{J^2 R^2 B^2 \chi} \left(\frac{\partial X}{\partial \psi} + v \frac{\partial X}{\partial \phi} \right)^2 + \frac{\sigma R^2 B^2 \chi}{J^2 B^2} \left(\frac{\partial U}{\partial \chi} + v \frac{\partial U}{\partial \phi} - X \frac{\partial v}{\partial \psi} \right)^2 \\ & + \tau_U B^2 \left[\frac{B^2 \chi}{R^2} \frac{\partial U}{\partial \phi} + \left(\frac{1}{J} \right) \frac{\partial}{\partial \psi}(JX) - \left(\frac{1}{J} \right) \frac{\partial}{\partial \chi} \left(\frac{RB\phi}{R^2} U \right) + \left(1 + \frac{\gamma}{\tau_0} \right) \left(\frac{\kappa \psi}{RB\chi} X - \frac{RB\chi \kappa}{B} U \right) \right]^2 \\ & + \sigma B^2 \left(1 + \frac{3\Gamma_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_0} \right) \left(\frac{\kappa \psi}{RB\chi} X - \frac{RB\chi \kappa}{B} U \right)^2 \\ & + \sigma j_{\parallel} \left[\frac{U}{JB} \left(\frac{\partial X}{\partial \chi} + v \frac{\partial X}{\partial \phi} \right) - \frac{X}{JB} \left(\frac{\partial U}{\partial \chi} + v \frac{\partial U}{\partial \phi} \right) + \frac{X^2}{JB} \frac{\partial v}{\partial \psi} \right]^2 \\ & - \left[X \frac{\partial}{\partial \psi}(p_{\parallel} + p_{\perp}) - \frac{RB\phi}{JB^2} U \frac{\partial}{\partial \chi}(p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa \psi}{RB\chi} X - \frac{RB\chi \kappa}{B} U \right) \} \quad . \end{aligned} \quad (D.1)$$

Because the system is axisymmetric, the different toroidal Fourier components of the perturbation are uncoupled. Therefore, we need only consider a single Fourier mode $\exp(-in\phi)$ of the displacement $\vec{\xi}_1$, where ϕ is the toroidal angle and n is the mode number. The energy principle then becomes

$$\begin{aligned}
\delta W_{DA} = \frac{1}{2} \int d^3x \{ & \frac{\sigma}{J^2 R^2 B^2} \left| \frac{\partial X}{\partial \psi} + \text{inv} X \right|^2 + \frac{\sigma R^2 B^2}{J^2 B^2} \left| \frac{\partial U}{\partial \chi} + \text{inv} U - X \frac{\partial v}{\partial \psi} \right|^2 \\
& + \tau_0 B^2 \left| \text{inv} \frac{B^2}{B^2} X U + \left(\frac{1}{J} \right) \frac{\partial}{\partial \psi} (JX) - \left(\frac{1}{J} \right) \frac{\partial}{\partial \chi} \left(\frac{RB}{B^2} \phi U \right) + \left(1 + \frac{\gamma}{\tau_0} \right) \left(\frac{\kappa \psi}{RB} X - \frac{RB}{B} \frac{\kappa}{s} U \right) \right|^2 \\
& + \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_0} \right) \left| \frac{\kappa \psi}{RB} X - \frac{RB}{B} \frac{\kappa}{s} U \right|^2 \quad (D.2) \\
& + \frac{\sigma j_{\parallel}}{2} \left[\frac{U^*}{JB} \left(\frac{\partial X}{\partial \chi} + \text{inv} X \right) - \frac{X^*}{JB} \left(\frac{\partial U}{\partial \chi} + \text{inv} U \right) + \frac{2|X|^2}{JB} \frac{\partial v}{\partial \psi} \right. \\
& \quad \left. + \frac{U}{JB} \left(\frac{\partial X^*}{\partial \chi} + \text{inv} X^* \right) - \frac{X}{JB} \left(\frac{\partial U^*}{\partial \chi} + \text{inv} U^* \right) \right] \\
& - \frac{1}{2} \left[X^* \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{RB}{JB^2} \phi U^* \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa \psi}{RB} X - \frac{RB}{B} \frac{\kappa}{s} U \right) \\
& - \frac{1}{2} \left[X \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{RB}{JB^2} \phi U \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa \psi}{RB} X^* - \frac{RB}{B} \frac{\kappa}{s} U^* \right) \} ,
\end{aligned}$$

where X^* and U^* are the complex conjugates of X and U , respectively. The first and second terms in Eq. (D.2) correspond to the field bending, the third is the field compression, the fourth is the anisotropic double adiabatic contribution, the fifth is the kink instability driving term, and the last two are drive ballooning and flute instabilities.

At this point, we note that the derivatives of the displacements X and U along the field lines are given by

$$(\vec{B} \cdot \vec{\nabla})X = \frac{1}{J} \left(\frac{\partial X}{\partial \chi} + \text{inv} X \right)$$

and

$$(\vec{B} \cdot \vec{\nabla})U = \frac{1}{J} \left(\frac{\partial U}{\partial \chi} + \text{inv} U \right) \quad (D.3)$$

-118-

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Appendix E

MINIMIZATION OF THE ENERGY PRINCIPLE

HIGH n PERTURBATIONS AND THE KINK TERM

We now consider only perturbations of large toroidal mode number n and expand the perturbations in powers of $(1/n)$.¹⁹ That is, we take $X = X_0 + \frac{1}{n} X_1 + \dots$ and $U = U_0 + \frac{1}{n} U_1 + \dots$. The double adiabatic energy principle [Eq. (D.2)] to lowest order in the $(1/n)$ expansion is:

$$\delta W_{DA} = \frac{1}{2} \int d^3x \, n^2 \left\{ \frac{\sigma}{J^2 R^2 B_X^2} |v X_0|^2 + \frac{\sigma R^2 B_X^2}{J^2 B^2} |v U_0|^2 + \tau_0 B^2 \left| \frac{B_X}{B} U_0 \right|^2 \right\},$$

which has contributions only from the field bending and the field compression. This becomes large and positive and, therefore, predicts stability in the limit $n \rightarrow \infty$ unless (1) the gradient along the field lines of the perturbation is of order unity, i.e.,

$$(\vec{B} \cdot \vec{\nabla}) X_0 \sim O(1) \quad ,$$

$$(\vec{B} \cdot \vec{\nabla}) U_0 \sim O(1) \quad ,$$

and (2) the divergence of the lowest-order component of the perturbation is also of order unity. An equivalent statement is that the perturbation is incompressible to lowest order. Then,

$$\vec{\nabla} \cdot \vec{\xi}_{10} = \ln \frac{B_X^2}{B^2} U_0 + \frac{1}{J} \frac{\partial}{\partial \psi} (J X_0) - \frac{1}{J} \frac{\partial}{\partial X} \left(\frac{R B}{R^2} U_0 \right) \sim O(1) \quad .$$

This can be put into the form

$$inU_o + \frac{\partial X_o}{\partial \psi} - \frac{RB\phi}{JB^2} \left(\frac{\partial U_o}{\partial \chi} + in\nu U_o \right) - \frac{U_o}{J} \frac{\partial}{\partial \chi} \left(\frac{RB\phi}{B^2} \right) + \frac{X_o}{J} \frac{\partial J}{\partial \psi} \sim 0(1) .$$

Knowing that $(\vec{B} \cdot \vec{\nabla})U_o$ is of order unity, it is clear that if $\vec{\nabla} \cdot \vec{\xi}_{10}$ is also to be of order unity, we must have to lowest order in $(1/n)$

$$U_o = \frac{i}{n} \frac{\partial X_o}{\partial \psi} . \quad (E.1)$$

Using this relation, we can write the kink contribution to the energy principle, which we shall call δW_J ,

$$\begin{aligned} \delta W_J = \frac{1}{2} \int d^3x \frac{\sigma j_{\parallel}}{2JB} \{ & - \frac{i}{n} \frac{\partial X_o^*}{\partial \psi} \left(\frac{\partial X_o}{\partial \chi} + in\nu X_o \right) - \frac{iX_o^*}{n} \left[\frac{\partial}{\partial \chi} \left(\frac{\partial X_o}{\partial \psi} \right) + in\nu \frac{\partial X_o}{\partial \psi} \right] + 2|X_o|^2 \frac{\partial \nu}{\partial \psi} \\ & + \frac{i}{n} \frac{\partial X_o}{\partial \psi} \left(\frac{\partial X_o^*}{\partial \chi} - in\nu X_o^* \right) + \frac{iX_o}{n} \left[\frac{\partial}{\partial \chi} \left(\frac{\partial X_o^*}{\partial \psi} \right) - in\nu \frac{\partial X_o^*}{\partial \psi} \right] \} . \end{aligned}$$

By algebraic manipulations, this expression can be reduced to

$$\delta W_J = \frac{i}{2} \int d\phi d\chi \int d\psi \frac{\sigma j_{\parallel}}{2nB} \frac{\partial}{\partial \psi} \left[X_o \left(\frac{\partial X_o^*}{\partial \chi} - in\nu X_o^* \right) - X_o^* \left(\frac{\partial X_o}{\partial \chi} + in\nu X_o \right) \right] .$$

An integration by parts over ψ yields

$$\delta W_J = - \frac{i}{2} \int d^3x \frac{\partial}{\partial \psi} \left(\frac{\sigma j_{\parallel}}{2nB} \right) [X_o (\vec{B} \cdot \vec{\nabla}) X_o^* - X_o^* (\vec{B} \cdot \vec{\nabla}) X_o] . \quad (E.2)$$

We immediately note that δW_J is of order $(1/n)$ because the derivatives of equilibrium quantities with respect to ψ and χ are of order unity. Since we will only keep terms of $O(1)$ in the energy principle, the driving term for kink instabilities will be neglected because it is of higher order and vanishes in the limit $n \rightarrow \infty$. The implication of this result is that current-driven instabilities do not display much structure in the ignorable direction of an axisymmetric toroidal device. This survival of an order $(1/n)$ contribution from the kink term depends on the particular form of the energy principle. In a different form, the kink term can be made to vanish exactly in a large n expansion.

The energy principle to order unity in the $(1/n)$ expansion is

$$\begin{aligned}
 \delta W_{DA} = \frac{1}{2} \int d^3x \{ & \frac{\sigma}{J^2 R^2 B_\chi^2} \left| \frac{\partial X_o}{\partial \chi} + i n v X_o \right|^2 + \frac{\sigma R^2 B_\chi^2}{J^2 B^2} \left| \frac{\partial U_o}{\partial \chi} + i n v U_o - X_o \frac{\partial v}{\partial \psi} \right|^2 \\
 & + \tau_o B^2 \left| i U_1 + \frac{1}{n} \frac{\partial X_1}{\partial \psi} - \frac{1}{J} \left[\frac{\partial}{\partial \chi} \left(\frac{R B_\chi}{B^2} U_o \right) + i n v \frac{R B_\chi}{B^2} U_o \right] + \frac{X_o}{J} \frac{\partial J}{\partial \psi} \right. \\
 & \left. + \left(1 + \frac{\gamma}{\tau_o} \right) \left(\frac{\kappa_\psi}{R B_\chi} X_o - \frac{R B_\chi \kappa_s}{B} U_o \right) \right|^2 \\
 & + \sigma B^2 \left(1 + \frac{3 p_\parallel}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left| \frac{\kappa_\psi}{R B_\chi} X_o - \frac{R B_\chi \kappa_s}{B} U_o \right|^2 \quad (E.3) \\
 & - \frac{1}{2} \left[X_o^* \frac{\partial}{\partial \psi} (p_\parallel + p_\perp) - \frac{R B_\chi}{J B^2} U_o^* \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) \right] \left(\frac{\kappa_\psi}{R B_\chi} X_o - \frac{R B_\chi \kappa_s}{B} U_o \right) \\
 & - \frac{1}{2} \left[X_o \frac{\partial}{\partial \psi} (p_\parallel + p_\perp) - \frac{R B_\chi}{J B^2} U_o \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) \right] \left(\frac{\kappa_\psi}{R B_\chi} X_o^* - \frac{R B_\chi \kappa_s}{B} U_o^* \right) \}.
 \end{aligned}$$

MINIMIZATION OF THE FIELD COMPRESSION TERM

The field compression term in δW_{DA} , which we refer to as δW_C , is to order unity

$$\delta W_C = \frac{1}{2} \int d^3x \left[\tau_0 B^2 \left| iU_1 + \frac{1}{n} \frac{\partial X_1}{\partial \psi} - \frac{1}{J} \left[\frac{\partial}{\partial \chi} \left(\frac{RB}{B^2} \phi U_0 \right) + i n v \frac{RB}{B^2} \phi U_0 \right] + \frac{X_0}{J} \frac{\partial J}{\partial \psi} \right. \right. \\ \left. \left. + \left(1 + \frac{\gamma}{\tau_0} \right) \left(\frac{\kappa \psi}{RB_\chi} X_0 - \frac{RB_\chi \kappa_s}{B} U_0 \right) \right|^2 \right].$$

We notice that the first-order component of the perturbation in the geodesic direction U_1 appears in the energy principle in the field compression term only. It does appear in the other terms, but only to higher order. Therefore, the field compression can be minimized algebraically to zero by taking U_1 to be

$$U_1 = \frac{i}{n} \frac{\partial X_1}{\partial \psi} - i \left(\frac{RB}{B^2} \phi U_0 \right) + \frac{i X_0}{J} \frac{\partial J}{\partial \psi} + i \left(1 + \frac{\gamma}{\tau_0} \right) \left(\frac{\kappa \psi}{RB_\chi} X_0 - \frac{RB_\chi \kappa_s}{B} U_0 \right).$$

Then using Eq. (E.1) for U_0 , the double adiabatic energy principle correct to order unity in $(1/n)$ is

$$\delta W_{DA} = \frac{1}{2} \int d^3x \left\{ \frac{\sigma}{J^2 R^2 B_\chi^2} \left| \frac{\partial X_0}{\partial \chi} + i n v X_0 \right|^2 + \frac{\sigma R^2 B_\chi^2}{J^2 B^2} \left| \frac{1}{n} \frac{\partial}{\partial \psi} \left(\frac{\partial X_0}{\partial \chi} + i n v X_0 \right) \right|^2 \right. \\ \left. + \sigma B^2 \left(1 + \frac{3p_\parallel}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_0} \right) \left| \frac{\kappa \psi}{RB_\chi} X_0 - \frac{i RB_\chi \kappa_s}{n B} \frac{\partial X_0}{\partial \psi} \right|^2 \right. \quad (E.4) \\ \left. - \frac{\kappa \psi}{RB_\chi} \frac{\partial}{\partial \psi} (p_\parallel + p_\perp) |X_0|^2 - \frac{R^2 B_\phi B_\chi \kappa_s}{n^2 J B^3} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) \frac{\partial X_0^*}{\partial \psi} \frac{\partial X_0}{\partial \psi} \right\}$$

$$\begin{aligned}
 & - \frac{i}{2n} \left[\frac{B \phi \kappa \psi}{J B^2 B_\chi} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{R B \chi \kappa s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] X_o^* \frac{\partial X_o^*}{\partial \psi} \\
 & + \frac{i}{2n} \left[\frac{B \phi \kappa \psi}{J B^2 B_\chi} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{R B \chi \kappa s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] X_o^* \frac{\partial X_o}{\partial \psi} \quad \} \quad ,
 \end{aligned}$$

where the first and second terms correspond to the field bending, the third to the anisotropic double adiabatic contribution, and the remaining terms to the drive flute and ballooning modes, the field compression term having been eliminated by the minimization of the energy principle with respect to the first-order component of the perturbation in the geodesic direction.

THE EULER EQUATION

To obtain an Euler equation from the guiding center fluid energy principle δW_{DA} , we need to minimize this term with respect to X_o^* . To perform this minimization, we would like only X_o^* and $\frac{\partial X_o^*}{\partial \chi}$ to appear, so we need to eliminate terms like $\frac{\partial X_o^*}{\partial \psi}$ and $\frac{\partial}{\partial \psi} \left(\frac{\partial X_o^*}{\partial \chi} \right)$ by means of an integration by parts in the ψ variable. As these integrations are done, we neglect contributions of order $(1/n)$, and we assume that the perturbation vanishes at the plasma boundary, which is the case if a wall is present.

The second of the two field bending terms, δW_{B2} , can be written as

$$\delta W_{B2} = \frac{1}{2} \int d^3x \frac{\sigma R^2 B_\chi^2}{J^2 B^2 n^2} \left[\frac{\partial}{\partial \psi} \left(\frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right) \right] \left[\frac{\partial}{\partial \psi} \left(\frac{\partial X_o^*}{\partial \chi} - \text{inv} X_o^* \right) \right] \quad .$$

Through integration by parts we get

$$\begin{aligned}
 \delta W_{B2} = & \frac{1}{2} \int d\phi d\chi \int d\psi \frac{\partial}{\partial \psi} \left[\frac{\sigma R^2 B_X^2}{J B_n^2} \left(\frac{\partial X_o^*}{\partial \chi} - i n v X_o^* \right) \frac{\partial}{\partial \psi} \left(\frac{\partial X_o}{\partial \chi} + i n v X_o \right) \right] \\
 & - \frac{1}{2} \int d\phi d\chi \int d\psi \frac{\partial}{\partial \psi} \left(\frac{\sigma R^2 B_X^2}{J B_n^2} \right) \left(\frac{\partial X_o^*}{\partial \chi} - i n v X_o^* \right) \frac{\partial}{\partial \psi} \left(\frac{\partial X_o}{\partial \chi} + i n v X_o \right) \\
 & - \frac{1}{2} \int d\phi d\chi \int d\psi \frac{\sigma R^2 B_X^2}{J B_n^2} \left(\frac{\partial X_o^*}{\partial \chi} - i n v X_o^* \right) \frac{\partial^2}{\partial \psi^2} \left(\frac{\partial X_o}{\partial \chi} + i n v X_o \right) .
 \end{aligned}$$

The first integral vanishes as the perturbation vanishes at the plasma boundary; the second term is of order $(1/n)$, so we neglect it; and only the third term survives, to make the field bending δW_B

$$\begin{aligned}
 \delta W_B = & \frac{1}{2} \int d^3x \left\{ \left[\frac{\sigma}{J^2 R^2 B_X^2} \left(\frac{\partial X_o}{\partial \chi} + i n v X_o \right) - \left(\frac{\sigma R^2 B_X^2}{J^2 B_n^2} \right) \frac{\partial^2}{\partial \psi^2} \left(\frac{\partial X_o}{\partial \chi} + i n v X_o \right) \right] \right. \\
 & \left. \left(\frac{\partial X_o^*}{\partial \chi} - i n v X_o^* \right) \right\} . \quad (E.5)
 \end{aligned}$$

The anisotropic double adiabatic term δW_E is given by

$$\begin{aligned}
 \delta W_E = & \frac{1}{2} \int d^3x \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left(\frac{\kappa_{\psi}}{R B_X} X_o - \frac{i R B_X \kappa_s}{n B} \frac{\partial X_o}{\partial \psi} \right) \frac{\kappa_{\psi}}{R B_X} X_o^* \\
 & + \frac{1}{2} \int d^3x \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left(\frac{\kappa_{\psi}}{R B_X} X_o - \frac{i R B_X \kappa_s}{n B} \frac{\partial X_o}{\partial \psi} \right) \frac{i R B_X \kappa_s}{n B} \frac{\partial X_o^*}{\partial \psi} .
 \end{aligned}$$

Integrating the second term of δW_E by parts and neglecting terms that are of order $(1/n)$ reduces δW_E to

$$\delta W_E = \frac{1}{2} \int d^3x \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_0} \right) \left[\left(\frac{\kappa_{\psi}}{RB_{\chi}} \right)^2 X_o - \frac{2i\kappa_{\psi}\kappa_s}{nB} \frac{\partial X_o}{\partial \psi} + \left(\frac{RB_{\chi}\kappa_s}{nB} \right) \frac{\partial^2 X_o}{\partial \psi^2} \right] X_o^* \quad (E.6)$$

Finally, the ballooning mode driving term δW_D is

$$\begin{aligned} \delta W_D = \frac{1}{2} \int d^3x \left\{ - \frac{\kappa_{\psi}}{RB_{\chi}} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) X_o X_o^* - \frac{R^2 B_{\phi} B_{\chi} \kappa_s}{n JB^3} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \frac{\partial X_o^*}{\partial \psi} \frac{\partial X_o}{\partial \psi} \right. \\ \left. - \frac{i}{2n} \left[\frac{B_{\phi} \kappa_{\psi}}{JB^2 B_{\chi}} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{RB_{\chi} \kappa_s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] X_o \frac{\partial X_o^*}{\partial \psi} \right. \\ \left. + \frac{i}{2n} \left[\frac{B_{\phi} \kappa_{\psi}}{JB^2 B_{\chi}} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{RB_{\chi} \kappa_s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] X_o^* \frac{\partial X_o}{\partial \psi} \right\} \end{aligned}$$

We now integrate the second and third terms by parts in δW_D , which contain $\frac{\partial X_o^*}{\partial \psi}$, keeping terms only to order unity. The resulting form for δW_D is

$$\begin{aligned} \delta W_D = \frac{1}{2} \int d^3x \left\{ - \frac{\kappa_{\psi}}{RB_{\chi}} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) X_o X_o^* + \frac{R^2 B_{\phi} B_{\chi} \kappa_s}{n JB^3} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \frac{\partial^2 X_o}{\partial \psi^2} \right. \\ \left. + \frac{i}{n} \left[\frac{B_{\phi} \kappa_{\psi}}{JB^2 B_{\chi}} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{RB_{\chi} \kappa_s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] \frac{\partial X_o}{\partial \psi} X_o^* \right\} \quad (E.7) \end{aligned}$$

Using Eqs. (E.5), (E.6), and (E.7), the double adiabatic energy principle can be written as

$$\delta W_{DA} = \frac{1}{2} \int d\phi d\psi \int d\chi \left[A_1 \frac{\partial X_o^*}{\partial \chi} + (-inv A_1 + A_2) X_o^* \right] , \quad (E.8)$$

where

$$A_1 \equiv \frac{\sigma}{JR^2 B_X^2} \left(\frac{\partial X_o}{\partial \chi} + inv X_o \right) - \left(\frac{\sigma R^2 B_X^2}{JB^2 n^2} \right) \frac{\partial^2}{\partial \psi^2} \left(\frac{\partial X_o}{\partial \chi} + inv X_o \right) ,$$

and

$$\begin{aligned} A_2 \equiv & J\sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left[\left(\frac{\kappa \psi}{RB_X} \right)^2 X_o - \frac{2i\kappa \psi \kappa_s}{nB} \frac{\partial X_o}{\partial \psi} + \left(\frac{RB_X \kappa_s}{nB} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right] \\ & - \frac{J\kappa \psi}{RB_X} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) X_o + \frac{R^2 B \phi_X^B \kappa_s}{n^2 B^3} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \frac{\partial^2 X_o}{\partial \psi^2} \\ & + \frac{iJ}{n} \left[\frac{B \phi_X \psi}{JB^2 B_X} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{RB_X \kappa_s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] \frac{\partial X_o}{\partial \psi} . \end{aligned}$$

Note that A_1 can alternatively be written as

$$\begin{aligned} A_1 = & \left(\frac{\sigma B}{R^2 B_X^2} \right) \frac{1}{JR} \left(\frac{\partial X_o}{\partial \chi} + inv X_o \right) - \left(\frac{\sigma R^2 B_X^2}{B n^2} \right) \frac{\partial^2}{\partial \psi^2} \left[\frac{1}{JB} \left(\frac{\partial X_o}{\partial \chi} + inv X_o \right) \right] \\ & - \left(\frac{\sigma R^2 B_X^2}{JB^2 n^2} \right) \frac{\partial (JB)}{\partial \psi} \frac{\partial}{\partial \psi} \left[\frac{2}{JB} \left(\frac{\partial X_o}{\partial \chi} + inv X_o \right) \right] - \left(\frac{\sigma R^2 B_X^2}{J^2 B^3 n^2} \right) \frac{\partial^2 (JB)}{\partial \psi^2} \left(\frac{\partial X_o}{\partial \chi} + inv X_o \right) . \end{aligned}$$

The third term is of order $(1/n)$ and the fourth term is of order $(1/n^2)$ compared to the first two, so we can neglect them. If we define the operator

$$ik_{\parallel} \equiv \frac{1}{JB} \left(\frac{\partial}{\partial \chi} + i\nu \right), \quad (E.9)$$

we then have the identity

$$\begin{aligned} \frac{\sigma}{JR^2 B_{\chi}^2} \left(\frac{\partial X_o}{\partial \chi} + i\nu X_o \right) - \left(\frac{\sigma R^2 B_{\chi}^2}{JB^2 n^2} \right) \frac{\partial^2}{\partial \psi^2} \left(\frac{\partial X_o}{\partial \chi} + i\nu X_o \right) = \left(\frac{\sigma B}{R^2 B_{\chi}^2} - \frac{\sigma R^2 B_{\chi}^2}{Bn^2} \frac{\partial^2}{\partial \psi^2} \right) ik_{\parallel} X_o \\ + O\left(\frac{1}{n}\right). \end{aligned} \quad (E.10)$$

By splitting the double adiabatic anisotropic contribution in A_2 — that is, by taking $\sigma B^2(1 + 3p_{\parallel}/\sigma B^2 - \gamma^2/\sigma\tau_o) = \sigma B^2(1 - \sigma/\tau) + \sigma B^2(\sigma/\tau + 3p_{\parallel}/\sigma B^2 - \gamma^2/\sigma\tau_o)$, where we recognize the first part as the anisotropic part and the second part as the double adiabatic term — and by redistributing the ballooning instability terms so that one can identify a purely poloidal pressure gradient contribution and a purely normal pressure gradient contribution, we obtain A_2 in the form

$$\begin{aligned} A_2 = -J \left(\frac{\kappa_{\psi}}{RB_{\chi}} X_o - \frac{iRB_{\chi} \kappa_s}{Bn} \frac{\partial X_o}{\partial \psi} \right) \left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{\kappa_{\psi}}{RB_{\chi}} \sigma B^2 \left(1 - \frac{\sigma}{\tau} \right) \right] \\ + J \left(\frac{i\kappa_{\psi}}{nRB_{\chi}} \frac{\partial X_o}{\partial \psi} + \frac{RB_{\chi} \kappa_s}{Bn^2} \frac{\partial^2 X_o}{\partial \psi^2} \right) \left[\frac{RB_{\chi}}{JB^2} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) - \frac{RB_{\chi} \kappa_s}{B} \sigma B^2 \left(1 - \frac{\sigma}{\tau} \right) \right] \\ + J \sigma B^2 \left(\frac{\sigma}{\tau} + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma\tau_o} \right) \left[\left(\frac{\kappa_{\psi}}{RB_{\chi}} \right)^2 X_o - \frac{2i\kappa_{\psi} \kappa_s}{nB} \frac{\partial X_o}{\partial \psi} + \left(\frac{RB_{\chi} \kappa_s}{nB} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right], \end{aligned} \quad (E.11)$$

where the first two terms combine the pressure driving contribution with the anisotropic contribution and the last term is the double adiabatic effect. The term $\sigma B^2(\sigma/\tau + 3p_{\parallel}/\sigma B^2 - \gamma^2/\sigma\tau_o)$ can be transformed by algebraic manipulation into $(p_{\perp}^2/3p_{\parallel}\tau_o\tau)(\gamma + 3p_{\parallel}\tau_o/p_{\perp})^2$.

We notice that this term must be positive as $\tau > 0$ for the equilibrium to be well posed.^{39,40} Otherwise, the equilibrium equation for the poloidal flux function is not elliptic. After further algebra, we may write $\sigma B^2(\sigma/\tau + 3p_{\parallel}/\sigma B^2 - \gamma^2/\sigma\tau_0) = (B^2/\tau)(p_{\perp}\sigma + 3p_{\parallel}\tau)^2/(p_{\perp}^2 + 3p_{\parallel}B^2\tau)$.

The minimization of Eq. (E.8) with respect to X_0^* can be accomplished using the standard techniques of the calculus of variations to yield the Euler equation

$$\frac{\partial A_1}{\partial \chi} + \text{inv}A_1 - A_2 = 0, \quad (\text{E.12})$$

which, using the previous relations, becomes

$$\begin{aligned} & B k_{\parallel} \left\{ \frac{\sigma B}{R^2 B_{\chi}^2} \left[1 - \left(\frac{R^2 B_{\chi}^2}{Bn} \right)^2 \frac{\partial^2}{\partial \psi^2} \right] k_{\parallel} X_0 \right\} \\ & - \left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{\kappa \psi}{RB_{\chi}} \sigma B^2 \left(1 - \frac{\sigma}{\tau} \right) \right] \left(\frac{\kappa \psi}{RB_{\chi}} X_0 - \frac{iRB_{\chi} \kappa_s}{Bn} \frac{\partial X_0}{\partial \psi} \right) \\ & + \left[\frac{RB_{\phi}}{JB^2} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) - \frac{RB_{\chi} \kappa_s}{B} \sigma B^2 \left(1 - \frac{\sigma}{\tau} \right) \right] \left(\frac{i\kappa \psi}{nRB_{\chi}} \frac{\partial X_0}{\partial \psi} + \frac{RB_{\chi} \kappa_s}{Bn^2} \frac{\partial^2 X_0}{\partial \psi^2} \right) \\ & + \frac{B^2 (p_{\perp}\sigma + 3p_{\parallel}\tau)^2}{\tau (p_{\perp}^2 + 3p_{\parallel}B^2\tau)} \left[\left(\frac{\kappa \psi}{RB_{\chi}} \right)^2 X_0 - \frac{2i\kappa \psi \kappa_s}{nB} \frac{\partial X_0}{\partial \psi} + \left(\frac{RB_{\chi} \kappa_s}{nB} \right)^2 \frac{\partial^2 X_0}{\partial \psi^2} \right] = 0. \end{aligned} \quad (\text{E.13})$$

From the equilibrium relation

$$\vec{\nabla} \left(p_{\perp} + \frac{B^2}{2} \right) = \hat{b} (\hat{b} \cdot \vec{\nabla}) \left(p_{\perp} + \frac{B^2}{2} \right) + \sigma B^2 \vec{\kappa}, \quad (\text{E.14})$$

we can obtain a relationship between the curvatures and the other equilibrium quantities. In particular,

$$\kappa_{\psi} = \frac{RB\chi}{\sigma B^2} \frac{\partial}{\partial \psi} \left(p_{\perp} + \frac{B^2}{2} \right) , \quad (E.15)$$

$$\kappa_s = \frac{B\phi}{\sigma B^3 J B_{\chi}} \frac{\partial}{\partial \chi} \left(p_{\perp} + \frac{B^2}{2} \right) .$$

If we now consider the guiding center fluid as a limiting case of the guiding center plasma, we can use the relation

$$(\hat{b} \cdot \vec{\nabla})_{p_{\perp}} = \frac{2p_{\perp} - C_1}{2B^2} (\hat{b} \cdot \vec{\nabla})_{B^2} , \quad (E.16)$$

which can be derived from Eqs. (3.4) and (3.5). When this relation is combined with the equilibrium relation

$$(\hat{b} \cdot \vec{\nabla})_{p_{\parallel}} = \frac{p_{\parallel} - p_{\perp}}{2B^2} (\hat{b} \cdot \vec{\nabla})_{B^2} ,$$

it will yield

$$\frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) = (\tau - \sigma) \frac{\partial}{\partial \chi} \left(\frac{B^2}{2} \right) = \left(1 - \frac{\sigma}{\tau} \right) \frac{\partial}{\partial \chi} \left(p_{\perp} + \frac{B^2}{2} \right)$$

after expanding in the orthogonal flux coordinate system.

Now, using the expression for the geodesic curvature in Eq. (E.15), we can obtain the relation

$$\frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) = \sigma B^2 \left(1 - \frac{\sigma}{\tau}\right) \frac{B J B_{\chi}}{B_{\phi}} \kappa_s \quad . \quad (E.17)$$

This implies that the third term of the Euler equation Eq. (E.13) is identically zero. Using Eq. (E.17) in the second term of Eq. (E.13) modifies the Euler equation to become

$$\begin{aligned} & B k_{\parallel} \left\{ \frac{\sigma B}{R^2 B_{\chi}^2} \left[1 - \left(\frac{R^2 B_{\chi}^2}{B_{\parallel}} \right)^2 \frac{\partial^2}{\partial \psi^2} \right] k_{\parallel} X_o \right\} \\ & - \left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{B_{\phi} (\kappa_{\psi} / \kappa_s)}{R B J B_{\chi}^2} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa_{\psi}}{R B_{\chi}} X_o - \frac{i R B_{\chi} \kappa_s}{B n} \frac{\partial X_o}{\partial \psi} \right) \\ & + \frac{B^2 (p_{\perp} \sigma + 3 p_{\parallel} \tau)^2}{\tau (p_{\perp}^2 + 3 p_{\parallel} B^2 \tau)} \left[\left(\frac{\kappa_{\psi}}{R B_{\chi}} \right)^2 X_o - \frac{2 i \kappa_{\psi} \kappa_s}{n B} \frac{\partial X_o}{\partial \psi} + \left(\frac{R B_{\chi} \kappa_s}{n B} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right] = 0 , \end{aligned} \quad (E.18)$$

where the first term corresponds to the field bending, the second term drives ballooning modes, and the last term is the double adiabatic addition. This equation is the analog of the ideal MHD equation first derived by Dobrott et al.,¹⁹ which they solved numerically assuming the perturbation vanished at the inner edge of the torus on every flux surface.

Appendix F

THE CONNOR-HASTIE-TAYLOR TRANSFORMATION

It is extremely difficult to accurately construct a perturbation that is periodic in the poloidal angle variable and that varies slowly along the field lines and rapidly across them in a magnetic field with shear. Even the choice of the perturbation in the eikonal form is inappropriate if the requirement of periodicity in χ is to be satisfied concurrently with $dq/d\psi \neq 0$. This obstacle can be overcome if the perturbation is chosen to vanish at the inner edge of the torus.^{19,21,22} But this arbitrary constraint on the displacement implies that the resulting eigenfunction does not necessarily represent the most unstable perturbation that could be constructed. An alternative approach is to introduce the transformation described by Connor, Hastie, and Taylor,²⁰ which allows us to extend the periodic domain of the poloidal angle into an infinite domain, so that periodicity conditions on the perturbation are no longer required. The perturbation takes the form

$$Y(\psi, \chi) = \frac{1}{2\pi} \sum_m \exp\left(-\frac{2\pi i m \chi}{\phi d\chi}\right) \int_{-\infty}^{\infty} dy \exp\left(\frac{2\pi i m y}{\phi d\chi}\right) \hat{Y}(\psi, y) \quad . \quad (F.1)$$

This transformation can be obtained by expanding the displacement $Y(\psi, \chi)$ in a Fourier series. The Fourier coefficients a_m represent a sequence of points in the complex plane which by analytic continuation can generate a function $a(m)$. The inverse Fourier transform of this function results in Eq. (F.1).

We next outline some of the properties of this transformation. The product of two functions are transformed as

$$a(\psi, \chi) Y(\psi, \chi) = \frac{1}{2\pi} \sum_m \exp\left(-\frac{2\pi i m \chi}{\phi d\chi}\right) \int_{-\infty}^{\infty} dy \exp\left(\frac{2\pi i m y}{\phi d\chi}\right) a(\psi, y) \hat{Y}(\psi, y) \quad , \quad (F.2)$$

where, say, $a(\psi, \chi)$ contains just equilibrium quantities and $Y(\psi, \chi)$ is the perturbation. The derivative of Y with respect to ψ transforms as

$$\frac{\partial}{\partial \chi} [Y(\psi, \chi)] = \frac{1}{2\pi} \sum_m \exp\left(-\frac{2\pi i m \chi}{\oint d\chi}\right) \int_{-\infty}^{\infty} dy \exp\left(\frac{2\pi i m y}{\oint d\chi}\right) \frac{\partial}{\partial y} [\hat{Y}(\psi, y)] \quad (F.3)$$

Using the properties described by Eqs. (F.2) and (F.3), we now proceed to transform the Euler equation, Eq. (3.8). We transform the field bending term as it appears in the left-hand side of Eq. (E.10), rather than in Eq. (3.8), and we have

$$\begin{aligned} & - \frac{1}{J} \left(\frac{\partial}{\partial y} + i n v \right) \left\{ \frac{\sigma B}{J R^2 B_\chi^2} \left[1 - \left(\frac{R^2 B_\chi^2}{B n} \right)^2 \frac{\partial^2}{\partial \psi^2} \right] \left(\frac{\partial \hat{X}_0}{\partial y} + i n v \hat{X}_0 \right) \right\} \\ & - \left[\frac{\partial}{\partial \psi} (p_\parallel + p_\perp) - \frac{B_\phi (\kappa_\psi / \kappa_s)}{R B J B_\chi^2} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) \right] \left(\frac{\kappa_\psi \hat{X}_0}{R B_\chi} - \frac{i R B_\chi \kappa_s}{B n} \frac{\partial \hat{X}_0}{\partial \psi} \right) \\ & + \frac{B^2 (p_\perp \sigma + 3 p_\parallel \tau)^2}{\tau (p_\perp^2 + 3 p_\parallel B^2 \tau)} \left[\left(\frac{\kappa_\psi}{R B_\chi} \right)^2 \hat{X}_0 - \frac{2 i \kappa_\psi \kappa_s}{n B} \frac{\partial \hat{X}_0}{\partial \psi} + \left(\frac{R B_\chi \kappa_s}{n B} \right)^2 \frac{\partial^2 \hat{X}_0}{\partial \psi^2} \right] = 0, \end{aligned} \quad (F.4)$$

in the infinite domain $-\infty \leq y \leq \infty$.

Because the transformed perturbation need not be periodic in the infinite domain, it can be expressed in a quasi-mode or eikonal form¹³

$$\hat{X}_0(\psi, y) = F(\psi, y) \exp\left(-i n \int^y v dy\right), \quad (F.5)$$

where F is a slowly varying function of ψ and y and all the rapid cross-field variations are contained in the exponent.

By using Eq. (F.5) for the perturbation, the field bending term of Eq. (F.4) can be written as

$$\begin{aligned}
 & - \frac{1}{J} \frac{\partial}{\partial y} \left\{ \frac{\sigma}{JR^2 B_X^2} \left[1 + \left(\frac{R^2 B_X^2}{B} \int^y dy \frac{\partial v}{\partial \psi} \right)^2 \right] \frac{\partial F}{\partial y} + \frac{i\sigma}{Jn} \left(\frac{R^2 B_X^2}{B^2} \int^y dy \frac{\partial v}{\partial \psi} \right) \frac{\partial^2 F}{\partial \psi \partial y} \right. \\
 & \quad \left. + \frac{i\sigma}{Jn} \left(\frac{R^2 B_X^2}{B^2} \int^y dy \frac{\partial^2 v}{\partial \psi^2} \right) \frac{\partial F}{\partial y} - \frac{\sigma R^2 B_X^2}{JB^2 n^2} \frac{\partial^3 F}{\partial \psi^2 \partial y} \right\} e^{-in \int^y v dy} .
 \end{aligned}$$

Here, the last three terms are of order $(1/n)$, $(1/n)$, and $(1/n^2)$, respectively, so they are neglected compared with the first term. The pressure gradient driving term and the double adiabatic contribution can be evaluated in a manner similar to the evaluation of the field bending term where any expression of order $(1/n)$ or higher is not kept.

The Euler equation then is given by

$$\begin{aligned}
 & \frac{1}{J} \frac{\partial}{\partial y} \left\{ \frac{\sigma}{JR^2 B_X^2} \left[1 + \left(\frac{R^2 B_X^2}{B} \int^y dy \frac{\partial v}{\partial \psi} \right)^2 \right] \frac{\partial F}{\partial y} \right\} \\
 & \quad \text{field bending} \\
 & + \left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{B_{\phi} (\kappa_{\psi} / \kappa_s)}{RB_X B_X J D_X} \frac{\partial}{\partial y} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa_{\psi}}{RB_X} - \frac{RB_X \kappa_s}{B} \int^y dy \frac{\partial v}{\partial \psi} \right) F \\
 & \quad \begin{array}{cc} \text{normal} & \text{poloidal} \\ \text{pressure} & \text{gradients} \end{array} \quad \begin{array}{cc} \text{normal} & \text{geodesic} \\ \text{curvatures} & \end{array} \\
 & + \frac{B^2 (p_{\perp} \sigma + 3p_{\parallel} \tau)^2}{r(p_{\perp}^2 + 3p_{\parallel} B^2)} \left(\frac{\kappa_{\psi}}{RB_X} - \frac{RB_X \kappa_s}{B} \int^y dy \frac{\partial v}{\partial \psi} \right)^2 F(\psi, y) = 0. \quad (\text{F.6}) \\
 & \quad \text{double} \quad \text{adiabatic}
 \end{aligned}$$

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Appendix G

EQUILIBRIUM EXPRESSIONS IN (ψ, B) SPACE

In orthogonal flux coordinates, we can express

$$\vec{\nabla} p_{\perp} = RB_{\chi} \hat{e}_{\psi} \frac{\partial p_{\perp}}{\partial \psi} \Big|_y + \frac{\hat{e}_{\chi}}{JB_{\chi}} \frac{\partial p_{\perp}}{\partial y} \Big|_{\psi} . \quad (G.1)$$

In nonorthogonal (ψ, B) space,

$$\vec{\nabla} p_{\perp} = RB_{\chi} \hat{e}_{\psi} \frac{\partial p_{\perp}}{\partial \psi} \Big|_B + \frac{1}{B} \frac{\partial p_{\perp}}{\partial B} \Big|_{\psi} \vec{\nabla} \left(\frac{B^2}{2} \right) . \quad (G.2)$$

The parallel pressure can be expanded in a similar manner. Consequently, making use of Eqs. (G.1) and (G.2), we get

$$\begin{aligned} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \Big|_y &= \frac{1}{RB_{\chi}} \hat{e}_{\psi} \cdot \vec{\nabla} (p_{\parallel} + p_{\perp}) \\ &= \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \Big|_B + \frac{1}{B} \frac{\partial}{\partial B} (p_{\parallel} + p_{\perp}) \Big|_{\psi} \frac{(\hat{e}_{\psi} \cdot \vec{\nabla}) R^2}{2RB_{\chi}} . \end{aligned} \quad (G.3)$$

Similarly,

$$\begin{aligned} \frac{1}{J} \frac{\partial}{\partial y} (p_{\parallel} + p_{\perp}) \Big|_{\psi} &= B_{\chi} \hat{e}_{\chi} \cdot \vec{\nabla} (p_{\parallel} + p_{\perp}) \\ &= \frac{1}{2B} \frac{\partial}{\partial B} (p_{\parallel} + p_{\perp}) \Big|_{\psi} (\vec{B} \cdot \vec{\nabla}) B^2 . \end{aligned} \quad (G.4)$$

The normal curvature is

$$\kappa_{\psi} = \frac{RB_{\chi}}{\sigma B^2} \frac{\partial}{\partial \psi} (p_{\perp} + \frac{B^2}{2}) \Big|_y = \frac{1}{\sigma B^2} \hat{e}_{\psi} \cdot \vec{\nabla} (p_{\perp} + \frac{B^2}{2}) .$$

Using Eq. (G.2), we obtain

$$\kappa_{\psi} = \frac{RB_{\chi}}{\sigma B^2} \frac{\partial p_{\perp}}{\partial \psi} \Big|_B + \frac{\tau}{2\sigma R^2 RB_{\chi}} (\vec{\nabla}_{\psi} \cdot \vec{\nabla}) B^2 . \quad (G.5)$$

The geodesic curvature is

$$\kappa_s = \frac{B_{\phi}}{\sigma B^3 JB_{\chi}} \frac{\partial}{\partial y} (p_{\perp} + \frac{B^2}{2}) = \frac{B_{\phi}}{\sigma B^2} (\hat{e}_{\chi} \cdot \vec{\nabla}) (p_{\perp} + \frac{B^2}{2}) ,$$

Using Eq. (G.2), this can be reduced to

$$\kappa_s = \frac{\tau I}{2\sigma B^3 RB_{\chi}} (\vec{B} \cdot \vec{\nabla}) B^2 . \quad (G.6)$$

Using Eqs. (G.1)-(G.6) and after some algebra, the term that drives instabilities in the ballooning mode equation [Eq. (3.11)] becomes

$$\left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \Big|_y - \frac{B_{\phi}}{RB_{\chi}^2 BJ} \frac{\partial}{\partial y} (p_{\parallel} + p_{\perp}) \right] = \left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \Big|_B - \frac{1}{\tau B} \frac{\partial}{\partial B} (p_{\parallel} + p_{\perp}) \frac{\partial p_{\perp}}{\partial \psi} \Big|_B \right] .$$

Since $\sigma = 1 - \frac{1}{B} \frac{\partial p_{\parallel}}{\partial B}$ and $\tau = 1 + \frac{1}{B} \frac{\partial p_{\perp}}{\partial B}$, we can get

$$\left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right]_y - \frac{B \phi \left(\frac{\kappa \psi}{\kappa_s} \right)}{RB_{\chi}^2 B_J} \frac{\partial}{\partial y} (p_{\parallel} + p_{\perp}) = \left(\frac{\partial p_{\parallel}}{\partial \psi} + \frac{\sigma}{\tau} \frac{\partial p_{\perp}}{\partial \psi} \right)_B \quad (G.7)$$

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Appendix H

EXPANSION OF THE SECULAR TERM AND THE CURVATURES FOR NUMERICAL TREATMENT

We try to reduce the numerical error as much as possible by minimizing the number of interpolations from the equilibrium grid onto the flux contours where the ballooning mode equation [Eq. (3.12)] is solved. We need to alter the form of this equation to facilitate the numerics using equilibrium relations. We can write

$$\int_{\chi_0}^y dy \frac{\partial v}{\partial \psi} = \int_{L_0}^{\ell} \frac{d\ell}{JB_{\chi}} \frac{\partial}{\partial \psi} \left(\frac{JB}{R} \phi \right) , \quad (H.1)$$

where ℓ is the poloidal arc length and L_0 is related to the initial angle of integration χ_0 . The operator $\vec{B} \cdot \vec{\nabla} = B_{\chi} \frac{\partial}{\partial \ell}$. The integral [Eq. (H.1)] can be very cumbersome to evaluate unless it is separated into a periodic and a secular part,

$$\int_{\chi_0}^y dy \frac{\partial v}{\partial \psi} = 2\pi N_c q' + \int_{L_0 + N_c \phi d\ell}^{\ell} \frac{d\ell}{JB_{\chi}} \frac{\partial}{\partial \psi} \left(\frac{JB}{R} \phi \right) , \quad (H.2)$$

where $\phi d\ell$ is the poloidal circumference,

$$\frac{\partial q}{\partial \psi} \equiv \frac{1}{2\pi} \oint \frac{d\ell}{JB_{\chi}} \frac{\partial}{\partial \psi} \left(\frac{JB}{R} \phi \right) , \quad (H.3)$$

$L_0 + N_c \phi d\ell \leq \ell \leq L_0 + (N_c + 1) \phi d\ell$, and N_c is the number of circuits of the poloidal domain of integration. The first term of Eq. (H.2) increases with N_c (secular), and the second term is periodic. This

considerably simplifies and reduces the number of operations to evaluate the integral defined by Eq. (H.1).

The integrands of Eqs. (H.2) and (H.3) can be written as

$$\frac{1}{JB_X} \frac{\partial}{\partial \psi} \left(\frac{JB_\phi}{R} \right) = \frac{B_\phi}{RB_X^3} \frac{1}{J} \frac{\partial}{\partial \psi} (JB_X^2) + B_X \frac{\partial}{\partial \psi} \left(\frac{B_\phi}{RB_X^2} \right) \quad (H.4)$$

Since the tensor pressure equilibrium equation is

$$\frac{R^2}{J} \frac{\partial}{\partial \psi} (JB_X^2) = - \frac{R^2}{\sigma} \frac{\partial p_{\parallel}}{\partial \psi} \Big|_B - \frac{1}{2\sigma^2} \frac{\partial}{\partial \psi} (\sigma I)^2 - \frac{1}{\sigma} \vec{\nabla} \psi \cdot \vec{\nabla} \sigma = \Delta^* \psi \quad (H.5)$$

this allows us to express

$$\begin{aligned} \frac{1}{JB_X} \frac{\partial}{\partial \psi} \left(\frac{JB_\phi}{R} \right) &= - \frac{(\sigma I)}{\sigma R (RB_X)^3} \left\{ \frac{(\sigma I)^2}{\sigma^3 B^2} \left[1 - \frac{\sigma^2 R^2 R_X^2}{(\sigma I)^2} \right] \frac{\partial p_{\parallel}}{\partial \psi} \Big|_B - \frac{2R^2 B_X^2}{\sigma B^2} \frac{\partial p_{\perp}}{\partial \psi} \Big|_B \right. \\ &\quad + \frac{1}{B^2} \left(\frac{\tau}{\sigma} - 1 \right) (\vec{\nabla} \psi \cdot \vec{\nabla}) B^2 + \frac{1}{R^2 B_X^2} (\vec{\nabla} \psi \cdot \vec{\nabla}) (\vec{\nabla} \psi)^2 \\ &\quad \left. + \frac{1}{2\sigma^2} \left[1 - \frac{\sigma^2 R^2 B_X^2}{(\sigma I)^2} \right] \frac{\partial}{\partial \psi} (\sigma I)^2 \right\} \quad . \end{aligned}$$

Using the equilibrium equation again to eliminate $\frac{\partial p_{\parallel}}{\partial \psi} \Big|_B$, we get

$$\begin{aligned} \frac{1}{JB_X} \frac{\partial}{\partial \psi} \left(\frac{JB_\phi}{R} \right) &= - \frac{(\sigma I)}{\sigma R (RB_X)^3} \left\{ \frac{2R^2 B_X^2}{(\sigma I)^2} \frac{\partial p_{\perp}}{\partial \psi} \Big|_B + \frac{\sigma^2 R^2}{2(\sigma I)^2} \left(\frac{\tau}{\sigma} - 1 \right) (\vec{\nabla} \psi \cdot \vec{\nabla}) B^2 \right. \\ &\quad \left. - \left[1 - \frac{\sigma^2 R^2 B_X^2}{(\sigma I)^2} \right] \Delta^* \psi + \frac{1}{R^2 B_X^2} (\vec{\nabla} \psi \cdot \vec{\nabla}) (\vec{\nabla} \psi)^2 \right\} \quad . \end{aligned}$$

Finally, substituting for $(\vec{\nabla}\psi \cdot \vec{\nabla})B^2$ from Appendix I and expanding $\Delta^*\psi$ and $\vec{\nabla}\psi$ in cylindrical coordinates, we obtain

$$\begin{aligned}
 \frac{1}{JB_\chi} \frac{\partial}{\partial \psi} \left(\frac{JB_\phi}{R} \right) &= \frac{1}{(\sigma I) R (RB_\chi)^5} \left(R^2 (R^2 B_\chi^2)^2 \frac{\sigma}{\tau} \frac{\partial p_\perp}{\partial \psi} \Big|_B \right. \\
 &- 4\sigma \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial x \partial z} \left[\frac{(\sigma I)^2}{\sigma^2} + \frac{1}{2} \left(1 - \frac{\sigma}{\tau} \right) R^2 B_\chi^2 \right] \\
 &+ \sigma(1 + \epsilon x) \frac{\partial}{\partial x} \left(\frac{1}{1 + \epsilon x} \frac{\partial \psi}{\partial x} \right) \left\{ \frac{(\sigma I)^2}{\sigma^2} \left[\left(\frac{\partial \psi}{\partial z} \right)^2 - \left(\frac{\partial \psi}{\partial x} \right)^2 \right] - R^2 B_\chi^2 \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \frac{\sigma}{\tau} \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \right\} \\
 &+ \sigma \frac{\partial^2 \psi}{\partial z^2} \left\{ \frac{(\sigma I)^2}{\sigma^2} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 - \left(\frac{\partial \psi}{\partial z} \right)^2 \right] - R^2 B_\chi^2 \left[\left(\frac{\partial \psi}{\partial z} \right)^2 + \frac{\sigma}{\tau} \left(\frac{\partial \psi}{\partial x} \right)^2 \right] \right\} \\
 &- \frac{\epsilon \sigma}{1 + \epsilon x} \frac{\partial \psi}{\partial x} \left\{ 2 \frac{(\sigma I)^2}{\sigma^2} \left(\frac{\partial \psi}{\partial x} \right)^2 - R^2 B_\chi^2 \left(1 - \frac{\sigma}{\tau} \right) \left[\frac{(\sigma I)^2}{\sigma^2} + \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \right\} \quad , \quad (H.6)
 \end{aligned}$$

where x and z have been normalized to the half-width a of the equilibrium rectangular box, $R \equiv 1 + \epsilon x$ and $\epsilon = a/R_0$. We evaluate this expression on all grid points within a grid spacing of the flux contours on which the ballooning equation is solved. We then interpolate the values obtained onto the flux surfaces and perform the integrations defined by Eqs. (H.2) and (H.3).

We notice from the driving term of the ballooning mode equation that we need to find expressions for $\kappa_\psi / (RB_\chi)$ and $RB_\chi \kappa_s / B$. Using the expression for the normal curvature written in Appendix G, we have

$$\frac{\kappa_\psi}{RB_\chi} = \frac{1}{\sigma B^2} \frac{\partial p_\perp}{\partial \psi} \Big|_B + \frac{\tau}{2\sigma B^2 R^2 B_\chi^2} (\vec{\nabla}\psi \cdot \vec{\nabla})B^2 \quad .$$

Substituting for $(\vec{\nabla}\psi \cdot \vec{\nabla})B^2$ from Appendix I, we get

$$\frac{\kappa_{\psi}}{RB_{\chi}} = \frac{1}{R^4 B_{\chi}^2} \left[\frac{(\sigma I)^2}{\sigma^2} \Delta^* \psi - R^2 B_{\chi}^2 \Delta \psi + \frac{1}{2} (\vec{\nabla} \psi \cdot \vec{\nabla}) (\vec{\nabla} \psi)^2 \right] .$$

We now expand $\Delta^* \psi$, $\Delta \psi$, and $\vec{\nabla} \psi$ in cylindrical coordinates to obtain

$$\begin{aligned} \frac{\kappa_{\psi}}{RB_{\chi}} = \frac{-1}{R^4 B_{\chi}^2} \left\{ \left(\frac{\partial \psi}{\partial z} \right)^2 \left[(1 + \epsilon x) \frac{\partial}{\partial x} \left(\frac{1}{1 + \epsilon x} \frac{\partial \psi}{\partial x} \right) \right] + \left(\frac{\partial \psi}{\partial x} \right)^2 \frac{\partial^2 \psi}{\partial z^2} \right. \\ \left. + \left[\frac{(\sigma I)^2}{\sigma^2} + \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \frac{\partial \psi}{\partial x} - 2 \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial x \partial z} \right\} . \quad (H.7) \end{aligned}$$

It is of interest to note that this expression is identical to that obtained in ideal MHD.

Similarly, using the expression for the geodesic curvature in Appendix G, we can write

$$\frac{RB_{\chi} \kappa_s}{B} = \frac{\tau(\sigma I)}{2\sigma^2 R^4} (\vec{B} \cdot \vec{\nabla}) B^2 .$$

Substituting for $(\vec{B} \cdot \vec{\nabla}) B^2$ from Appendix I, we obtain

$$\begin{aligned} \frac{RB_{\chi} \kappa_s}{B} = \frac{\sigma I}{(\sigma R B^2) \left[\frac{(\sigma I)^2}{\sigma^2} + \frac{\sigma R^2 B_{\chi}^2}{\tau} \right]} \left\{ \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z} \left[\frac{\partial^2 \psi}{\partial z^2} - (1 + \epsilon x) \frac{\partial}{\partial x} \left(\frac{1}{1 + \epsilon x} \frac{\partial \psi}{\partial x} \right) \right] \right. \\ \left. + \frac{\partial^2 \psi}{\partial x \partial z} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 - \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \right. \\ \left. + \frac{\epsilon}{1 + \epsilon x} \frac{\partial \psi}{\partial z} \left[\frac{(\sigma I)^2}{\sigma^2} + \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \right\} . \quad (H.8) \end{aligned}$$

Expressions (H.7) and (H.8) are interpolated onto the flux contours in the same manner described earlier in this Appendix.

Appendix I

EVALUATION OF $(\vec{\nabla}\psi \cdot \vec{\nabla})B^2$ AND $(\vec{B} \cdot \vec{\nabla})B^2$

We can express

$$(\vec{\nabla}\psi \cdot \vec{\nabla})B^2 = (\vec{\nabla}\psi \cdot \vec{\nabla}) \left\{ \frac{1}{R^2} \left[\frac{(\sigma I)^2}{\sigma^2} + (\vec{\nabla}\psi)^2 \right] \right\} ,$$

or

$$\begin{aligned} (\vec{\nabla}\psi \cdot \vec{\nabla})B^2 &= -\frac{2B^2}{R} (\vec{\nabla}\psi \cdot \vec{\nabla})R - \frac{2(\sigma I)^2}{\sigma^3 R^2} (\vec{\nabla}\psi \cdot \vec{\nabla}\sigma) + \frac{B^2}{\sigma^2} \frac{\partial}{\partial \psi} (\sigma I)^2 \\ &\quad + \frac{1}{R^2} (\vec{\nabla}\psi \cdot \vec{\nabla}) (\vec{\nabla}\psi)^2 . \end{aligned} \quad (I.1)$$

Using the results of Appendix G, we can expand $\vec{\nabla}\psi \cdot \vec{\nabla}\sigma$ as

$$\vec{\nabla}\psi \cdot \vec{\nabla}\sigma = \frac{R^2 B^2}{B} \frac{\partial}{\partial \psi} (p_{\perp} - p_{\parallel}) + \frac{1}{2B^3} \frac{\partial}{\partial B} (p_{\parallel} + p_{\perp}) (\vec{\nabla}\psi \cdot \vec{\nabla})B^2$$

with the pressures expressed as functions of ψ and B . Since $\sigma = 1 - \frac{1}{B} \frac{\partial p_{\parallel}}{\partial B}$ and $\tau = 1 + \frac{1}{B} \frac{\partial p_{\perp}}{\partial B}$, this expression can be written as

$$\vec{\nabla}\psi \cdot \vec{\nabla}\sigma = \frac{R^2 B^2}{B} \frac{\partial}{\partial \psi} (p_{\perp} - p_{\parallel}) + \frac{1}{2B^2} (\tau - \sigma) (\vec{\nabla}\psi \cdot \vec{\nabla})B^2 . \quad (I.2)$$

Consequently, substituting Eq. (I.2) into Eq. (I.1), we get

$$\begin{aligned}
 \left[1 + \frac{(\sigma I)^2}{\sigma^2 R^2 B^2} \left(\frac{\tau}{\sigma} - 1 \right) \right] (\vec{\nabla} \psi \cdot \vec{\nabla}) B^2 &= - \frac{2B^2}{R} (\vec{\nabla} \psi \cdot \vec{\nabla}) R + \frac{2(\sigma I)^2 B^2 \chi}{\sigma^3 B^2} \frac{\partial}{\partial \psi} (p_{\parallel} - p_{\perp}) \\
 &+ \frac{B^2 \chi}{\sigma^2} \frac{\partial}{\partial \psi} (\sigma I)^2 + \frac{1}{R^2} (\vec{\nabla} \psi \cdot \vec{\nabla}) (\vec{\nabla} \psi)^2 .
 \end{aligned}
 \tag{I.3}$$

Using Eq. (I.2), the tensor pressure equilibrium equation can be written as

$$\frac{(\sigma I)^2}{\sigma^3 B^2} \frac{\partial p_{\parallel}}{\partial \psi} = - \Delta^* \psi - \frac{R^2 B^2 \chi}{\sigma B^2} \frac{\partial p_{\perp}}{\partial \psi} - \frac{1}{2\sigma^2} \frac{\partial}{\partial \psi} (\sigma I)^2 - \frac{1}{2B^2} \left(1 - \frac{\tau}{\sigma} \right) (\vec{\nabla} \psi \cdot \vec{\nabla}) B^2 .
 \tag{I.4}$$

Substituting Eq. (I.4) into Eq. (I.3), with some additional algebra we get

$$\frac{\tau}{\sigma} (\vec{\nabla} \psi \cdot \vec{\nabla}) R^2 = - \frac{2B^2}{R} (\vec{\nabla} \psi \cdot \vec{\nabla}) R - \frac{2R^2 B^2 \chi}{\sigma} \frac{\partial p_{\perp}}{\partial \psi} - 2B^2 \Delta^* \psi + \frac{1}{R^2} (\vec{\nabla} \psi \cdot \vec{\nabla}) (\vec{\nabla} \psi)^2 .$$

Since $\Delta^* \psi = \Delta \psi - (\vec{\nabla} \psi \cdot \vec{\nabla} R)/R$, then

$$(\vec{\nabla} \psi \cdot \vec{\nabla}) B^2 = - \frac{2\sigma}{\tau} \left[\frac{R^2 B^2 \chi}{\sigma} \frac{\partial p_{\perp}}{\partial \psi} + B^2 \Delta \psi - \frac{(\sigma I)^2}{\sigma^2 R^2} \Delta^* \psi + \frac{1}{2R^2} (\vec{\nabla} \psi \cdot \vec{\nabla}) (\vec{\nabla} \psi)^2 \right] .
 \tag{I.5}$$

Similarly, the evaluation of $(\vec{B} \cdot \vec{\nabla})B^2$ starts with

$$(\vec{B} \cdot \vec{\nabla})B^2 = (\vec{B} \cdot \vec{\nabla}) \left[\frac{(\sigma I)^2}{\sigma^2 R^2} + \frac{(\vec{\nabla}\psi)^2}{R^2} \right],$$

which becomes

$$(\vec{B} \cdot \vec{\nabla})B^2 = -\frac{2B^2}{R} (\vec{B} \cdot \vec{\nabla})_R + \frac{1}{R^2} (\vec{B} \cdot \vec{\nabla}) (\vec{\nabla}\psi)^2 - \frac{2(\sigma I)^2}{\sigma^3 r^2} (\vec{B} \cdot \vec{\nabla})_\sigma.$$

Expanding the last term of the expression above using Eq. (G.2) of Appendix G, we get

$$(\vec{B} \cdot \vec{\nabla})_\sigma = -\frac{1}{2B^3} \frac{\partial}{\partial B} (p_{\parallel} - p_{\perp}) (\vec{B} \cdot \vec{\nabla})B^2 + \frac{p_{\parallel} - p_{\perp}}{B^4} (\vec{B} \cdot \vec{\nabla})B^2.$$

Since $\frac{\partial p_{\parallel}}{\partial B} = \frac{p_{\parallel} - p_{\perp}}{B}$, then

$$(\vec{B} \cdot \vec{\nabla})_\sigma = \frac{1}{2B^3} \frac{\partial}{\partial B} (p_{\parallel} + p_{\perp}) (\vec{B} \cdot \vec{\nabla})B^2 = \frac{1}{2B^2} (\tau - \sigma) (\vec{B} \cdot \vec{\nabla})B^2.$$

Thus,

$$\left[1 + \frac{(\sigma I)^2}{\sigma^2 R^2 B^2} \left(\frac{\tau}{\sigma} - 1 \right) \right] (\vec{B} \cdot \vec{\nabla})B^2 = -\frac{2B^2}{R} (\vec{B} \cdot \vec{\nabla})_R + \frac{1}{R^2} (\vec{B} \cdot \vec{\nabla}) (\vec{\nabla}\psi)^2. \quad (I.6)$$

The operator $(\vec{B} \cdot \vec{\nabla}) = B_{\chi} \hat{e}_{\psi} \cdot (\hat{e}_{\phi} \times \vec{\nabla})$. Then, expanding (in cylindrical coordinates) $\hat{e}_{\phi} = \hat{e}_x \times \hat{e}_z$, we obtain

$$\begin{aligned}
 (\vec{B} \cdot \vec{\nabla})B^2 &= \frac{2}{R^3 \left[1 + \frac{(\sigma I)^2}{\sigma^2 R^2 B^2} \left(\frac{\tau}{\sigma} - 1 \right) \right]} \left\{ \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z} \left[\frac{\partial^2 \psi}{\partial z^2} - (1 + \epsilon x) \frac{\partial}{\partial x} \left(\frac{1}{1 + \epsilon x} \frac{\partial \psi}{\partial x} \right) \right] \right. \\
 &\quad + \frac{\partial^2 \psi}{\partial x \partial z} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 - \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \quad (I.7) \\
 &\quad \left. + \frac{\epsilon}{1 + \epsilon x} \frac{\partial \psi}{\partial z} \left[\frac{(\sigma I)^2}{\sigma^2} + \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \right\} ,
 \end{aligned}$$

where x and z are normalized to the half-width a of the equilibrium rectangular box, ϵ is the inverse aspect ratio, and $R \equiv 1 + \epsilon x$.

REFERENCES

1. D. W. Swain et al., in Sixth Eur. Conf. on Controlled Fusion and Plasma Physics, Oxford, 1979.
2. L. A. Berry et al., in Sixth Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Berchtesgaden (IAEA, Vienna, 1976), Vol. 1, p. 49.
3. Equipe TFR, in Sixth Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Berchtesgaden (IAEA, Vienna, 1976), Vol. 1, p. 69.
4. J. W. M. Paul et al., in Sixth Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Berchtesgaden (IAEA, Vienna, 1976), Vol. 1, p. 69.
5. V. S. Vlasenkov et al., in Sixth Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Berchtesgaden (IAEA, Vienna, 1976), Vol. 1, p. 85.
6. H. P. Eubank et al., Bull. Am. Phys. Soc. 23, 745 (1978).
7. H. P. Eubank, Princeton Plasma Physics Laboratory, private communication, 1978.
8. J. T. Hogan, Nucl. Fusion 19, 753 (1979), and private communication, 1979.
9. L. Dupas, Bull. Am. Phys. Soc. 23, 914 (1978) (post-deadline paper).
10. G. Tonon et al., Commission de l'Energie Atomique Report EUR-CEA-FC 914 (1977).
11. O. Eldridge, Oak Ridge National Laboratory, private communication, 1979.
12. D. J. Sigmar and H. C. Chan, Nucl. Fusion 18, 1569 (1978).
13. K. V. Roberts and J. B. Taylor, Phys. Fluids 8, 315 (1965).
14. M. N. Rosenbluth and C. L. Longmire, Ann. Phys. 1, 120 (1957).
15. D. Berger et al., in Sixth Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Berchtesgaden (IAEA, Vienna, 1976), Vol. 2, p. 411.
16. A. M. M. Todd et al., Phys. Rev. Lett. 38, 826 (1977).
17. G. Bateman and Y-K. M. Peng, Phys. Rev. Lett. 38, 829 (1977).

18. C. H. An and G. Bateman, Phys. Fluids 22, 1517 (1979).
19. D. Dobrott et al., Phys. Rev. Lett. 39, 938 (1977).
20. J. W. Connor, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. 40, 396 (1978).
21. B. Coppi, Phys. Rev. Lett. 39, 938 (1977).
22. P. Rutherford et al., in Third Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Novosibirsk (IAEA, Vienna, 1968), Vol. 1, p. 376.
23. G. Bateman and D. B. Nelson, Phys. Rev. Lett. 41, 1804 (1978).
24. M. S. Chu et al., Phys. Rev. Lett. 41, 247 (1978).
25. P. J. Fielding and F. A. Haas, Phys. Rev. Lett. 41, 801 (1978).
26. P. H. Rutherford, L. Chen, and M. N. Rosenbluth, Princeton Plasma Physics Laboratory Report PPPL-1418 (1978).
27. C. Mercier and M. Cotsaftis, Nucl. Fusion 1, 12 (1961).
28. J. W. Connor and R. J. Hastie, Phys. Fluids 19, 1727 (1976).
29. J. D. Callen and R. A. Dory, Phys. Fluids 15, 1523 (1972).
30. J. G. Cordey and M. J. Houghton, Nucl. Fusion 13, 215 (1973).
31. U. Buneman, Stanford University Report SUIPR 294 (1969).
32. H. Grad, Phys. Fluids 9, 225 (1966).
33. M. D. Kruskal and C. R. Oberman, Phys. Fluids 1, 275 (1958).
34. M. N. Rosenbluth and N. Rostoker, Phys. Fluids 2, 23 (1959).
35. G. F. Chew, M. L. Goldberger, and F. E. Low, Proc. R. Soc., London, Ser. A, 236, 112 (1956).
36. D. B. Nelson, G. O. Spies, and C. L. Hedrick, Phys. Fluids 21, 1742 (1978).
37. P. J. Fielding and F. A. Haas, Nucl. Fusion 19, 855 (1979).
38. A. Sestero and A. Taroni, Nucl. Fusion 16, 164 (1976).
39. H. Grad, in Symposia in Applied Mathematics (American Mathematical Society, 1967), Vol. 18, p. 162.
40. H. Grad, Phys. Fluids 9, 498 (1966).

41. J. G. Cordey, Nucl. Fusion 16, 499 (1976).
42. D. B. Nelson, Office of Fusion Energy, U. S. Dept. of Energy, private communication, 1979.
43. J. W. Connor and R. J. Hastie, Phys. Rev. Lett. 33, 202 (1974).
44. L. F. Shampine and M. K. Gordon, Computer Solutions of Ordinary Differential Equations: The Initial Value Problem (W. H. Freeman and Co., San Francisco, 1975).
45. D. Lortz and J. Nuhrenberg, Phys. Lett. A, 68, 49 (1978).
46. A. Sykes et al., in Seventh Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Innsbruck (IAEA, Vienna, 1978), Vol. 1, p. 625.
47. C. Mercier, Plasma Phys. 21, 589 (1979).
48. B. Coppi et al., Nucl. Fusion 19, 715 (1979).
49. D. A. Monticello et al., in Proc. 1979 Sherwood Meeting on Theoretical Aspects of Controlled Thermonuclear Research (1979), p. 2B12.
50. A. Ferreira et al., in Proc. 1979 Sherwood Meeting on Theoretical Aspects of Controlled Thermonuclear Research (1979), p. 2A1.
51. A. Sykes, J. A. Wesson, and S. J. Cox, Phys. Rev. Lett. 39, 757 (1977).
52. J. L. Dunlap et al., Bull. Am. Phys. Soc. (1979).
53. L. A. Charlton et al., Phys. Rev. Lett. 43, 1395 (1979).
54. G. E. Guest and C. L. Hedrick, Oak Ridge National Laboratory Report ORNL/TM-3943 (1972).
55. J. B. Taylor and R. J. Hastie, Phys. Fluids 8, 323 (1965).

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